On Some Vistas Disclosed by Mathematics to the Russian Avant-Garde: Geometry, El Lissitzky and Gabo

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ABSTRACT

Generally speaking, all avant-garde movements have had one characteristic in common: belief in the new. It is also true that all of those movements were aware of changes, progress and advances in science. As a consequence, non-Euclidean geometry was considered a manifesto for revolution in the arts. This article discusses the visualization of mathematics—the process of transferring concepts from mathematics to works of art—with examples from the artworks and writings of El Lissitzky and Naum Gabo.

Diderot, great character of the Enlightment, is one of the more remarkable scientific and humanistic symbols of the eighteenth century. His work, performed in collaboration with d'Alembert, *Encyclopédie* (1751–1765) [1] summarizes, although in hundreds of pages, the triumphs of the intellect and of the age—and of these two visionary men of genius. D'Alembert, the other half of that editorial adventure, in his article entitled "Dimension", alluded to the fourth dimension, in contrast to traditional learning [2]. And in Diderot's "Traité du Beau", contained in Volume II, we find the first example of an analogy of beauty with a mathematical theorem concerning a curve [3]. Nearly a century later, in the second stanza from the fifth song in Lautréamont's evil treatise *Les Chants de Maldoror,* the same analogy (although he does not refer to Diderot's work) is repeated: "beau comme un memoire sur la courbe que decrit un chien en courant apres son maitre" (as beautiful as the memory of the curve described by a dog running after its master) [4].

Note the view, especially popular among mathematicians, that beauty can be found in mathematical discourse. Of course, this idea could not occur to anyone without knowledge of mathematics, as Diderot and, to a lesser degree, Lautréamont had. However, the beauty of the visual image was not their business. Nevertheless, in the eighteenth and nineteenth centuries, the study of bizarre and mysterious curves became important.

It was only in the last half of the nineteenth century that an interest in visualizing contemporaneous mathematical ideas began. I shall make no attempt to describe that epoch called Modernism, whose genesis and strength are still matters under discussion [5]. However, there was at that time a general momentum that carried with it art, poetry, science, fashion, advertising and architecture. We can establish some interplay between these distinct types of activities. Consider the influence of mathematics upon the arts. It is my opinion that mathematics had three roles in the visual arts. First, it was a metaphor for progress. Second, it provided a language of forms and shapes. And third, mathematical concepts could enlighten, modify and penetrate art notions that were then reflected in the visual arts. My discussion of these roles will be best considered in the climate of the Russian avant-garde (Fig. 1), which constitutes one vigorous, optimistic and neat example of Modernism, and which perhaps served as a forerunner of some of the present attitudes in visual mathematics.

THE METAPHOR OF PROGRESS

In the 1920s Russian writer Iouri Tynianov, referring to Futurist poet Velimir Khlebnikov, wrote "the poet Khlebnikov becomes the Lobachevskii of words" [6]. In this sentence Tynianov is using Nicolai Lobachevskii as a metaphor for a founder of a new system or a novel theory—for this is what Lobachevskii had done. From Tynianov's point of view, the old theory was Euclidean geometry, the deductive theory founded upon Euclid's five postulates. The first four of Euclid's postulates are self-evident, and the fifth can be paraphrased as "there exists only one parallel to a given straight line through a given point". Tynianov's metaphor refers to Lobachevskii's idea about this postulate, which Lobachevskii introduced in his lecture delivered at the University of Kazan on 12 February 1826. That lecture replaced Euclid's fifth postulate without affecting the coherence of the geometrical discourse. Also proposed independently by J. Bolyai, the new postulate allowed the existence of an infinite number of parallels to a line through a given point. However, the general acknowledgment of Lobachevskii's ideas came many years later.

Meanwhile, another lecture questioned the dominant role of Euclidean Geometry. In fact, on 10 June 1854, Bernhard Riemann addressed the topic *On the Hypotheses which Lie at the Basis of Geometry* [7]. Indeed, this lecture, published 13 years later, introduced important mathematical concepts, such as the concept of manifolds, mentioned others (such as the fourth dimension), and contributed strongly to the philosophy of geometry [8]. Riemann also

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Fig. 1. First Russian Art Exhibition, Van Diemen Gallery, Berlin, 1922. Two of Lissitzky's pictures are barely visible on the left wall. In the corner of the room is Gabo's *Head of a Woman*, and at the centre is Gabo's sculpture *Torso*. Rodchenko's *Hanging Spatial Construction* hangs from the ceiling.

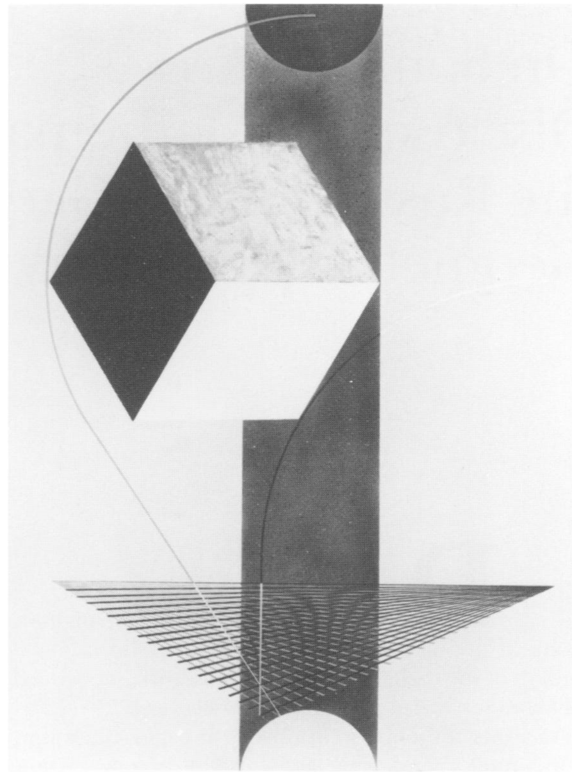


Fig. 2. El Lissitzky, *Proun 99*, oil on wood, 129.4 × 99 cm, 1923. (Courtesy of Yale Univ. Art Gallery, Gift of the Société Anonyme, New Haven, Connecticut) Lissitzky followed the logic of the avant-garde movements: to plunge into the waters of scientific progress. If the geometry of curved surfaces led to new conceptions of space, how is it possible that pictorial space looks so traditional?

discussed the geometry of spherical surfaces on which straight lines correspond to great circles, called equators. As a consequence, there is a geometry (popularly known as Riemann's Geometry) with no parallel lines, for two equators always meet at two points: the poles.

Riemann's ideas together with Lobachevskii's geometry have constituted the subject matter of philosophical and scientific debates since 1860. Thus, in the last half of the nineteenth century, the dominance of Euclidean geometry ended. A revolutionary change with respect to tradition had been accomplished.

