The Late-Twentieth Century Resolution of a Mid-Nineteenth Century Dilemma Generated by the Eighteenth-Century Experiments of Ernst Chladni on the Dynamics of Rods

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I. The Eighteenth-Century Experiments of Ernst Chladni

In musical parlance, an organ pipe about 2 feet long and open at both ends will sound at ut_4 or middle C. An organ pipe closed at one end is known musically as a "gedakt." If such a pipe is approximately as long as a 2-foot pipe open at both ends, it will sound one octave below middle C, at ut_3.[1] In 1787 ERNST CHLADNI[2] was convinced that to understand the dynamics of rods, it was essential to demonstrate in the laboratory that the longitudinal vibration of a solid rod clamped at one end and free at the other would parallel the difference in response between the gedakt and diapason stops of the organ, that is, it would sound an octave lower than a rod of the same length with both ends free.[3]

A rod with both ends free will sound its fundamental longitudinal frequency when held at its center and struck axially at one end. There is very different effect when one end of the rod is fixed by clamping and the other end is struck axially. In concept, the sound emitted should resemble that for the organ pipe with one end closed, by sounding one octave lower. Actually, a rod clamped at one end and struck axially at the other emits only a dull clunk.

Again exercising the ingenuity he had shown when he had introduced sand figures on vibrating plates, CHLADNI firmly grasped the clamped rod in a cloth impregnated with rosin and rubbed it lengthwise at an antinode, causing it to

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[1] This paper is based upon a lecture given at the *First Rutgers Conference on Theoretical Mechanics: The Dynamics of rods*; August 24–27, 1990.

[2] CHLADNI [1787, 1802].

[3] As LORD RAYLEIGH and others discovered a century later, at the open end of an unflanged tube, impedance introduces what is now known as an "end leakage correction factor". Given the wave speed C, to obtain a "theoretical" frequency, $f = C/2L$, requires a small adjustment in the length L that depends upon the radius. In order for the closed-end pipe of the gedakt stop to sound an octave below that of an "adjusted" open-end pipe, lengths differ by an amount again dependent solely upon the radius of the open-end pipe. In contrast, within the definition of a rod, the fundamental longitudinal frequency of the solid rod free at both ends does not depend upon its radius.

sing. It sang at a fundamental longitudinal frequency that indeed was one octave below the lowest frequency obtained when a rod of the same length and free at each end is struck axially.

As CHLADNI explained in describing his invention of a musical instrument known as a *euphon*, wet fingers were sufficient to set in motion a glass rod securely clamped at one end. The euphon, described by contemporaries as melodious, was constructed of 42 horizontal glass rods.

> "In the back part of the assembly of rods was a sounding board divided in the middle through which the tubes passed. The whole notes were of dark green glass while the half tones were of a kind of milky white glass."[4]

For what was, in fact, the first direct measurement of the modulus[5] E of an elastic bar by means of longitudinal vibration, CHLADNI in 1787, for his rods with one end fixed, chose a length the same as that of an organ pipe with one end closed that emits sound at ut_3, *i.e.*, approximately 0.6 meter or two feet. He tabulated his data in musical notation, comparing the velocity of sound in each solid to that in air. The nearly 30 solids he examined ranged from whalebone and several woods, to copper, brass, and steel. This, of course, was not a comparison of free-field wave speeds in the two media.

In 1816 JEAN BAPTISTE BIOT in his classic treatise on physics[6] converted CHLADNI's velocity ratios to E moduli of a linearly elastic bar. BIOT's calculations required that he estimate the density of CHLADNI's diverse specimens tested three decades earlier. Table I gives CHLADNI's list of measurements in musical terms, from which BIOT made his calculation of EULER's bar modulus E.[7]

That a rod clamped at one end and free at the other would sound one octave below a rod with both ends free was what CHLADNI had expected to hear. What he did not expect, gives his experiment on a rod an importance that is of great interest even now, namely, that a rubbed rod with one end fixed and the other end free sounds not only an octave below that of a free rod, but also emits a second frequency far below that for either the rod with both ends free or the rod with one end clamped.

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4 For a detailed description of the Euphon, see *Magazin für das Neueste aus der Physik, IX*, Part 4, p. 100, (1796) or see *The Philosophical Magazine, II*, pp. 391–398 (1798).

5 For a given solid the constant E of a linearly elastic body in modern form depends neither upon the area of the individual specimen nor upon its density. Without historical accuracy or experimental foundation, in British tradition the modulus E has been attributed to THOMAS YOUNG, presumably based upon his treatise in 1807. In fact, in that course of lectures, YOUNG introduced a "height of the modulus" which depends upon the density of the body, and a "weight of the modulus" which depends upon the cross-section of the area. YOUNG never claimed that he had introduced the elastic constant E. The modern form, in fact, is attributable to LEONARD EULER in conception and GIORDANO RICCATI in experiment, both writing many decades before THOMAS YOUNG's treatise appeared. See BELL [1973] *Handbuch der Physik*, VIa/1, Springer-Verlag, pp. 184–191.

6 BIOT [1816] *Traité de Physique Expérimental et Mathématique*, Paris.

7 *Ibid.*

Table I

Type of rod	N' (sound in rod)	Relative velocities[a] N'/ut_3	Type of rod	N' (sound in rod)	Relative velocities[a] N'/ut_3
Whalebone	la_5	6.66	Mahogany approx.	$\mathrm{si}_6^{\flat}$	14.40
Tin	si_5	7.50	Ebony approx.	$\mathrm{si}_6^{\flat}$	14.40
Silver	re_6	9.00	Hornbeam approx.	$\mathrm{si}_6^{\flat}$	14.40
Walnut	fa_6	10.66	Elm approx.	$\mathrm{si}_6^{\flat}$	14.40
Yew[b]	fa_6	10.66	Alder approx.	$\mathrm{si}_6^{\flat}$	14.40
Brass	fa_6	10.66	Birch approx.	$\mathrm{si}_6^{\flat}$	14.40
Oak	fa_6	10.66	Linden almost	si_6	15.00
Plum	fa_6	10.66	Cherry	si_6	15.00
	mi_6	10.00	Willow	ut_7	16.00
Pipe stem	and		Pine[c]	ut_7	16.00
	sol_6	12.00	Glass	ut_7	16.66
Copper about	sol_6	12.00	Iron or Steel	ut_7	16.66
Pear	$\mathrm{sol}^{\sharp}{}_6$	12.50	Fir	$\mathrm{ut}^{\sharp}{}_7$	
Copper beach	$\mathrm{sol}^{\sharp}{}_6$	12.50	or almost	re_7	18.00
Maple	la_6	13.33			

[a] ut_3 is the tone of the air column.

