A

PALIMPSEST

on the

Electronic Analog Art

Being a collection of reprints of papers & other writings which have been in demand over the past several years

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A Palimpsest on the Electronic Analog Art

Being a collection of reprints of papers and other writings which have been in demand over the past several years. Some of the entries are discreetly annotated. Newer material has been included to reduce obsolescence, but there is no claim to definitive or exhaustive treatment.

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FOREWORD

The title of this compilation derives from the Greek παλιμπσεστος (palimpsestos) meaning "scraped again" and referring to parchments or tablets which have been erased and used over again two or three times. The term has come to mean documents which have been rewritten or re-engraved. A rough rendering of the term would correspond to the common slang "rehash".

In less than a decade, the electronic analog art has passed through a rapid evolution, the early stages of which now seem of mere historical interest. Development of new tools and application to new problem areas is proceeding at an accelerating pace. For the novitiate technologist, however, some perspective on this bedusted gladiatorial arena is needed.

Thus, we have happily undertaken this compilation of lore to meet a continuing demand for introductory and fundamental material needed by teachers, students and practitioners alike.

The reader will immediately observe that our source material has merely been "dusted off and stapled up" into a veritable "coat of Joseph", with somewhat less than the minimum amount of editing, recomposing and other customary editorial concommitants. The editor humbly begs the reader's forebearance, leaning on the excuse that such short cuts have saved time and expense and have reduced the possibility of introducing new errors in processing.

Our touchstone for the material which we have included is quite simple: mere pertinence.

We believe in each case that the text has one or more of the following attributes: historical interest, soundness of doctrine, authentic novelty, pedagogical value.

Without doubt, each of the authors involved would fondly wish his work could have been revised and modernized. But besides the improbability of such an undertaking, much historical flavor would be lost in the process.

Rather than including a separate, less complete bibliography we felt that the several hundred references distributed among the individual papers serve this purpose admirably.

Lastly, your editor apologizes specifically for the inclusion of no small part of our own writings. Mayhap they were less dusty and assuredly did they persist in popping before our eyes.

We close, then, with some powerful words of Kepler:

". . . and I cherish more than anything else the Analogies, my most trustworthy masters. They know all the secrets of Nature . . ."