Looking back on the turn of the century, we find signs of fundamental changes in science, in literature, in technology, in fashion—in short, everywhere. In the arts, the attacks on the role of representation followed one after another, from Impressionism to Cubism, which was the deepest criticism of the role of visual imagery as representations of reality. A never-ending story traverses all these 'isms'. Most simply, we can regard them as the emblems of change and of denial of tradition.

For artists there was a widespread feeling that behind these changes, science was the ultimate cause of this transforming world. And progress legitimated the process. Thus, a Futurist manifesto says: "Comrades, we tell you now that the triumphant progress of science makes profound changes in humanity inevitable, changes which are hacking an abyss between those docile slaves of past tradition and us free moderns" [9].

In these lines we note the assured volume of confidence in progress implanted by the contemporaneous scientific avalanche. These manifestos are evidence of how science imposed itself on artistic thought.

This tendency was common to several avant-garde movements bearing different banners. Cubism and Futurism in the West were no less involved with science as a metaphor of progress than were the avant-garde artists in Russia. For example, I refer to a passage from Vladimir Markov's *Principles of New Art* (1912): "It must be noticed that contemporary Europe which had done great conquests in the scientific and technological domains is very poor with respect to the evolution of plastic principles inherited from the past" [10].

This sentence could not have been written some years later. Indeed, the following years were years of increasing renewal in visual art, which achieved its chief goal in Constructivism.

Constructivism was the confluence of several diverse aspects of the avant-garde. As a collective ideology, it grew up in the years of 1917 to 1920. The Russian Revolution had provided the optimistic atmosphere sympathetic to new formulations in art. This was a period of discussion and of revolution extended to all spheres. Many explanations have been proposed for this historical avant-garde [11]. For our present purpose it suffices to scan the bundle of mathematical ideas involved in visual art practice. On the one hand, advance in mathematics contributed to support the metaphor of progress. On the other hand, mathematics represented a new way to approach visual arts problems and also created an appropriate place to look for non-naturalistic shapes according to the ideals of both Constructivism and Suprematism.

The Visualization of Abstract Mathematical Notions

El Lissitzky's essay *Art and Pangeometry* is an essential document for studying the mathematical issues discussed by the Russian avant-garde [12]. Although it was published in 1925 in Germany, from its first line we note it deals with our present situation: "In the period between 1918 and 1921 a lot of old rubbish was destroyed. In Russia we also dragged Art off its sacred throne" [13].

This tone, a typical denial of the past under Dadaist influence, is a rhetorical detour from the essay's aim of describing the parallel development of art and science—geometry—by means of analogies. For example, perspectival space, the representation of space that originated in the Renaissance, corresponds to the laws of three-dimensional Euclidean geometry. However, "in the meantime science undertook fundamental reconstructions" [14], for Euclid's laws had been destroyed by Lobachevskii, Gauss and Riemann. And, in the arts, Cubism had replaced perspective [15]. So said Lissitzky, and so wrote Apollinaire, 12 years earlier [16].

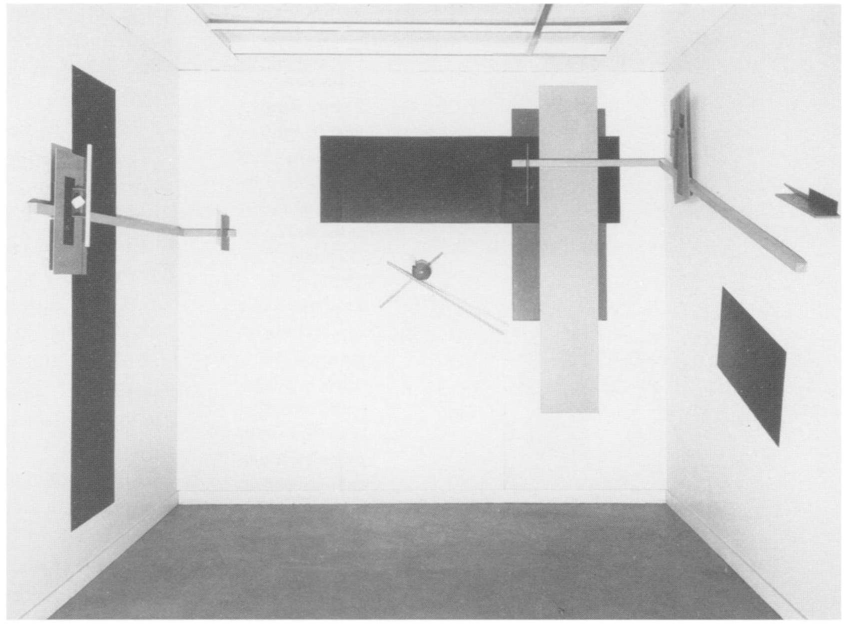


Fig. 3. El Lissitzky, *Proun Space*, painted wood, 300 × 300 × 260 cm, 1965 reconstruction of the 1923 original. (Courtesy of Stedelijk Van Abbemuseum, Eindhoven, The Netherlands) From theoretical speculation to a real environment: "The series of analogies which I am going to bring to your attention", writes Lissitzky, referring to analogies between art and mathematics, "is put forward not to prove, for the works themselves are there for that, but to clarify my views" [60].

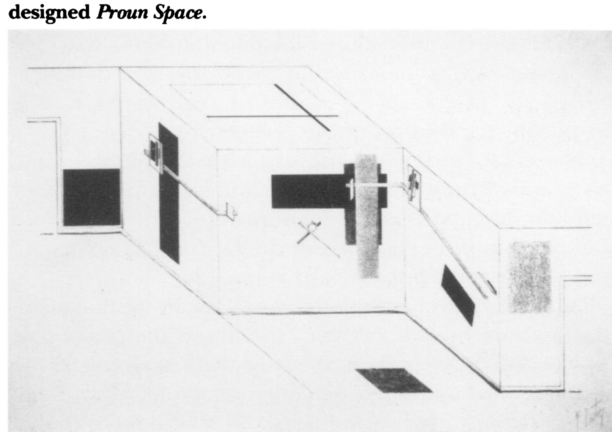
The spatial conception of Suprematism is expressed by the phrase 'irrational space' [17]. In order to explain it, Lissitzky began with an inquiry into non-Euclidean geometries and Gaussian curvature (Fig. 2). Let us pause to sketch this point. Descartes' translation of geometry into algebra allows us to state geometrical properties in terms of functions involving the coordinates of the points concerned. In this way, the study of purely geometrical, and to some extent visual, properties of figures is reduced to the study of functions. Since the seventeenth century this approach has been to state and solve problems about curves. Consider the equality of shapes among figures. All will agree that for planar figures bounded by straight lines, equality of shape means equality of corresponding angles. But what if the borders are curved lines?

We can naively think that curvature at a point is, in some unprecise sense, like an angle. Thus, equality of shape would mean equality of curvature at the corresponding points. However much we stay on informal ground and understand curvature at a point as the index of the deviation of the curve from its tangent line in that point, or, in the case of surfaces, from its tangent plane, nevertheless a precise formal definition must refer to the functional translation of curves and surfaces.