[b] If the fibers of this wood had been exactly straight, the sound would have been a bit higher.

[c] If the fibers of this wood had been less straight, the sound would have been one-third lower.

This second frequency later became known as the "deep tone." It sounds at the same time or sometimes intermittently with the longitudinal vibration in tension and compression. It has a frequency approximately two octaves below that of the rod with both ends free; thus it sounds approximately one octave below the frequency of the longitudinal vibration of the same rod with one end clamped and the other end free. That this dual response was heard only in a rubbed rod clamped at one end and free at the other, puzzled two generations of experimentists before 1850. In 1851, one effort to explain the origin of the "deep tone" provided the mid-nineteenth century dilemma to which I refer in the title of this article.

When a strong grasping force from a cloth impregnated by rosin is directed axially along a rod with one end tightly clamped and the other end free, a stopped motion of dry friction is introduced. Such a combination provides an initial condition which induces the axially symmetric radial displacement that characterizes a longitudinal shear wave in a rod, irrespective of the shape of the cross-section. In the eighteenth century, initial conditions were found in the laboratory for inducing flexural, longitudinal, and torsional modes of vibration. They gave the impetus for a now legendary wealth of accompanying analysis. If, in that same century, the deep tone had been recognized both by experimentists and by theorists as laboratory evidence for still another simple mode of vibration, it is reasonable to conclude that theory and experiment on the dynamics of rods in the nineteenth century, and textbooks in the twentieth century, would have included an equally interesting, fourth elemental state of vibration, *i.e.*, a simple, longitudinal shear wave propagating in the axial direction with axially symmetric radial displacement.

With a little care as to the choice of a fine sand or powder and the manner of adapting a violin bow so as to vibrate the edge of a thin plate, and with knowledge of where to apply it, anyone interested in repeating precisely CHLADNI's celebrated experiments on plates can do so without much trouble. One is rewarded with the dramatically sudden appearance of the "CHLADNI figures" that fascinated audiences during CHLADNI's lecture tours in the late eighteenth and early nineteenth centuries, and that at the turn of the twenty-first century still are celebrated in the history of experiment.

In marked contrast to the easy production of sound when a wet finger is rubbed around the rim of a partially full wine glass, and resembling the difficulty experienced when a dry finger replaces the wet one, it can be frustratingly difficult to try to repeat CHLADNI's experiment on a clamped rod rubbed axially by a cloth impregnated with rosin.

In an issue of the *Annales de Chimie et de Physique* in 1820, the experimentist FÉLIX SAVART, unable himself to perform CHLADNI's experiment on a rod with one end fixed and the other end free, publicly accused CHLADNI of having fabricated his data and, despite earlier published descriptions and available musical commentary on the instrument, claimed that CHLADNI's glass-rod euphon was mythical.[8] Said SAVART, on rejecting CHLADNI's parallel between the dynamics

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[8] The euphon was but an extension of a well known principle. The harmonica, which produced musical sounds by means of drinking glasses touched with moistened fingers, is said to have been introduced by an Irishman named RICHARD PUCKERIDGE before the middle of the 18th century. The glasses were placed on a table, their pitch fixed by the quantity of water each contained. "There are glimpses, too, of family music. In [BENJAMIN] FRANKLIN's household Scotch songs were sung to the accompaniment of his daughters on the harpsichord and of his own 'armonica.'" [Chapter on "Pre-Revolutionary Culture," from *The Revolutionary Generation*, 1763–1790, by EVARTS BOUTELL GREENE, p. 151: *A History of American Life*, *IV*, edited by ARTHUR M. SCHLESINGER and DIXON RYAN FOX. Macmillan, (1943).] On July 13, 1762, BENJAMIN FRANKLIN, in a letter to Father GIAMBATISTA BECCARIA, professor of physics at the University of Torino, provides a description of his invention of an improved version of PUCKERIDGE's glass harmonica: "The advantages of this instrument are, that its tone is incomparably sweet beyond those of any other; that they may be swelled and softened at pleasure by stronger or weaker pressure of the finger, and continued to any length; and that the instrument, being once well tuned, never again wants tuning. In honor of your musical language, I have borrowed from it the name of this instrument, calling it the Armonica." The range, FRANKLIN tells us, was three octaves with the lowest note at G below middle C. (See *The Writings of Benjamin Franklin*, ALBERT SMYTH, editor, N.Y. *III*, p. 163, *V*, p. 451 [1907]. There is an Italian translation of this in *Memorie Istoriche intorno gli studi del Padre Giambatista Beccaria* [1783].) Drawing from the same sources, writers near the end of the nineteenth century tell us again about this modification of the primitive harmonica: "Instead of fixing the glasses he [FRANKLIN] made them rotate round a spindle set in motion by the player's foot by means of a treadle. The edge of the glasses

of rods and the acoustics of organ stops: "I admit that however careful I was, I was never able to repeat this experiment even when using quite long rods."[9]

Such a sequence of unfounded verbal attacks on experimental discoveries in continuum mechanics has been all too common; originality in the laboratory often has been rebuffed or disparaged.

History abounds with examples of unwarranted rejection of valid experiments. Only in hindsight do we learn that a good nonlinear ear is required to hear HERMANN HELMHOLTZ's summation tones in musical acoustics, an acoustical property universal among musicians but obviously not a common characteristic of the ears of many, but fortunately not all, physicists since the 1850's.[10] Only in hindsight, too, do we learn that patience and knowledge, ignored by his numerous contemporary discreditors, were required to reproduce LÉON FOUCAULT's pendulum experiment in the mid-nineteenth century.[11] Another

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by the same means passed through a basin of water, the pitch henceforth being determined by the size of the glasses [hemispheres] alone. The player touched the brims of the revolving glasses with his finger, his task being further facilitated by the scale of colour which FRANKLIN adopted in accordance with the musical gamut. Thus C was red, D orange, E yellow, F green, G blue, A indigo, and B violet. The black keys of the piano were represented by white glasses. The instrument thus improved became very fashionable in England, and a Miss DAVIS, a relation of FRANKLIN's, became a celebrated harmonica player who performed at numerous concerts with great applause. It is interesting to know that the great composer GLÜCK was a virtuoso on the musical glasses in their earlier form, which he played, according to a contemporary advertisement, at the Haymarket Theatre, April 23, 1746. He even seems to have claimed the instrument as his own invention, and promises to perform upon it whatever may be done on a violin or harpsichord.'" [From *Encyclopaedia Britannica*, ninth edition, *XI* (with American revisions and additions) Chicago: R. S. Peale Company [1892], p. 481.]