The Electronic Analog Computor as a Lab Tool

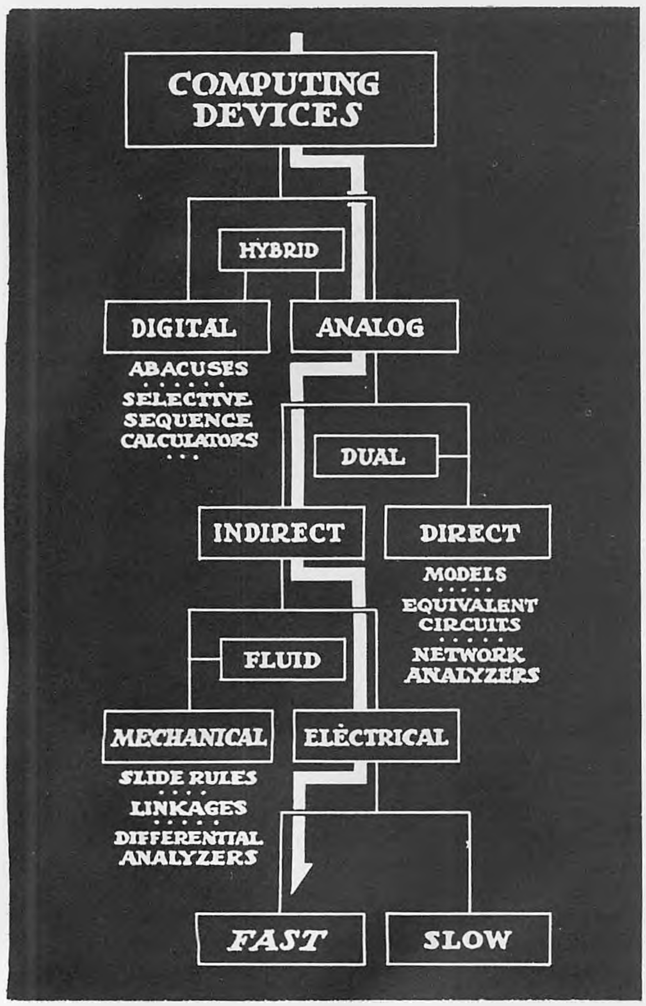
INDUSTRIAL LABORATORIES—May, 1952

By George A. Philbrick,
Geo. A. Philbrick Researches, Inc.
Boston Mass.
and
Henry M. Paynter,
Massachusetts Institute of Technology

Would you drive a spike with your fist? No? Then why bruise your brain on an armor-plated differential equation? Man is a user of *tools*, and is helpless without them. His aboriginal club and spear outdid the beast; and now the superiority of his implements measures the material advance of civilization. All *instruments*, however elaborate and refined, are just tools along with hoes and hammers. They extend the power of our muscles, the length of our arms, and the sharpness and range of our perceptions. At the upper stages, culturally, come the tools which extend the powers of the intellect. And chief among these we count computing devices, or *Computors.*

The computing machine is becoming as essential to our modern existence as electricity and the automobile. Already with the help of computors, innovations have been wrought that would assuredly have been unthinkable without their use; recent rapid advances in the fields of nuclear science, supersonic flight and rocketry offer strong testimony to their value in analysis, design, development.

Not only have computors been used to obtain increased knowledge and more accurate, rapid prediction of the behavior of physical processes, but on an ever greater scale computing equipment is being incorporated as an essential element in the control itself of such processes. Present examples of such applications are the well known computing gunsights and other fire-control equipment, as well as the less well known but equally successful uses of computors to operate machine tools, and for continuous automatic process and quality control. Potential developments of this sort seem unlimited.



Digital and Analog

Computors come in all sizes, all speeds, all prices. Here we shall concentrate on the *fast electronic analog*, but first we must place this type properly in the hierarchy of all computors. The two major categories, nowadays, are called *digital and analogical*. The one deals in numbers only, the other in continuous physical variables. The first computes by repeatedly refining an approximation, and its accuracy is potentially unlimited; in the second type some sort of physical model is set in operation to generate a solution which is valid by virtue of an analogy to the problem at hand. This latter type, the analog computor, is limited in accuracy by the physical elements of the model.

It is frequently constructive to employ more than one type of computor on any given problem, since each type may be most effective for certain phases. Thus analog and digital methods may complement one another when applied in sequence, the first pointing the way to an answer and the second leading to any desired precision of the final results. As to all the types and combinations of computors which are in use and in development, there is no pretense here of complete generality or inclusiveness. Space limitations, alone forbid such scope.

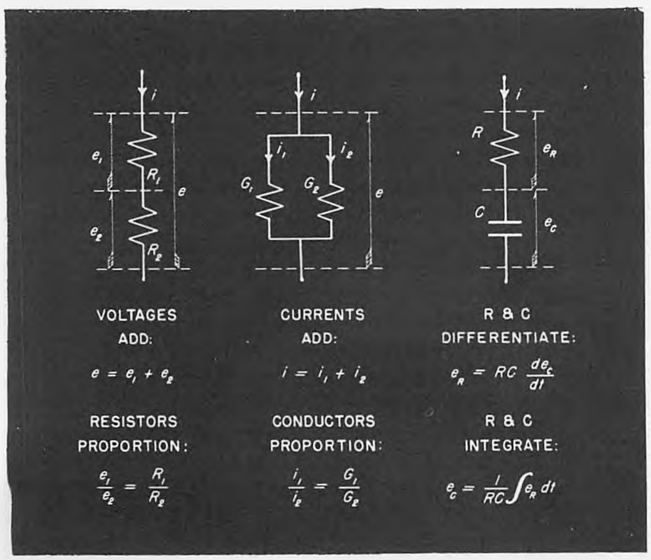
We leave the digital types now (to their proponents), and proceed in more detail to break down the analog category into its several branches. One important branch, not usually called "computors," is represented by true *scale models*. In these a structure or phenomenon is reproduced by changing its physical dimensions without discarding the general physical form. To this branch belong such models as the tiny experimental airplane in a wind tunnel, the miniature dam and dammed lake on a table top, and even model transmission lines reduced to laboratory size. Somewhat less direct models are those in which the medium is altered: for example the electrolytic tank used in the study of hydromechanical field problems. Still less direct, but models nevertheless, are the semi-mathematical types normally identified as analog computors. The physical elements of such computors make available a set of *operations*, which may be chosen either because they are mathematically fundamental, or because they recur as dynamic relations in the structures being studied.