For plane curves, curvature intuitively is the degree to which a curve is bent at each point. Consider the simplest plane curve, the circle. Because it is equally curved throughout, its curvature is constant and is measured by the reciprocal of the radius. So the smaller the radius, the larger the curvature. In all other curves the amount of curvature varies from point to point, therefore it must be measured with infinitesimals. Thus arises the necessity of using the functional translation of geometrical figures in order to deal with infinitesimals. Now consider a given plane curve and a point P on it. Let Q and R be two neighboring points of the curve. Therefore, there exists a circle through P, Q and R. If Q, R approach P, then the curvature of the circle approaches a limiting number. This number is defined as the curvature of the given curve at P [18].

The measure of curvature for surfaces can now be reduced to the computation of the curvatures for plane curves. The method for determining the curvature of a surface is, briefly, as follows. Given a point on a surface, the lines tangent to this point lie on a plane. Draw the planes perpendicular to that plane through the point. Each of these planes will intersect the surface in a plane curve. As they are plane curves, their curvatures can be calculated in the way just shown. Thus we determine a set of real numbers, in which each number corresponds to the curvature of one of the plane sections. However this set has a minimum and a maximum, called the *principal curvatures* of the surface at the

Fig. 4. El Lissitzky, *Proun*, lithograph, 60.5 × 44.5 cm, 1923. (Published by Kestner Gessellschaft, Hanover, 1923) One of Lissitzky's purposes was the visualization of abstract concepts from mathematics. To do so, and strictly adhering to his much-quoted belief in the creation of a new conception of space, he designed *Proun Space*.



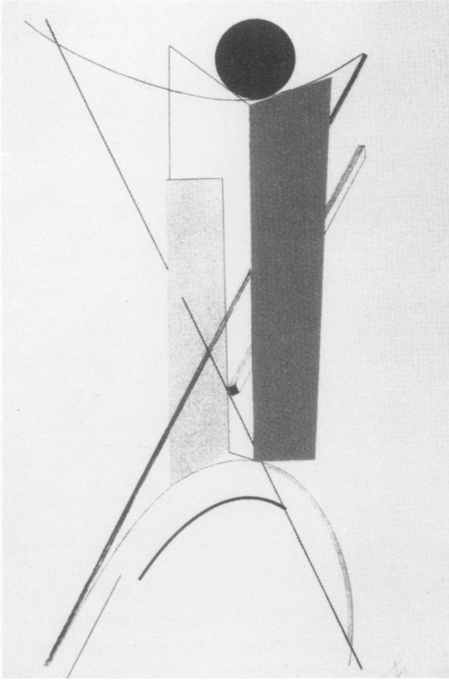


Fig. 5. El Lissitzky, *Proun*, lithograph, 60.5 × 44.5 cm, 1923. (Published by Kestner Gesellschaft, Hanover, 1923) Lissitzky's use of sections of geometrical objects might well have descended from popular ideas that considered material objects as sections of four-dimensional entities.

point considered [19]. The product of the principal curvatures is called the *Gaussian curvature of the surface* at the point under consideration [20]. This definition originated with the work of Gauss in 1827 [21].

It is by means of the Gaussian curvature, actually a number, that a very elegant characterization of non-Euclidean geometries can be formulated. In fact, if the sum of the internal angles of a triangle lying on a surface is less than two right angles (180°), then the surface has negative curvature. If the sum is greater than two right angles, then the surface has positive curvature. Euclid's law, which says that the sum of the internal angles of a triangle equals two right angles (Euclid, I. 32), holds in surfaces of zero curvature [22].

Now, let us come back to Lissitzky's essay. Its discussion refers to spaces of non-zero curvature [23]. More precisely, it says that spaces in which Euclid's postulates hold are the only spaces we can visualize. For spaces of non-zero curvature "only a mirage can simulate this" [24]. This constitutes Lissitzky's criticism of irrational space, which, for him, was the spatial concept of Suprematism. What does irrational space mean? Lissitzky's explanation can be easily and accurately expressed in mathematical terms [25]. For irrational space is a four-dimensional manifold. This was one of the fundamental notions introduced by Riemann's 1854 lecture, although Lissitzky does not mention it.

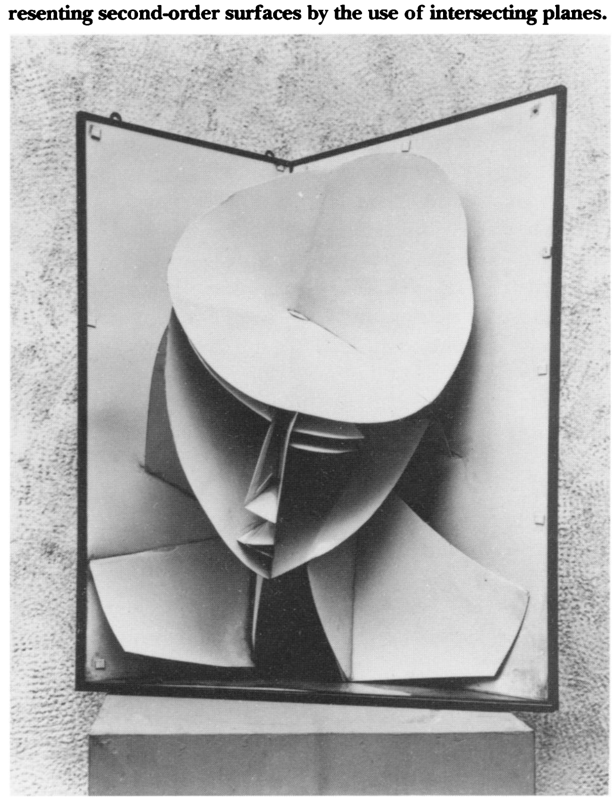
The definition of the concept of manifold is a difficult task [26]. Briefly, an n-dimensional manifold is a space M, which near each point is like the Euclidean space of dimension n, i. e. the set of all n-ples of real numbers [27]. Most geometrical forms whose points may be defined by n parameters are n-dimensional manifolds. Of course, Euclidean space of dimension n is the simplest n-dimensional manifold. Also, perceptible color qualities form a manifold of dimension three by virtue of the fact that all colors are produced by mixing three basic colors [28].

Riemann's 1854 lecture begins with the advice that manifolds are rare in ordinary life: "Color and the position of sensible objects are perhaps the only simple concepts whose instances form a multiply extended manifold" [29]. Compare Riemann's advice with Lissitsky's ideas about irrational space: " In this space the distances are measured only by the intensity and the position of the strictly defined color areas" [30]. He thus continues with several remarks that simply yield to the coincidence of Suprematist or irrational space with a four-dimensional manifold [31].