9 SAVART [1820].

10 Aural harmonics are a subjective measure of the phenomena of summation and difference tones. As a small sample of the difference of opinion among physicists I quote titles from the "Letters to the Editor" section of a single issue in June 1957 of *The Journal* of the Acoustical *Society of America*: "Aural harmonics are Fictitious"; "On the Inadequacy of the Method of Beats as a Measure of Aural Harmonics"; "In Support of an 'Inadequate' Method for Detecting 'Fictitious' Aural Harmonics."

11 See PAUL ACLOQUE's monograph [1981] appraising FOUCAULT's research. (And see the review of this book by BELL, *Isis*, **73**: 3: 268, [1982]). FOUCAULT presented the results of his experiment to the French Academy on February 3, 1851 and demonstrated the experiment to the general public in the Pantheon in May, 1851. FOUCAULT's experiment which demonstrated the rotation of the earth aroused his contemporaries to publish over 60 papers during that same year. There were debates among the theorists who adopted opposing analytical approaches. There was discord among the experimentists, some of whom, not appreciating the demanding requirments of FOUCAULT's experiment on the pendulum, obtained conflicting results. Thus was generated the heated controversy that dominated the remaining 17 years of FOUCAULT's life of only 47 years. Before publishing his first data on the pendulum, FOUCAULT had determined with great precision the influence of errors introduced by the use of wires that were elastically anisotropic, imperfectly circular, or improperly mounted at their point of attachment. He had painstakingly ascertained the proper conditions by the ingenious use of an experiment anal-

post hoc historical perspective: we have to reexamine data from his often ingenious, unfamiliar approaches to experiment in order to understand why GUILLAUME WERTHEIM's many different measurements of POISSON's ratio disposed of the POISSON-CAUCHY uniconstant theory of elasticity. For his nineteenth century contemporaries in experiment and theory who resented the suggestion of any impediment to a popular theory, WERTHEIM's experiments demanded an insight that their preconceptions precluded.[12] In 1794, a quarter of a century before SAVART's attack upon the validity of his experiments on the dynamics of rods, CHLADNI had suffered an earlier interval of vilification.[13] He was accused of being a combination of atheist and heretic because he had had the temerity to propose in print that "shooting stars" and meteoric rocks were two manifestations of the same phenomenon. Nearly fifty years before it became regarded as an incontrovertible fact, CHLADNI had proposed that the phenomenon had galactic origins. In 1794 he first presented the arguments that eventually, by the late 1830's, convinced the scientific community and gradually the public in general, that "shooting stars" were not a further example of BENJAMIN FRANKLIN's "electricity in the sky", nor were meteorites a form of rock ejected either from Mt. Vesuvius or, as PIERRE LAPLACE and others insisted, from volcanoes on the moon. Instead, said the perceptive CHLADNI, both were galactic debris from interstellar space.

In response to adverse criticism from scholar and non-scholar alike, CHLADNI summed his position in 1796:

> "Some critics, as well as others, have ridiculed my singular hypothesis, or condemned it altogether; but no one has yet confuted my principles, or given any other explanation that corresponds as well with the facts."[14]

Clearly, in 1794 CHLADNI foresaw a skeptical reaction to his theses about celestial matters. In his second article he stated:

> "For this reason, after I had written the Treatise on the Mass of Iron discovered by Professor Pallas, I hesitated whether I should publish it, because I expected I should meet with considerable opposition."[15]

We may mark still another pertinent episode in CHLADNI's stormy career as a truly outstanding experimentist in continuum mechanics. In 1811, and repeatedly during the next forty years, CHLADNI's experiments of 1787 on the dynamics of rods were injustly challenged[16] on the basis of an experiment by JEAN BAPTISTE

ogous to his experiment on the pendulum. The details of that elegant experiment, more easily controlled in the laboratory than the experiment on the pendulum, were published posthumously through the combined efforts of his widow and his mother. The experiment dealt with the stability of a vibrating cantilever beam with a rotating base.

[12] See BELL [1973] Section 3.16 and *circa*.

[13] CHLADNI [1794].

[14] CHLADNI, Professor VOIGT's *Magazin für das Neueste aus der Physik*, *IX*, [1796]. Quotation is from *Philosophical Magazine II*, 229 [1798].

[15] CHLADNI, Professor VOIGT's *Magazin für den Neuesten Zustand der Naturkunde*, [1797]. Quotation from *Philosophical Magazine*, *II*, 345, [1798].

[16] BIOT [1809].

BIOT in 1807. Controversy arose primarily from the fact that CHLADNI's ratio of measured velocities of sound in iron to those in air was 16.66, while a ratio of 10.5 was obtained from measured travel times, and hence wave speeds, by BIOT on a kilometer of Parisian cast iron water pipe.[17]

Today we know that both measurements were accurate; the materials were not the same as had been tacitly assumed.[18] Despite the lasting acclaim for his celebrated "CHLADNI figures" in his experiments on the vibration of thin plates and the impact they had in the nineteenth century in motivating theories for the vibration of elastic plates, controversies over his observations on the dynamics of rods engaged successive participants for decades. In no small measure the controversies caused CHLADNI's accurate and detailed experiments on the dynamics of rods in 1787 to lie in limbo well before the end of the nineteenth century.

In 1822, two years after SAVART's published assault, in the same journal CHLADNI bitterly and in vain responded to the unfounded criticism[19]. He described the occasions in which he had demonstrated his experiments on the dynamics of rods to savants such as Professor GILBERT of GILBERT's *Annalen der Physik und Chemie*. CHLADNI further informed his readers that although one of his euphons had been shattered some years earlier in a disastrously jolting stagecoach traveling to Paris, where he was to give a musical performance, a euphon still was available for SAVART to inspect in CHLADNI's home.

In 1837, finally having mastered CHLADNI's technique that utilized a cloth impregnated with rosin and having learned how to clamp a glass rod properly, with mention neither of CHLADNI nor of CHLADNI's published advice to him regarding the latter, SAVART described the discovery of the deep tone as if it were his own!

Almost parenthetically, SAVART referred to an experimentist named SAINT-ANGE who in 1820 had remarked that a low sound was emitted at the moment of fracture produced by rubbing a glass rod. It is more than a little ironic that SAVART, ignoring the comments he had made years earlier on the mythical nature of CHLADNI's euphon, instead dwelt at length upon a detailed description of his [SAVART's] own use of wet fingers to produce vibrations in glass rods clamped at one end and free at the other.