Various Media

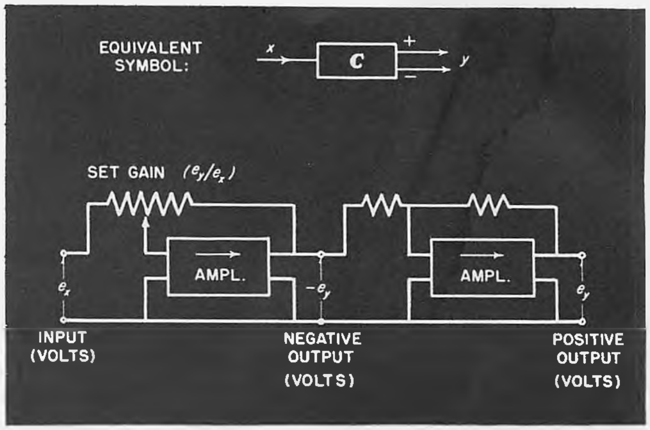
Indirect analog computors generally carry out or assist in the solution of equations, algrebraic and differential, linear and nonlinear. An extremely familiar and simple example is the *slide rule*, which is entirely algebraic. At the other extreme of elaboration is the *differential analyzer,* which may be a roomful in itself, and which can take some formidable differential equations in its stride. Actually this general class of computor is represented in many physical media, even including pneumatic and hydraulic; the predominant realms, however, are the mechanical and electrical. The differential analyzer already mentioned, being based on a mechanical disk-type integrator mechanism, belongs to the mechanical group. This instrument is very far from being outmoded, but as a practical matter it cannot be operated at a conveniently high speed for certain applications.

We turn now to the *electrical* (and electronic) *analog computors.* Some machines in this class are often referred to as "differential analyzers," but this is perhaps an impropriety in view of the connotations of that term.

Electrical analogs of the indirect class normally employ high-gain amplifiers, which are applied in feedback loops to assure accuracy in terms of a few highly consistent circuit elements. By such means it is possible to take advantage of the precision and other desirable properties of those elements, and accordingly to perform quite strict mathematical operations among the variables in an artificial system. Let us take a look at some fundamentals, beginning with purely conductive or resistive elements. If such elements are arranged in series, the voltage across an entire set is equal to the sum of the voltages across the individual elements. Again, if conductors converge, or branch from a single conducting element, the current in the single element is precisely the algebraic sum of the currents in the branches. Either of these physical truths may be applied to make a reliable *adder.* Look further at the property of resistance, or at the inverse property of conductance. In any given electrical path, these properties fix the proportionality between current and voltage in a precise way. For example if a pair of resistors have equal currents, the voltages they experience are in proportion to their resistances. In an appropriate feedback loop, this fact may be used to magnify or diminish a computed signal in a ratio which may be accurately determined and adjusted. Thus for example one can "multiply by a constant" of any magnitude, or multiply voltages by each other.



Another electric circuit element available in pure form is the capacitor, which is able to store a charge. The charge it stores is equal to the product of its capacitance and the voltage across its terminals. Thus the charge is measurable by means of this voltage, and being quantity of electricity its rate of change is simply the current supplied to the capacitor element. Conversely, the voltage is proportional to the time integral of the incoming current, and the proportionality is directly dictated by the value of the capacitance. These facts lead to a means for *integrating* and *differentiating* with respect to time.