There is no evidence that Lissitzky had read Riemann's lecture or any other book that contains such kinds of ideas. However, these ideas had become part of the philosophical and scientific knowledge of the time [32]. For instance, in *The Foundations of Geometry* (1897) Bertrand Russell wrote two passages dealing with color as an example of manifold [33]. Another account of Riemann's ideas was given by H. V. Helmholtz in *On the Origin and Significance of Geometrical Axioms* [34]. This was the first attempt to expose manifolds and curved spaces to an audience knowing only "the amount of geometry taught in our gymnasia" [35].

Despite the impossibility of determining the exact origin upon which Lissitzky built his explanation of irrational space, it is still possible to make some comments. To start with, manifolds, even if they are not explicitly mentioned, form the underlying mathematical concept that gives meaning to Lissitzky's account of Suprematist space. Second, it seems reasonable that the concepts are of a mathematical kind, for the essay is full of advice to artists not to use 'advanced' scientific concepts without a deep understanding of the corresponding theories. Third, as they are mathematical spaces, "Our minds are incapable of visualizing this, but that is precisely the characteristic of mathematics—that it is independent of our powers of visualization" [36]. From this Lissitzky concludes that those spaces, "cannot be conceived,

Fig. 6. Naum Gabo, *Head of a Woman*, construction in celluloid and metal, 62.2 × 48.9 × 35.4 cm, 1916–1917. (Courtesy of the Museum of Modern Art, New York) Gabo's Constructivist emphasis on economy of materials and his rejection of mass volumes challenged the solid-space tradition of sculptural forms. The technical device he used to make these sculptures is analogous to representing second-order surfaces by the use of intersecting planes.



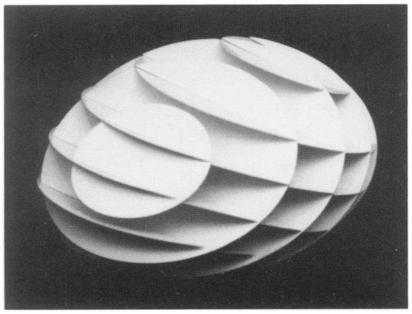


Fig. 7. A multifaceted model of the ellipsoid: several circles create a curved surface built with planes. (Model and photo: Gonzalo Puga)

cannot be represented; in short, it is impossible to give them material form" [37].

This last sentence contains the refutation of a then-current belief that had oversimplified and confused the discussion of the interplay between art and mathematics during the avant-garde period. I am referring to the belief that avant-garde visual art is, ultimately, nothing more than a transference from mathematics or from a mathematical approach to relativist space to the visual arts [38]. But, Lissitzky's lines touch upon a concept essential to an understanding of the role of mathematics in avant-garde visual arts—the concept of mathematical visualization. Thus, visual images with mathematical notions underlying them do not depend on models or representations of those notions. Visual works may be generated from purely abstract mathematical notions that, of course, have no three-dimensional representation.

Malevich's *Black Square* illustrates the effect of abstract mathematical notions upon the arts [39]. Insightful critics have shown that behind this artwork are thoughts about the fourth dimension [40]. But *Black Square* is a picture—it can be seen, it can be photographed. This is not possible with the fourth dimension—it is a mathematical concept, it cannot be photographed. At most we can obtain designs of representations of objects belonging to four-dimensional space in spaces of fewer dimensions, but not designs of the abstract concept of dimension itself [41].

When Lissitzky refers to *Black Square,* he says it "has now started to form a new space" [42]—indeed, irrational space [43]. He then points out the impossibility of visualizing those new spaces. Here Lissitzky approaches visualization in its more straight meaning, which, he says, neither Malevich's *Black Square* nor Suprematist painting had achieved. How to visualize irrational or Suprematist space? Or, to use mathematical terms, How to visualize four-dimensional manifolds?

In Lissitzky's opinion, he has answered these last questions in *Proun Space* (Fig. 3) [44]. The trick, influenced by motion pictures and advertisements, was to transform the surfaces of an almost-cubic room by displaying objects on them. The movement of the viewer and daylight changes in this environment produced the temporal coordinate. The objects were very simple—parallelepipeds, cubes and spheres—and established a relationship with the walking viewer. Clearly, all this comes from the mathematical approach to relativity [45]. Lissitzky designed a series of six lithographs in *Proun* [46] . One of the lithographs (Fig. 4) is a perspectival view of the *Proun Space*; another shows one of the objects (the object on the left wall in Fig. 3). Four other lithographs present planar sections of geometrical figures (Fig. 5). This probably comes from popular ideas that three-dimensional objects are sections of four-dimensional entities [47].

Proun Space was Lissitzky's visualization of a four-dimensional manifold—in his own words, the creation of 'imaginary space'. Whether or not he achieved it may be a subject of discussion. Yet his method of visualizing abstract mathematical notions was coherent, even if from these works alone it seems impossible to comprehend the mathematical concepts involved.

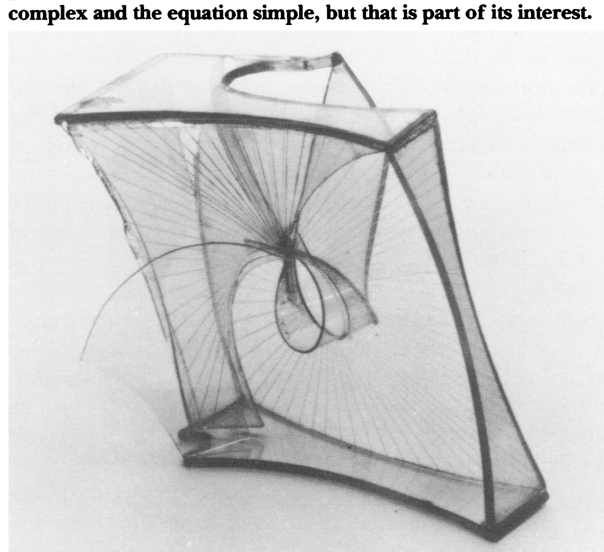
The Language of Naum Gabo

Although Lissitzky's Suprematist ideas were in the sphere of a Constructivist tendency, his approach, similar to Malevich's, did not allow any kind of direct transference from mathematical concepts to visual artworks [48]. In order to discuss the analogical visualization of mathematical notions we must draw our attention outside Suprematism and appeal to Constructivism. A glance at Rodchenko's or Tatlin's works immediately reveals clear geometrical patterns. This does not mean that Constructivist works can be generally characterized by their resemblance to geometrical shapes.

Since the 1920s the controversy concerning who and what belongs to Constructivism has given rise to declarations, debates and writings. Rodchenko's spatial constructions, for example his *Hanging Spatial Construction* (see Fig. 1), and Tatlin's 'counter-reliefs' (a term coined by Tatlin in 1913 to describe assemblages of industrial materials) may have many elements in common with Pevsner's and Gabo's sculptures. Pevsner and Gabo declared their work to be Constructivist art; however, Gan called Rodchenko and Tatlin Constructivist artists at the same time that he excluded Pevsner and Gabo. It is a complex and many-sided topic. Art issues were, as always, confluent with social, ideological and political affairs [49]. However, for most artists, mathematics had an important role in the genesis of visual images. The role could be subtle, as was the case for Lissitzky and Malevich, or direct, as in Gabo's sculptures.