Incredible as it may seem, SAVART infers that except for SAINT ANGE's passing comment on fracture 17 years earlier, he himself in 1837 not only had provided the first laboratory study of the deep tone, but had discovered its existence.[20]

In 1851 WERTHEIM described the deep tone as a phenomenon "known to everyone working in this type of experiment." He observed further "that the deep tone was a common source of fracture when glass or crystal was rubbed longitudinally."[21] It is indeed difficult and, in fact, impossible for SAVART or anyone else to believe that from 1787 until long after the turn of the century, during CHLADNI's

[17] BELL [1973], section 3.8, pp. 191–196.
[18] *Ibid*, pp. 404–405.
[19] CHLADNI [1822].
[20] SAVART [1837].
[21] WERTHEIM [1851]; see also BELL [1973], Section 3.18 and *circa*.

extensive and detailed study of the vibration of rods with one end fixed and the other end free, he had never heard the deep tone. When glass rods are rubbed with wet fingers, it is more difficult to avoid the deep tone than to produce it.

The failure of glass rods during rubbing is described by CHLADNI as a prime hazard for the performer when playing the euphon. Such a fracture in glass rods, as all who have performed CHLADNI's rod experiments have noted, is commonly accompanied by either the sudden appearance of the deep tone or a sudden increase in its intensity. Rubbed axially, glass tubes fracture when a shear mode of deformation increases, a mode that corresponds to the presence of an axially symmetric, longitudinal shear wave.

Because the measured deep tone has the same frequency for rods of circular or of rectangular cross-section, it is curious that SAVART would speculate on the existence of an unknown lateral mode of vibration, somehow not dependent upon the lateral dimensions of the rod. In fact, whether the cross-section is circular or rectangular, the frequency of the deep tone is the same!

Also omitted from SAVART's discussion was why the initial condition of rubbing was essential to excite the oscillation at the lower frequency. The deep tone was not heard in a struck rod, whatever was the direction of the applied force.

When in the 1990's we look back on SAVART's study of 1837, we see that SAVART's only contributions to understanding this dilemma of the midnineteenth century that had been generated by the elegant experiments of CHLADNI 50 years before, were threefold: his reference to the harsh sound sometimes emitted by the deep tone; his observation that when a rod is fixed at one end and free at the other, the deep tone often is accompanied by a diminution in the intensity of the longitudinal tone an octave above; and his prescription for making glass rods fracture by increasing the intensity of the deep tone. As we shall see, the dissonant tone arises from the fact that POISSON's ratio does not have the same value for all solids. From WERTHEIM's detailed research, based upon his experiments reported in 1851, we know that only for solids having a POISSON's ratio of 1/3, the combination of the axial tone and the deep tone (the former produced by longitudinal tension and compression) will sound at a true musical octave.

II. The Mid-Nineteenth Century Dilemma

In 1849 GUILLAUME WERTHEIM, the foremost experimentist in solid mechanics in the nineteenth century, characteristically acknowledging his debt to the genius of CHLADNI six decades earlier, undertook the first direct laboratory study of the deep tone emitted while rubbing rods with one end fixed and the other end free, as in CHLADNI's experiment. Using needles to record oscillations on a coated glass plate for rods having a square cross-section, WERTHEIM measured the motion perpendicular to all four sides. He found that the deep tone indeed is related to lateral motion in the rod. By comparing measurements in orthogonal directions, WERTHEIM demonstrated that, contrary to what would be expected for a flexural mode of vibration as proposed by SAVART, the radial displacements for the deep tone are axially symmetric.

In 1851, on the basis of these experiments, WERTHEIM suggested that there was a previously unsuspected free mode of vibration in rods.[22] It was a longitudinal shear wave, propagating in the axial direction with a speed 1/2 that of the longitudinal wave in a bar, namely, $[E/\varrho]^{\frac{1}{2}}$, and having axially symmetric radial displacement. This longitudinal shear wave proposed 64 years after the laboratory observation of the initial condition that generated it, is kinematically *unrelated* to the long-studied torsional shear wave in an elastic rod; the origins of the latter in both experiment and theory date from CHARLES COULOMB in 1784.[23]

Referring to the known relations among elastic constants, one finds that for an isotropic, linearly elastic solid, the ratio of the free-field transverse wave speed, C_2, to the free-field dilatational wave speed, C_1, may be determined from the value of POISSON's ratio alone: $C_2/C_1 = [(1-2\nu)/(1+\nu)]^{\frac{1}{2}}$. If $\nu = \frac{1}{3}$, this ratio has the value $\frac{1}{2}$.

For axial wave propagation in a rod, WERTHEIM assumed that the same ratio, C_2/C_1, holds for the speed of this new longitudinal shear wave and the speed of the familiar longitudinal wave of tension and compression.

If POISSON's ratio differs from $\frac{1}{3}$ as, for example, in steel, for which $\nu = 0.296$, or brass, for which $\nu = 0.350$, the deep tone becomes a sharp or flat octave of the longitudinal frequency, *i.e.*, a mis-tuned octave consistent with SAVART's observation of dissonance.

For glass, however, with an average POISSON's ratio of $\nu = 0.225$ provided by the definitive optical experiments of CONSTANTIN STRAUBEL in 1899,[24] the ratio of the speeds of the deep tone and the longitudinal tone is $\frac{2}{3}$, musically a just fifth! An interval of a just fifth produces an *aural* harmonic in the form of a difference tone known as Tartini's beat, which provides an additional faint sound one octave below the deep tone itself. We thus have a harmonic triad of the fundamental, a just fifth, and an octave, compatible with the reputed melodious timbre of the euphon.

In a report on WERTHEIM's experiments written by AUGUSTIN CAUCHY[25] that CAUCHY, HENRI REGNAULT, and JEAN DUHAMEL published in the *Comptes Rendus* of the French Academy in 1851, all three accepted WERTHEIM's experiment of 1849 on the deep tone as demonstrating that in a rod there could indeed exist a longitudinal shear wave with axially symmetric, radial displacement. In the face of such distinguished support one may wonder why such an important new development in the dynamics of rods — a hitherto unknown elemental mode of vibration — did not arouse the immediate interest of other informed theorists, and why, within less than a decade, the experiments, a proposed explanation for them, and even the existence of the deep tone, ceased to interest anyone. I have been unable to find any reference to the deep tone after 1851 in either experiment or theory.

Explanation for the demise does not lie in the quality of the experiments or the experimentists, for both CHLADNI and WERTHEIM had been admired scholars,

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[22] *Ibid.*

[23] COULOMB [1784].

[24] STRAUBEL [1899] These measurements provide a classic example of excellence.

[25] CAUCHY [1851].

regarded as outstanding laboratory savants in solid mechanics during their respective times. We already have suggested that the controversy that accompanied CHLADNI's experiments on rods was one factor in obscuring the phenomenon. A similar fate awaited WERTHEIM's explanation for the deep tone, as well as his revealing experiment. WERTHEIM was disregarded despite the fact that his reputation as an outstanding experimentist was affirmed (albeit reluctantly) by one of his most biased and vituperative adverse critics, KARL PEARSON, who stated: "We now reach a scientist whose labours in the field of physical elasticity are among the most important we have to deal with in this period."[26]

PEARSON, whose understanding of eighteenth and nineteenth century experiments in solid mechanics was minimal, and in many instances distorted, was so committed to the POISSON-CAUCHY uniconstant theory of elasticity that he summarily dismissed WERTHEIM's observations. Upon finding nothing adverse to say about the actual experiments, PEARSON gratuitously, and erroneously, wrote of a man who earlier in his career had studied mathematics with JACOBI, STEINER, and DIRICHLET: "The memoir [on WERTHEIM's experiments definitively demonstrating that POISSON's ratio varies from that predicted by POISSON's and CAUCHY's uniconstant theory] is very instructive as shewing the dangers into which a physicist may fall who has not throughly grasped the steps of a mathematical process."[27] In fact, the Memoirs of WERTHEIM that PEARSON was reviewing for the *History of the Theory of Elasticity* dealt with purely experimental matters and, as was invariably characteristic of WERTHEIM's papers, were designed to stimulate development of physically sound theory, rather than to promote any analysis of his own.[28]

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[26] TODHUNTER & PEARSON [1886], *I*, section {1292}.

[27] *Ibid*, section {1319}.

[28] Said WERTHEIM: "MM. LAMÉ and MAXWELL admit that the ratio above defined [POISSON's ratio], or what comes to the same thing, the ratio between the cubic and linear compressibilities, may vary in different substances. Experiment alone can determine whether this is the case, as I have not failed to remark both in my original memoir, and several of those I have since published ... Several distinguished geometricians, without repeating my experiments, and without disputing their accuracy, have endeavored to bring them in accordance with the ancient theory by various, and unfortunately also, very arbitrary hypotheses. I shall shortly mention and discuss these hypotheses before describing my new experiments on this subject." This is from an English translation in the *Philosophical Magazine*, [1861], fourth series, **21**, pp. 447–451, of WERTHEIM's last published Memoir that appeared in the *Comptes Rendus*, [1860], **51**, p. 969. The new manuscripts to which he referred, unfortunately fell into the hands of MARCEL VERDET. Like several of his contemporaries to whom the uniconstant theory seemed indisputable, VERDET was one of WERTHEIM's adverse critics. Nevertheless, it was he who in 1861 wrote WERTHEIM's obituary in *L'Institut*, wherein he states that the condition of those manuscripts was such that he would not attempt to publish them. (This is difficult to believe. WERTHEIM's papers are a model for clarity in the presentation of experimental results.) My several years of effort to locate those manuscripts among VERDET's papers at the Ecole Polytechnique in Paris, where VERDET had been a member of the faculty, in Tours where WERTHEIM died, and elsewhere in France, have been unavailing. VERDET [1861].

Some of the less vituperative adherents of the now abandoned POISSON-CAUCHY uniconstant theory included several outstanding theorists and most of the contemporary experimentists in elasticity. In general, they were more moderate in their remarks, which, nonetheless, aimed to diminish the significance of WERTHEIM's numerous experiments on POISSON's ratio and thus preserve the sanctity of a popular theory. Positive evaluations by CAUCHY, MAXWELL, and LAMÉ, among others, were not enough.

In 1861, at age 45, WERTHEIM leapt to his death from the Cathedral of Tours. This great loss for solid mechanics was compounded by the rapid decline in the influence of his research, including his research on the deep tone.[29]

By the end of the nineteenth century when the uniconstant theory of linear elasticity had been discarded, thus corroborating WERTHEIM's definitive observations half a century earlier, WERTHEIM the man, and his magisterial research in the laboratory, were almost totally forgotten.

In less than a decade after WERTHEIM's death appeared an additional factor contributing to the loss of interest in CHLADNI's deep tone. It was the shift of emphasis in research on the dynamics of rods that occurred after 1867, the year ST. VENANT introduced his theory for axial impact in linearly elastic rods.[30] LUDWIG BOLTZMANN, on the basis of his experiments in 1881, became adversely critical of ST. VENANT's results. Indeed, until well into the twentieth century a long series of experimentists on the impact of rods continued to seek the source of the observed discrepancies.[31] Experiment and theory thus evolved in a direction in which the deep tone does not appear.

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[29] Not occasionally, experimental discovery can be far ahead of any theory. WERTHEIM's experiments on the small torsional deformation of rods provided a measured nonlinear response that he refused to approximate as elastic moduli. From the same data on nonlinear response he discovered normal stress effects a century ahead of their rediscovery and extensive study as the "POYNTING effect" in the twentieth century. Both PEARSON and ST. VENANT lacked the insight to appreciate the significance of WERTHEIM's discovery. ST. VENANT's remarks on WERTHEIM's torsion data in a lecture to the French Academy in 1856 bordered upon ridicule; such data were contrary to ST. VENANT's linear theory of torsion. PEARSON sums this lopsided debate in which the only problem was that even after the passage of 40 years, for PEARSON, WERTHEIM was still too far ahead of his time:

"Reference is also made to SAINT VENANT, whose work WERTHEIM seems totally to have misunderstood, probably to a great extent through insufficient analytical knowledge ... we hold WERTHEIM to have been in the wrong throughout, and occasionally, we fear, influenced by the dread that SAINT VENANT's brilliant theoretical achievements would throw into shade his own very valuable experimental research." TODHUNTER & PEARSON [1893], *II*, Part I, section {800}.

[30] ST. VENANT [1867], *J. Math. Pures Appl.*, second series, **12**, 236–376.

[31] Of particular importance was the seminal experiment in 1881 on the axial collision of rods in bifilar suspension, by the distinguished theorist LUDWIG BOLTZMANN. Late nineteenth century and early twentieth century experiments on the dynamics of rods were designed to question facets of the ST. VENANT theory. BOLTZMANN's experiment set a pattern for the laboratory study of the dynamics of rods that continued to 1940. The experimentists WOLDEMAR VOIGT, VICTOR HAUSMANINGER, MAX HAMBURGER,

The deep tone, forgotten for 115 years, was resurrected in my laboratory during the late 1960's. Questions rose to my mind, while I was examining source material and repeating a number of nineteenth century experiments in preparation for writing a treatise, historically-based, for the *Handbuch der Physik*, on the foundations of experimental solid mechanics.[32] Those questions stimulated a quest that has been off and on for many years: to find a quantitatively precise way to decouple and study WERTHEIM's longitudinal shear wave in an experiment, preferably one that did not entail the use of wet fingers or cloths impregnated by rosin.