Gabo's proposal was stated in his *Realistic Manifesto* (1920), signed together with his brother Anton Pevsner [50]. The 1920s meant new days because of the Revolution, new knowledge thanks to science and technology and, as a

Fig. 8. Naum Gabo, *Construction in Space: Crystal*, celluloid, 7.6 × 7.6 × 3.8 cm, 1937. (Courtesy of the Tate Gallery, London) This piece is artistic, although it looks mathematical. The shape is a bit complex and the equation simple, but that is part of its interest.



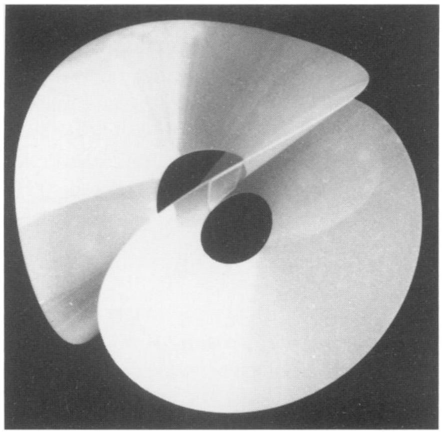


Fig. 9. The shape of Enneper's Minimal Surface provides the basic form for many different Gabo sculptures. (Model and photo: Gonzalo Puga)

consequence, new art for the new epoch. However, new art cannot be founded upon old principles. Accordingly, Gabo's manifesto presents five fundamental principles for Constructivist technique. Two of them, relevant to our discussion, reject mass volumes as spatial elements and explore the construction of volumes by means of planes, which, moreover, results in economic use of materials.

The sculptures made by Gabo in the period previous to the publication of his manifesto followed those principles. I refer to works he made in Norway, *Head of a Woman* (1917) (Fig. 6), *Torso* (1917) (see Fig. 1) and two works both entitled *Bust* (1915–1916). As, according to Gabo, "older sculpture was created in terms of solids; the new departure was to create in terms of space" [51], it was necessary to develop a construction device. The four sculptures I have mentioned show its use. They are constructed by the use of intersecting planes following a craft principle. Let me explain it with a simple example. If we take four cardboard squares and make slots in each, then the intersection at right angles of two parallel squares with the remaining two will produce a cube. By this principle it is very easy to build a sphere, for it suffices to take two families of concentric circles and fit them together in such a way that the planes of the circles intersect at right angles. What happens if we hinge one family of circles? We get a surface, called an ellipsoid, which has the same relation with the ellipse as does the sphere with the circle (Fig. 7). Namely, the plane sections of the ellipsoid are ellipses just as the plane sections of the sphere are circles.

The ellipsoid thus built fulfills one of the requirements of the *Realistic Manifesto,* the construction of volumes by means of planes. Gabo's technique was founded upon this particular way of constructing surfaces. He had studied physics and certainly had become acquainted with mathematics. In those years the use of models was the standard method to illustrate properties of surfaces, for in visualizing them, one gained the intuitive background needed to discuss more abstract notions [52]. The model of the ellipsoid we have described is a particular case of a second-order surface constructed with cardboard circles.

Second-order surfaces, also called quadrics, are surfaces satisfying a second-order equation in three Cartesian coordinates. In the eighteenth century Euler classified them into nine different types according to their equations. Another criterion for classification of quadrics is whether they intersect a plane in a circle. In this way we get two classes. The first sort consists of quadrics that do not intersect any plane in a circle. They are the parabolic and hyperbolic cylinders and the hyperbolic paraboloid. The second sort includes the elliptical cylinder, the elliptical paraboloid, the hyperboloids of one and two sheets, the cone and the ellipsoid. In other terms, surfaces of this kind have circular plane sections that, in turn, allow the construction of models for these surfaces with the use of cardboard circles, as was the case for the ellipsoid [53].

As we see, the Constructivist principle of Gabo's 1916–1917 sculptures rests upon free artistic variations of circle models for second-order surfaces. Gabo remained linked to the Moscow avant-garde and to Constructivism until he left Russia in 1922. Even if his subsequent works are not Constructivist in the historical sense, they are characterized by the use of geometrical forms. The basis of most of Gabo's sculptures can be traced back to his *Spheric Theme* (1936) and *Construction in Space: Crystal* (1937) (Fig. 8). These sculptures have analogous mathematical representations. For example, *Crystal* was inspired by a model of the cubic ellipse. This is a space curve defined by a third-degree algebraic equation. Furthermore, it can be proved that it is obtained from the intersection of two quadrics [54]—indeed, from the intersection of an elliptical cylinder and a cone [55].

Spherical Theme is, by far, Gabo's most popular sculpture and the basis upon which he constructed numerous works. Again, surprisingly, it corresponds to a mathematical representation of a surface, even if Gabo never made this claim. Indeed, its form coincides with the shape of the three-dimensional representation of Enneper's Minimal Surface (Fig. 9) [56].

Gabo's sculptures may be considered words in the artist's language, an alphabet containing mathematical forms and shapes. Thus, in my mind, mathematics supplies visual arts with a marvelous and almost infinite catalogue of mysterious, enigmatic and unknown figures. And art gives pleasure to us.

The Teachings of Apollinaire

Gabo's sculptures illustrate one of the roles mathematics plays in the visual arts. Visual mathematics supplied an alphabet of new forms that the epoch needed. For the sake of estimating the strength of scientific influence on Gabo's sculptures, we can cite a precise remark by Herbert Read:

> The creative construction which the artist presents to the world is not scientific, but poetic. It is the poetry of space, the poetry of time, of universal harmony, of physical unity. Art—it is its main function—accepts this universal manifold which science investigates and reveals, and reduces it to the concreteness of a plastic symbol [57].

This remark goes to the heart of our difficulty. How can we describe the process of visualizing mathematics? As we have seen there is not just one answer to this question. Gabo started from visual representations of mathematical elements and ended with sculptures. Lissitzky theorized concepts of art by means of mathematical notions that, in turn, resulted in visual works [58]. Both are attitudes resulting in visual mathematics. They are the more brilliant and extreme examples of an attitude common to the Russian avant-garde, and therefore to Modernism, for the Russian avant-garde was a compendium of Modernism.

Today's equivalent of cardboard, plaster and wire models are computer-generated images. They propose to us, among other things, a catalogue of forms and shapes, as models did to Gabo. Nondeterminist ideas explain a whole world of phenomena and can be the source of fresh insights—in the same way that Lissitzky reached an understanding of Suprematist space by appealing to Riemann's ideas.