In the title of this presentation, the reference to "a late-twentieth century resolution of a mid-nineteenth century dilemma" relates to my finding initial conditions for axial impact capable of generating the longitudinal shear waves proposed in 1851.[33] Other than a difference in initial conditions, *i.e.*, rubbing *vs* impact, an equally important difference between WERTHEIM's nineteenth century experiment and mine in the twentieth century is that in the 1990's, observation is not confined to the special situations of infinitesimal strain and linear elasticity.

For 75 years after ST. VENANT had introduced his theory of linearly elastic waves in axially colliding rods, laboratory study, while raising serious questions about various facets of the theory, was curtailed by being limited to the peripheral examination of secondary effects. When not content merely to speculate on the character of a wave from observing only side-effects, we must measure with precision the profiles of waves during wave propagation. Only in this way, from actual measurement in the laboratory, can we critically evaluate the physical significance of any theory of wave propagation, including that of ST. VENANT.

In 1940, when a method was developed to determine strain by variations in electrical resistivity ("electric resistance strain gages"), it became possible to measure, directly, profiles of waves in a solid. With such an experiment it would be feasible to re-examine physically WERTHEIM's longitudinal shear wave in the very experiment CHLADNI had introduced in 1787. However, time, controversy, and analytical preference had set the problem aside.[34]

J. E. SEARS, BERTRAM HOPKINSON, J. E. WAGSTAFF, W. A. PROWSE, and many others, all studied the problem of linear elastic waves in rods in a context adversely critical of ST. VENANT's analysis. BELL [1973], Sections 3.33–3.36, pp. 308–331.

[32] BELL [1973].

[33] BELL [1987a]; [1990b].

[34] The experiment in 1940 that introduced a direct measurement of wave profiles used an electric resistance method to measure infinitesimal dynamic strain; it was introduced in 1936 by E. SIMMONS and was developed between 1936 and 1940 by A. RUGE, D. S. CLARK, and A. DE FOREST. In 1940 R. FANNING & W. V. BASSETT made the first use of this method to provide a direct observation of linearly elastic wave fronts in the context of the ST. VENANT theory. Their experiments included the study of reflection from free and semi-fixed boundaries in a rod. *Trans. Am. Soc. Mech. Engrs.* **62**, A-24-A-28, [1940]; and see BELL [1973], Section 3.36, pp. 329–331.

The measurements in 1940 did address and resolve the source of the adverse, *laboratory*-based criticism of St. Venant's theory for the impact of linearly elastic rods. Observation and theory had not been in accord primarily because St. Venant's theory failed to account for the decomposition of the shock wave in plane strain that emanates from impacted surfaces of a rod.[35]

During the waste of war between 1940 and 1945, specific solutions to the one-dimensional, nonlinear wave equation for finite strain were provided independently by six different theorists, one from England, three from the United States, and two from the Soviet Union.[36] The English and American solutions were not released from wartime secrecy until 1948; they predicted that wave fronts of finite amplitude would be highly dispersive in the solids Chladni and Wertheim had studied.

For annealed metals and most hardened metal alloys, to observe the predicted dispersion in a wave front for finite strain would require the use of gage lengths of the order of $1/10^{\text{th}}$ millimeter. Such small gage lengths and the accurate measurement of large dynamic strain were beyond the scope of electric resistance strain gages. A different way to measure wave profiles had to be found.

As to finite amplitude waves in solids of nonlinear material, eight years elapsed between the publication in the open literature of the English and American suggestions of analytical solutions for a nonlinear theory of waves in rods and my finding an experiment to evaluate their physical pertinence. This experiment introduced high density diffraction gratings that measured finite strain precisely during wave propagation.[37] Thus, after 1956 it became feasible to pick up the nineteenth-century thread, not in terms of the quantitative limitations of rubbing a rod with rosin, but in an experiment for axial impact resembling St. Venant's proposal in 1867. First, however, the problem Chladni had uncovered in 1787 had to be re-discovered. Three more decades elapsed before a laboratory experiment in the twentieth century resolved Wertheim's mid-nineteenth century dilemma.

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[35] See Bell [1960d], "The initial development of an elastic strain pulse propagating in a semi-infinite bar," and Bell [1973] 3.38, pp. 347–351.

[36] G. I. Taylor, T. von Kármán, M. P. White, L. van Griffis, K. Rakhmatulin, and G. S. Shapiro.

[37] In 1956 I described a new laboratory method employing ruled diffraction gratings to determine strain and the angle of the surface during the passage of a wave in a solid. By 1958 I had spent nine years building a ruling engine capable of producing cylindrical diffraction gratings with a density in excess of 31000 lines per inch. The sensitivity was sufficient to provide precise optical measurements in microsecond time. During wave propagation, dynamic strains could be measured from small strains to 20% or more, with gage lengths as short as 1/10 millimeter — two orders of magnitude smaller than previously had been possible for the direct measurement of dynamic strain. The change in the spacing of lines during the passage of a wave front produced changes in the angles of diffraction. Observation of these changed angles, now, for the first time, made it possible to obtain an optically precise, direct analysis of dispersive wave fronts at large finite strain. Bell [1956] [1958] [1960a] [1960b] [1960c].

III. The Late Twentieth-Century Resolution of the Mid-Nineteenth Century Dilemma

In contrast to the elastic oscillatory motion CHLADNI had observed in his experiment for the deep tone, in the modern nonlinear version of his experiment, when a longitudinal shear wave is generated in a rod, there is no unloading. Both a target rod and an identical projectile rod are prepared from an annealing recipe that ensures an initial state free of stress.[38] During loading, analogous statements may be inferred either in nonlinear parabolic elasticity or in nonlinear parabolic plasticity.

The experiment I developed to address the resolution of the nineteenth-century dilemma utilizes the axial collision of rods in the sense of ST. VENANT but differs from the nineteenth-century version in three respects: 1) nonlinear response functions replace HOOKE's law; 2) the strains to be measured are large rather than infinitesimal; and 3), a third rod is introduced to facilitate generation of the desired shear wave.[39] So that no unloading occurs during the measurement, this third rod, referred to as a "transmission rod," is many times longer than either the projectile rod or the target rod. Unlike the finite strain and the parabolic response function that apply to both projectile and target, the strain for the transmission rod remains infinitesimal and HOOKE's law governs the response to very large stress. This transmission rod is affixed in axial alignment to the far end of the target rod. The interface between the rods is coated with a thin film of a silicone lubricant that is designed for high pressure. The projectile and target rods are aligned axially at the moment of impact. After the axial collision between the ground and polished flat ends, identical wave fronts propagate away from the common impact face. Projectile velocities range from 90 to 300 miles per hour.

Experiments I performed in the early 1960's revealed that in a circular cylinder the plane wave front that emanates from a normal impact on a flat surface reflects from the side walls. While traversing a distance from the impact face equal to the diameter of the rod, it decomposes into an uniaxial stress wave. Whatever be the projectile velocity, downstream it becomes a highly dispersive coupled wave. The work per unit volume in the undeformed reference configuration is equally divided between the longitudinal and radial components of this coupled wave.[40]

When such a coupled wave front reflects from a lubricated interface between

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[38] In 1959, by a long process of trial and error, I sought for and found heat treatments for aluminum and for copper that would place in a state free of stress a fully annealed solid of small sized grains. For aluminum, this recipe, which I have used for 30 years in over 2000 tests, is as follows: aluminum specimens are annealed at a temperature of 1100°F for two hours, cooled in the furnace to room temperature, removed from the furnace in a state free of stress, and checked for small grain size. (The temperature 1100°F is 90% of the melting point.) BELL [1960a].

[39] BELL [1987a]; [1990b].

[40] BELL [1960b]; [1961a]; [1961b]; [1962]; [1963a].

a rod with a relatively soft parabolic response function and a hard linearly elastic rod, the longitudinal and radial components decouple into two separate reflected wave fronts. As will be shown below, the reflected radial component is a nonlinear version of the longitudinal shear wave that WERTHEIM proposed to explain the source of CHLADNI's deep tone.

To demonstrate in the laboratory that this reflected radial component is indeed a longitudinal shear wave with lower wave speeds and radial displacement that is axially symmetric, we must have precise knowledge of the details of the coupled nonlinear wave front that impinges upon the lubricated interface. Fortunately, some 30 years ago the properties of such a coupled wave front were a prime objective in my laboratory analysis of the propagation of waves of finite amplitude in rods.

Between 1960 and 1963, diffraction grating measurement of wave profiles of strain and of surface angle for nonlinear waves in rods revealed that the initial development of a wave front in the vicinity of the colliding surfaces is as important for wave fronts of finite amplitude in nonlinear solids as it is for wave fronts of infinitesimal amplitude after the axial impact of rods in linear elasticity.[41] All material constants for the known nonlinear response functions were directly measured at strain rates from 10^{-4} to 10^{4} per sec.

In 1960, to study the growth of wave fronts in rods with flat ends, diffraction gratings of 31 000 lines per inch with a gage length of 0.12 mm were located from 0.50 mm to 25 mm from the impact face. Profiles of strain and surface angle of the collapsing initial shock wave were compared with the average of the stress over the impact face during the first few microseconds. The latter were measured by piezo-crystals, 0.12 mm in thickness, dynamically calibrated and placed on the flat end of the target rod so as to lie between the colliding impact faces.[42]

Near the impact face, the initial wave front of plane strain, in accord with HUYGENS' principle, reflects from the side walls of the cylinder. In the fall of 1961 Professor JERALD ERICKSEN pointed out that my experiments which provided detail relating stress at the impact face and wave speeds and strain at successive positions near that face, correlated with analytical results published that same year by our colleague at The Johns Hopkins University, Professor CLIFFORD TRUESDELL in his study, "General and Exact Theory of Waves in Finite Elastic Strain."[43] Given the velocity of the projectile, I could follow every detail from the time the initial elastic shock front in plane strain was reflected from the side walls of the cylinder near the impact face to the time the coupled wave front propagated down the target rod toward the interface. When given the velocity of the projectile and the material of the rod, we know precisely the wave speeds, strains, particle velocities, maximum stress, and surface angles, for both of the coupled components.[44]

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[41] *Ibid.*

[42] BELL [1962].

[43] TRUESDELL [1961]; BELL [1962].

[44] BELL [1963b]; [1968].

IV. One Illustrative Example of a Decoupled Longitudinal Shear Wave

The arbitrarily chosen example is a summary of measurements for 21 tests in fully annealed aluminum. The average projectile velocity of 1524 in/sec, is the same for all tests. For perspective we note that this corresponds to an axial collision at about 90 miles per hour. The illustration may be divided into two parts: the first describes the kinematics of the living wave, and the second views its corpse after unloading. In the development here, we assume that compression components have a positive sign. The total elapsed time before unloading is of the order of 500 microseconds.

This initially stress-free, fully annealed solid has a parabolic response function for uniaxial stress in a rod. Whether the loading is quasi-static or dynamic, the parabola coefficient[45] in equation (1), determined from experiment, is $\beta^* = 5.6 \times 10^4$ psi $= 39.4$ kg/mm^2,

$$\sigma = (\operatorname{sgn} E)\, \beta^* (E)^{\frac{1}{2}}, \tag{1}$$

where $\sigma =$ force/(undeformed area) and $E = \Delta L/L_0$ are the unaxial stress and finite strain in the original, undeformed reference configuration. For the linear elastic transmission rod, $E^* = 10 \times 10^6$ psi $= 7031$ kg/mm^2.

From equation (2), the one-dimensional wave equation,

$$\varrho\, \partial V/\partial t = \frac{\partial \sigma}{\partial x}, \tag{2}$$

where V is a particle velocity, and $\varrho = 0.000253$ lb-sec^2/in^4 is the mass density, we have

$$V = \int C(E)\, dE \quad \text{where} \quad C(E) = [(d\sigma/dE)/\varrho]^{\frac{1}{2}}. \tag{3}$$

Substituting equation (1) into (3), we obtain

$$E = [(9\varrho)/8\beta^*]^{\frac{2}{3}} (V)^{\frac{4}{3}}. \tag{4}$$

The work per unit volume in the undeformed reference configuration is given by

$$U = \int \sigma\, dE = (\tfrac{2}{3})\, \sigma E = (\tfrac{2}{3})\, \beta^* (E)^{\frac{3}{2}}. \tag{5}$$

For an equipartition of the maximum incident work per unit volume in the undeformed reference configuration U_I, we have $\overline{U} = U_I/2$. Upon substitution of $\overline{U}$ in equation (6) we obtain $\overline{E}$, the greatest longitudinal compression in the incident wave:

$$\overline{E} = [\tfrac{3}{2}\, \overline{U}/\beta^*]^{\frac{2}{3}} \tag{6}$$

After reflection, from measurements with diffraction gratings we find that the total maximum strain, E_T, (in close accord with the average postdeformation

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[45] BELL [1968], Chapter II, pp. 7–50.

value for 21 tests) has an average of $E_T = 0.055$. Thus the amplitude of the strain of the reflected radial component, E_{rR}, becomes

$$E_{rR} = \tfrac{1}{2}[E_T - E_{LR}] + E_{rI} = -0.015 - 0.010 = -0.025. \tag{7}$$

The 21 tests in the illustrative example are on fully annealed aluminum rods initially free of stress. The projectile rod and the target rod are identical. In a symmetrical impact, the maximum velocity V_I of the particles in the target rod is one-half the velocity of the projectile: $V_I = 762$ in/sec.