The effect of this convergence of art and science has a long history. Yet, in the age of Modernism a great period of direct mathematical influence upon the visual arts began. Mathematics provided the arts with an example of continuous progress—an encyclopedia of deep ideas, a cartography of forms. Apollinaire, perhaps the spokesperson of avant-garde movements, while meditating on art, concluded that art must advance as mathematics had done. He advised, "Geometry is to the plastic arts what grammar is to the art of the writer" [59]. Why not believe this idea to still be valid?

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8. For a detailed account of the influence of Riemann's lecture on the philosophy of geometry, see R. Torretti, *Philosophy of Geometry from Riemann to Poincaré* (Dordrecht: D. Reidel, 1978) pp. 67–109.

9. U. Boccioni, C. Carra, L. Russolo, G. Balla and G. Severini, "Manifesto of the Futurist Painters 1919", in U. Apollonio, ed., *Futurist Manifestos* (London: Thames and Hudson, 1973) pp. 24–25.

10. *Art et Poesie Russes 1900–1930* [6] p. 54. French translation by Manuel Corrada.

11. For the first systematic presentation of Constructivism, see C. Gray, *The Great Experiment in Russian Art, 1862–1922* (London: Thames and Hudson, 1962) chapter VII. A more recent and comprehensive account is C. Lodder, *Russian Constructivism*, S. Barron and M. Tuchman, eds. (New Haven, CT: Yale Univ. Press, 1983). The complexities involved in studying the Russian avant-garde are explained in Stephanie Barron, "The Russian Avant-Garde: A View from the West", in *The Avant-Garde in Russia 1910–1930* (Los Angeles, CA: Los Angeles County Museum of Art, 1980) pp. 12–17.

12. El Lissitzky, "K. und Pangeometrie", in *Europa-Almanach*, C. Einstein and P. Westhein, eds. (Potsdam: Gustav Kiepenheuer, 1925). Quotations and references are from El Lissitzky, "Art and Pangeometry", in Sophie Lissitzky-Küppers, *El Lissitzky: Life, Letters, Texts* (London: Thames and Hudson, 1968) pp. 348–353. On Lissitzky's writings, see A. Birnholz, "El Lissitzky's Writings on Art", *Studio International* **183**, No. 942, 90–92 (1972).

13. Lissitzky-Küppers [12] p. 348.

14. Lissitzky-Küppers [12] p. 349.

15. Lissitzky-Küppers [12] p. 349.

16. G. Apollinaire, *The Cubist Painters, Aesthetic Meditations 1913*, L. Abel, trans. (New York: Wittenborn, Schultz, 1949) pp. 13–14.

17. Lissitzky-Küppers [12] p. 350.

18. There are other ways to define curvature of a curve at a given point. However all the definitions are equivalent. For details, see D. Hilbert and S. Cohn-Vossen, *Geometry and the Imagination* (New York: Chelsea, 1952) pp. 172–178.

19. The existence of the principal curvatures is due to Euler (1760). For a proof of this fact see Theorem 1 in Spivak [7] p. 2.

20. This is not Gauss's definition of curvature, but it is equivalent to it. There are many different definitions of curvature, but they are all equivalent. For a discussion of the advantages of various definitions of curvature, see Torretti [8] p. 74.

21. C. F. Gauss, *General Investigations of Curved Surfaces*, A. Hiltebeitel and J. Morehead, trans. (New York: Raven Press, 1965) p. 9f.

22. Hilbert and Cohn-Vossen [18] p. 246.

23. Gauss applied the measure of curvature only to surfaces. Later, Riemann extended this term to space of n dimensions. However, while for surfaces it corresponds to an intuitive property, the measure of curvature of the space is a purely analytical expression. Furthermore, to speak of measure of curvature of the space is sheer nonsense unless the space is provided with a manifold structure (see [27] below). Note, however, that Lissitzky recognizes that curvature, when referred to space, is an abstract concept (Lissitzky-Küppers [12] p. 350).

24. Lissitzky-Küppers [12] p. 351.

25. For example, Lissitzky's irrational space is compared to the set theoretical construction of the real numbe ➜ E. Levinger, "Art and Mathematics in the Thought of El Lissitzky: His Relationship to Suprematism and Constructivism", *Leonardo* **22**, No. 2, 227–236 (1989).

26. According to B. Russell, "there is, if I am not mistaken, considerable obscurity in the definition of a manifold." See B. Russell, *An Essay on The Foundations of Geometry* (New York: Dover, 1956) p. 66. In fact, Riemann's lecture was read to a nonmathematical audience, and all formal notions were omitted. For this reason, it is very difficult to understand what Riemann said even today. However, a formal definition that agrees with what Riemann had in mind when he spoke of manifold can be made (see [27] below). M. Spivak explained how this precise definition expresses Riemann's concept of manifold: see "What did Riemann say?", in Spivak [7] pp. 4B–1f. For a fuller discussion of the difficulties of understanding Riemann's concept of manifold, see Torretti [8] pp. 85–107.

27. An n-dimensional manifold is a metric space with the property that for each element x of M there is some neighborhood U of x such that U is homeomorphic to R^n. This means that manifolds are locally like n-dimensional space. For further details, see the entry "Manifolds", in S. Iyanaga and Y. Kawada, eds., Mathematical Society of Japan, *Encyclopedic Dictionary of Mathematics* (Cambridge: MIT Press, 1977) pp. 819–823. However, to deal with the geometry of curves and surfaces conceived as manifolds, a restricted kind of manifolds, differentiable manifolds, is required. Yet this new definition does not give meaning to the measure of curvature of the space of n dimensions. To this end, an even more restrictive class of manifolds must be considered: the so-called Riemannian manifolds. See Spivak [7] pp. 4B-3–4B-5; and M. do Carmo, *Differential Geometry of Curves and Surfaces* (Englewood Cliffs, NJ: Prentice-Hall, 1976) pp. 425–443.

28. On color space, see H. Weyl, *Philosophy of Mathematics and Natural Science* (New York: Atheneum, 1963) pp. 70–71.

29. Riemann [7] p. 4A-6.

30. Lissitzky-Küppers [12] p. 350.

31. In order to consider irrational space as a four-dimensional manifold, let the elements in the manifold be 4-ples of real numbers. The first three correspond to the intensities of blue, red and yellow, and the fourth coordinate is a real number that depends on the position of the element with respect to the canvas: negative in depth, positive in the frontal direction and zero in the picture's surface.

The perusal of Lissitzky's essay supports this interpretation. Art objects generate a space, the art space. Space conceptions precede forms of art (Lissitzky-Küppers [12] p. 353), and properties of the space constitute its geometry. But one of Riemann's ideas (explicitly expressed by Helmholtz, see Helmholtz [34] p. 656, below) was that the geometry of a space is determined by the notion of distance. Accordingly, "in this space [writes Lissitzky] the distances are measured only by the intensity and the position of strictly defined colour areas" (Lissitzky-Küppers [12] p. 350). Certainly Lissitzky was familiar with the geometry and the perceptual theory of the color space, for he wrote that "new optical experience has taught us that two areas of different intensities, even when they are lying in one plane, are grasped by the mind as being at different distances from the eye" (Lissitzky-Küppers [12] p. 350). Thus, three coordinates of the elements of irrational space are obtained. The fourth agrees with the following passage from Lissitzky's essay: "Suprematistic space may be formed not only forward from the plane but also backward in depth. If we indicate the flat surface of the picture as 0, we can describe the direction in depth by – (negative) and the forward direction by + (positive), or the other way around. We see that Suprematism has swept away from the plane the illusions of two-dimensional planimetric space, the illusions of three-dimensional perspective space, and has created the ultimate illusion of *irrational* space, with its infinite extensibility into the background and foreground" (Lissitzky-Küppers [12] p. 350).

According to Yve-Alain Bois, Lissitzky wanted to destroy the spectator's certainty and deconstruct any visual apprehension of depth. "In his *Prouns,* Lissitzky wanted to invent a space in which orientation is deliberately abolished: the viewer should no longer have a base of operations, but must be made continually to choose the coordinates of his or her visual field, which thereby become variable", writes Bois in "El Lissitzky: Radical Reversibility", *Art in America* **76**, No. 4, 160–181 (1988). The interpretation of irrational space as a four-dimensional manifold agrees with Bois's suggestion.

32. Torretti [8] pp. 67–68. For an excellent presentation of the general knowledge on geometry during that epoch, see the section "The Rise of Popular Interest in the New Geometries", in L. D. Henderson, *The Fourth Dimension and Non-Euclidean Geometry in Modern Art* (Princeton, NJ: Princeton Univ. Press, 1983) pp. 10–43.

33. Russell [26] pp. 15, 66–68. Russell attributes to Herbart's writings the comparison of space with the color series and "many of Riemann's epoch-

making speculations" (Russell [26] p. 63). Note that Herbart and Gauss were the only authors quoted in Riemann's 1854 lecture (Riemann [7]).

34. H. von Helmholtz, "Ueber den Ursprung und die Bedeutung der geometrischen Axiome", in Helmholtz, *Ueber Geometrie* (Darmstadt: Wissenschaftliche Buchgesellschaft, 1868) pp. 1–31. (This was a lecture delivered by Helmholtz in 1870 and first published in 1884.) Quotations and references are from the English translation, "On the Origin and Significance of Geometrical Axioms", in J. R. Newman, ed., *The World of Mathematics* (New York: Simon and Schuster, 1956) pp. 647–668.

35. Helmholtz [34] p. 648. Helmholtz was the most widely read writer on the foundations of geometry (see Russell [26] p. 23), and his article "On the Origin and Significance of Geometrical Axioms", the most influential source for the philosophical debates on geometry from the 1870s onward (see Torretti [8] p. 155). Peter Nisbet has suggested that most features of Lissitzky's writings in the early 1920s are drawn from the first volume of Spengler's *Decline of the West;* see P. Nisbet, "An Introduction to El Lissitzky", in *El Lissitzky 1890–1941* (Cambridge, MA: Harvard Univ. Art Museums, Bush-Reisinger Museum, 1987) p. 29. Spengler (in O. Spengler, *The Decline of the West*, Vol. 1, C. F. Atkinson, trans. [London: George Allen and Cluwin, 1926]) explains Riemann's spherical geometry (p. 88), *n*-dimensional manifolds (p. 74, p. 90) and many other mathematical subjects. According to him, Helmholtz is a "savant of the calibre of Gauss and Humboldt" (p. 424) whose lecture of 1870 "has become famous" (p. 377). It is interesting to compare Lissitzky's and Helmholtz's essays. There are numerous similarities between them. To start with, Helmholtz three times emphasizes that abstract mathematical concepts cannot be represented. Examples are: the fourth dimension (Helmholtz [34] p. 650 and p. 664) and the measure of curvature of the space (Helmholtz [34] p. 656). The same remark with the same examples is found in Lissitzky's article: "The space in our physical science is three-dimensional" (Lissitzky-Küppers [12] p. 351); and "we cannot change the degree of curvature of our space in a real way" (Lissitzky-Küppers [12] p. 351). Other similarities concern Riemann. Helmholtz's main aim was to expose and comment on Riemann's conception of space (Helmholtz [34] p. 655f). Considerations on *n*-fold extended aggregates, the color manifold, the measure of curvature of the space and the importance of distance all play important roles in Helmholtz's discussion of Riemann's ideas—and also in Lissitzky's essay (for example, see [31]). Thus, although we do not know whether Lissitzky read Helmholtz's essay, we suspect that he did.

36. Lissitzky-Küppers [12] p. 351.

37. Lissitzky-Küppers [12] p. 351.

38. See, for example, B. Zevi, *Architecture as Space (How to Look at Architecture)* M. Gendel, trans., J. A. Barry, ed., (New York: Horizon Press, 1957) p. 26: "The Cubist conquest of the fourth dimension is of immense historical importance", p. 26: "the mind of man discovered that a *fourth* dimension existed in addition to the three dimensions of perspective. This was the Cubist revolution in the concept of space, which took place shortly before the first World War".

39. Kazimir Malevich's *Black Square*, oil on canvas, 79 × 79 cm, 1913, was shown for the first time in The Last Futurist Exhibition of Pictures: 0–10, Petrograd, 19 December 1915–19 January 1916. *Black Square* entered the Tretiakov Gallery in Moscow in 1929 (No. 11225) and is still there. For a good photograph of this painting and commentaries, see J. C. Marcade, "Kazimir Malevich, Quadrangle, 1915", *Art Press* **111** (1987) pp. 84–85.

40. See Henderson [32] pp. 287–291; and S. Compton, "Malevich and the Fourth Dimension", *Studio International* **187**, No. 965, 190–195 (1974).

41. As an example, consider polyhedra of 4-dimensional space (they are also called polytopes). Their representations into spaces of fewer dimensions depend on the center of projection and, eventually, the image plane. Therefore the designs of these polyhedra could be significantly different. For details, see Hilbert and Cohn-Vossen [18] pp. 145–157.

42. Lissitzky-Küppers [12] p. 350.

43. Through the interpretation of irrational space given in [31] above, Malevich's *Black Square* corresponds to the point (0, 0, 0, 0) of the manifold. This agrees with the view of Lissitzky: "It is only now in the twentieth century that the square is being acknowledged as a plastic value, as 0 in the complex body of Art. The solidly coloured square stamped out in rich tone on a white surface has now started to form a new space" (See Lissitzky-Küppers [12] p. 350).

44. PROUN is an acronym for *Proekt utverzhdeniya novogo* (Project for the Affirmation of the New). *Proun Space* was included in the exhibition Grosse Berliner Kunstausstellung, Berlin, 1923. Lissitzky's description of *Proun Space* is included in "Proun Room, Great Berlin Art Exhibition (1923)", in Lissitzky-Küppers [12] p. 361. For a detailed discussion of *Proun Space,* see A. C. Birnholz, "El Lissitzky: From Passivity to Participation", in Barron [11] pp. 98–101. For a detailed discussion of Lissitzky's works known under the neologistic acronym PROUN, see Nisbet [35] pp. 17–25.