Substituting this value of V_I into equation (4), we obtain $E_I = 0.0203$, the largest axial strain of the incident wave. Substituting this maximum strain E_I into equation (1), we obtain the calculated maximum unaxial stress of the incident wave $\sigma_I = 8032$ psi. Substituting σ_I and E_I into equation (5), at the maximum of the incident wave we obtain a value $U_I = 109$ in-lbs for the work per unit volume in the undeformed reference configuration. For an equipartition of this work per unit volume at the maximum of the longitudinal component, we have $\bar{U} = U_I/2 = 54.5$ in-lbs. Substituting this value of $\bar{U}$ into equation (6), we obtain $\bar{E} = 0.0125$ for the largest strain of the longitudinal component of the incident wave. In turn, substituting $\bar{E}$ in equation (1), we obtain $\bar{\sigma} = 6331$ psi for the corresponding maximum stress of the longitudinal component of the incident wave. (As I have noted, $\bar{E}$ is determinable independently from diffraction gratings of high density; it is the strain at which we observe a maximum in the measured surface angle of the wave.)[46]

Upon reflection at either a lubricated or a glued interface, the reflected *longitudinal* strain component doubles. (Experiment reveals that the entire wave front doubles at a glued interface; the radial component is negligible.) Thus, for the longitudinal compression component after reflection, we have: $E_{LR} = 2\bar{E} = 0.025$. Substituting E_{LR} in equation (1), we obtain a maximum stress of $\sigma_{LR} = 8854$ psi for the reflected longitudinal component. The measured maximum of the longitudinal compression stress in the transmission rod is $\sigma = 8770$ psi, the average for 21 tests. The measured value in the linearly elastic transmission rod differs by less than 1% from the calculated σ_{LR}!

For the target rod, since the maximum stress is equal on both sides of the interface, we substitute this measured σ into equation (1). This substitution provides the reflected strain of 0.025, identifiable as the calculated reflected longitudinal component $E_{LR} = 2\bar{E} = 0.025$.

As expected, the reflected radial component contributes nothing to the maximum axial stress of the reflected wave front.

In the reflected radial component the measured amplitude of strain given in equation (7) is $E_{rR} = -0.025$. From the general theory,[47] a parabola coefficient for pure shear is known, *i.e.*, 2.46×10^4 psi. From the maximum of the shear strain $E_{rL} = -0.025$, we obtain for the corresponding largest stress $\sigma_{rL} =$

[46] Bell [1963a].

[47] Bell [1987b]; [1988]; [1989]; [1990a].

3 880 psi. We thus may calculate the final work per unit volume in the undeformed reference configuration. Adding the calculated work for the reflected radial component to that for the reflected longitudinal component, and allowing for the small amount of work transferred to the transmission rod in which the strain remains infinitesimal, we obtain a close correlation with the work available in the incident wave front. The dispersive wave speeds are known for both reflected wave fronts. For the same absolute value of strain, the speed of the decoupled longitudinal shear wave is $\frac{2}{3}$ that for the longitudinal compression wave.

The reflected radial component from a lubricated interface both quantitatively and qualitatively meets the conditions characterizing WERTHEIM's longitudinal shear wave. The wave is a decoupled single front propagating along the rod with *measured* lower wave speeds and *measured* axially symmetric radial displacement.

When time has elapsed (500 microseconds) sufficient for unloading to occur, permanent plastic set provides a frozen record of the reflected wave fronts. Unlike the axial motion of the reflected longitudinal wave of compression, the radial motion of the reflected longitudinal shear wave is not transmitted to the projectile specimen through the polished surface of the impacting faces. Since we know the wave speeds of the coupled incident wave and of both the decoupled reflected waves, we can estimate when the maximum strain of the longitudinal shear wave in the target rod will reach the original impact face. When we reduce the length of the target specimen in successive tests, as shown in Figure 1, we find that as the length of the target specimen decreases, there is a difference in the diameters of the projectile and the target specimens. The reflected longitudinal shear wave propagates in the target rod and not in the projectile. Because of the additional radial displacement, the diameter of the target rod exceeds that of the projectile. The difference in their diameters provides evidence that the decoupled longitudinal shear wave has arrived in the region of impact.

For those who still may harbor a nineteenth-century view of the deep tone, namely, that it should remain in limbo, it is worth while to examine further the corpse of the decoupled wave by comparing Figures 1 and 2. The diameters after impact by an identical incident wave reflected from a glued interface are shown in Figure 2. Near the impact face the radial restraint on the target specimen results in a situation which is the reverse of that for the lubricated interface. In successive tests, measurements of specimens of decreasing length demonstrate that their diameters are indeed less than the diameter of the projectile.

In the late twentieth-century the resurrection of the "deep tone," that was ignored in theory and in experiment for more than 140 years, adds a physically observable, longitudinal shear mode to the long-studied flexural, longitudinal, and torsional response in the dynamics of rods.

Except for the vagaries of history, CHLADNI's experiments in 1787 on the dynamics of rods might have been celebrated no less than the experiments, published that same year, in which he discovered the "CHLADNI figures" that characterize the lateral vibration of thin plates. Given a different sequence of events in the two centuries, a sequence less dominated by preconception, the impetus that the "CHLADNI figures" gave to the nineteenth century theory of plates could have been matched by the impetus of the "deep tone" on the theory of rods.

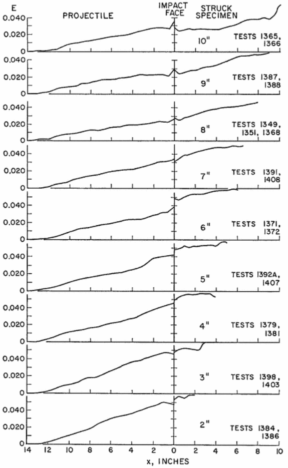

Fig. 1. Measured diameters of projectile and target rods after deformation. The length of the target rods, each with a lubricated interface at the far right end, was varied to reveal the appearance of the longitudinal shear wave as the length decreased.

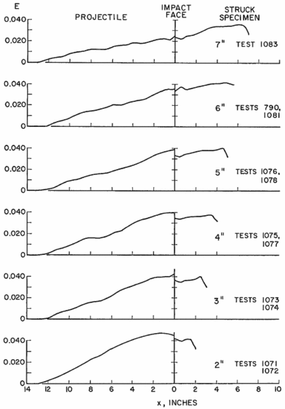

Fig. 2. Measured diameters of projectile and target rods after deformation. The right end of each target rod was securely glued to a long, hard transmission rod. For all the initial lengths referred to in Figure 1, this glued end condition constrained the diameters of the target rod.

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