45. In "Nasci", Lissitzky writes that "modern art, following a completely intuitive and obvious course, has reached the same results as modern science" because both have concluded that "every form is the frozen spontaneous image of a process. Thus a work is a stopping-place on the road of becoming and not the fixed goal" (from El Lissitzky, "Nasci", in Lissitzky-Küppers [12] p. 347). Clearly, this is a relativistic view. Further, it is identical with the last paragraph in chapter 3 of A. N. Whitehead, *The Concept of Nature* (Cambridge: Cambridge Univ. Press, 1920).

According to P. Nisbet (Nisbet [35] p. 30), this sentence, cited as a central idea in Lissitzky's thought, is copied verbatim from Raoul Heinrich Francé's *Kie Pfanze als Erfinder.* Nisbet comments on the fact that some of Lissitzky's ideas are drawn from Spengler and Francé (see Nisbet [35] pp. 28–30).

46. El Lissitzky, *Erste Kestnermappe,* color lithographs (Hannover: Kestner Gesellschaft, 1923). Reproduced in *El Lissitzky 1890–1941,* exh. cat. (Hannover: Sprengel Museum, 1987–1988) pp. 160–163.

47. See Henderson [32] pp. 139–140. Henderson traces this view back to Poincaré and Jouffret.

48. See Gray [11] chapters 5 and 8, and A. Birnholz [12].

49. See the Introduction to Lodder [11]. For example, according to E. Levinger (Levinger [25] p. 231), Lissitzky initiated to study the correspondences between art and mathematics as a way to understand the denial of absoluteness. Lissitzky's ultimate aim was to install a new social order, as art and mathematics had done.

50. N. Gabo and A. Pevsner, "The Realistic Manifesto", in J. E. Bowlt, ed., *Russian Art of the Avant-Garde: Theory and Criticism, 1902–1934* (New York: Viking Press, 1976) pp. 209–214. For Gabo's views on art and science, see N. Gabo, "Art and Science", in *The New Landscape: In Art and Science,* G. Kepes, ed. (Chicago, IL: Paul Theobald, 1956) pp. 60–63.

51. I quote from page 17 of H. Read, "Constructivism: The Art of Naum Gabo and Antoine Pevsner", in *Naum Gabo—Antoine Pevsner,* exh. cat. (New York: The Museum of Modern Art, 1948) pp. 7–13. The relationship between Gabo's scientific background with his use of planes to describe volumes is discussed by Martin Hammer and Christina Lodder in "Naum Gabo and the Constructive Idea of Sculpture", in *Naum Gabo: The Constructive Idea,* exh. cat. (London: South Bank Centre, 1987) pp. 41–51. See also pp. 13–15 of S. A. Nash, "Sculptures of Purity and Possibility" in *Naum Gabo: Sixty Years of Constructivism,* S. A. Nash and J. Merkert, eds. (Munich: Prestel-Verlag, 1985) pp. 11–46.

52. At that time there were two big suppliers of mathematical models: Martin Schilling in Leipzig and B. G. Teubner in Leipzig and Berlin. They published catalogs: *Catalog mathematisher Modelle fur den Höheren mathematishen Unterrich* (Leipzig: Martin Shilling, 1911), and *Verzeichnis von H. Wieners und P. Treutleins Sammlungen Modelle fur Hochschulen, Höhere Lehranstalten und Techniche Fachschulen* (Leipzig and Berlin: B. G. Teubner, 1912).

53. For a discussion on circle models, see Hilbert and Cohn-Vossen [18] pp. 17–19. The analytical proof that the quadrics mentioned can be constructed with circles is in D. M. Y. Sommerville, *Analytical Geometry of Three Dimensions* (Cambridge: Cambridge Univ. Press, 1959) pp. 205–206. The construction of the ellipsoid using cardboard circles is explained in H. M. Cundy and A. P. Rollet, *Mathematical Models* (Oxford: Clarendon Press, 1954) pp. 154–156.

54. This result is due, mainly, to Möbius and Cremona. For a proof of it, see G. Salmon, *A Treatise on the Analytic Geometry of Three Dimensions,* Vol. I (New York: Chelsea, 1927) p. 345. The resemblance of *Crystal* to the cubic ellipse was suggested by A. Hill on p. 144 of "Constructivism—The European Phenomenon", *Studio International,* Vol. 171, No. 876, (1966) pp. 140–147. Compare also S. A. Nash [51] pp. 34–35, in particular marginal note 114.

55. Space curves can be determined as the intersection of two surfaces given by equations in homogeneous coordinates $F(x, y, z, w) = 0$ and $G(x, y, z, w) = 0$. In case of the cubical ellipse take the cone $F(x, y, z, w) = y^2 + z^2 - zx = 0$ and the elliptical cylinder $G(x, y, z, w) = y^2 + z^2 - yw = 0$.

56. Enneper's surface is the surface given in parametrized form as:

$$x = u - u^3/3 + uv^2$$
$$y = v - v^3/3 + u^2v$$
$$z = u^2 - v^2.$$

The principal curvatures are:

$$k_1 = 2/(1 + u^2 + v^2)^2 \text{ and } k_2 = -2/(1 + u^2 + v^2)^2.$$

Therefore $H = (k_1 + k_2)/2 = 0$. H is called the mean curvature. And $H = 0$ is the condition for minimal surfaces. See J. C. C. Nitsche, *Vorlesungen uber Minimalflachen* (Berlin: Springer, 1975) pp. 75–81. Because $H = 0$, it follows that $k_1 = -k_2$. Hence the Gaussian curvature of the surface, i. e. the product k_1k_2, is negative. This point made a difference between Gabo's *Spheric Theme* and Enneper's Minimal Surface, because Gabo's form is made with two flat circles (see Fig. 9), and, as a consequence, its Gaussian curvature is zero. However, from a purely visual point of view, both look very similar because their shapes are identical. For an account of minimal surfaces in art, → M. Emmer, "Soap Bubbles in Art and Science: From the Past to the Future of Mathematical Art", *Leonardo* **20**, No. 4, 327–334 (1987). A computer-generated image of Enneper's surface appears in D. Hoffman, "The Computer-Aided Discovery of New Embedded Minimal Surfaces", *The Mathematical Intelligencer* **9**, No. 3, 8–21 (1987).

57. Read [51] p. 11.

58. See Levinger [25]. Levinger argues that Lissitzky's analogy between pictorial space and mathematical concepts led to the foundations of the theoretical bases for nonobjective art, different from both Suprematism and Constructivism. Nonobjective art is discussed in Levinger [25] pp. 228–230.

59. Apollinaire [16] p. 13.

60. Lissitzky-Küppers [12] p. 348.