It is convenient, however, instead of dealing with both currents and voltages, to employ one or the other type of variable alone for the inputs and outputs of computing operations. By applying regulatory—or *feedback*— techniques, the above elementary circuit arrangements may be adapted in such a way that the relevant operations take place between applied currents and responding currents, or between applied voltages and responding voltages. Either currents or voltages may serve in practice as computing signals: the choice is up to the designer. It happens that voltages, applied and measured with respect to a common conductor (or "ground"), have become most popular. In any event, one is led by these techniques to devices in which the operations of addition, multiplication by adjustable factors, and differentiation or integration with respect to time are automatically performed between inputs and outputs; and generally these latter are voltages with respect to a convenient reference. In a computing or analog assembly, of course, these voltages correspond to the *variables* of the equations or systems represented.

Combining of Operations

Thus far it has been shown how to *add,* to *proportion,* to *integrate* and *differentiate,* and how such operations may be embodied in individual instruments (or components) to serve as building blocks for fashioning a computor. It should also be pointed out how general these few operations really are. Note first that the proportioning operation will permit multiplying by negative as well as positive factors, and in particular by minus one. This enables the extension of adding, for example, to include the operation of *subtracting.* These two operations, in combination with proportioning to establish coefficients, and with full freedom of repetition and sequencing, will be seen able to reproduce and represent any *first-degree arithmetic* operation, and hence may be applied to "solve"sets of linear algebraic equations. The technique is simply to interconnect the operational components as dictated by the equations to be solved. Solution is then performed by the assemblage with no further outside ministrations, provided the power is on and all connections are properly established. One reads off the answer by measurement of the voltages corresponding to the unknown variables.

Usually more than one arrangement is possible for satisfactory solving a given problem with operational components such as those described above. But whatever the arrangement, certain rules must be observed. Of course it is evident that the known variables are fed in from without, and that the known or assumed characteristics are set in by adjustment of the various components. In general there will be several unknown variables involved, and there may be a choice as to how the corresponding voltages in the computor are allowed to affect one another. In these relations we recognize a *causal* property. For any component of the above type there is a definite direction from input to output, coinciding with the direction from cause to effect. This implies a *causal order* among the variables in the problem as set up on the computor, an order which is not explicit in the mathematical formulation. Thus there is some room for imagination in setting up a computor, and often there are many approaches leading to success. We cannot cover all the techniques of computing in this space, but it may be pointed out at least that in general, *loops* of causal activity are formed in all but the simplest cases. As a result the question of stability arises, as with causal loops in other connections, and much attention is given this subject by those who design and deal with such instruments. Suffice it to say that stability may always be achieved

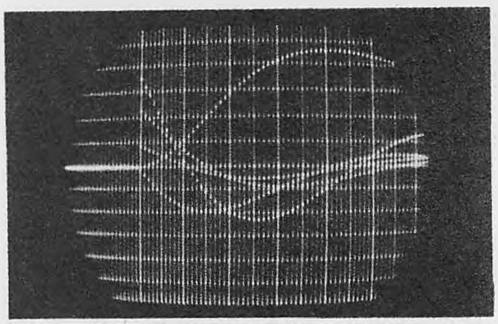
When time and rate of change are added as variables, algebraic problems turn into *calculus.* We have, for example *differential equations*. The addition of either differentiating or integrating components enables the computor to handle the linear class of such equations. It has turned out, for reasons of stability as already mentioned, that integrating components are usually more practical, but it is not uncommon to employ both types in the same problem. At least ideally one type (the integrator say) will cover all cases in the solution and study of linear phenomena, and any linear system may be represented by combinations of the three components for adding, proportioning, and integrating. Parenthetically: for evident reasons, the proportioning component is often called the *coefficient* component. A bit later we shall discuss techniques and equipment for nonlinear applications.

Slow or Fast?

The various electric analog computors discussed to this point have not been identified as to speed of operation. They may be designed to operate exceedingly fast, completing a solution in one microsecond or less. They may also be built to run very slowly indeed, taking days to "figure out" just one problem. An optimum speed, between these limits, is generally dictated by the economics of the application. Sometimes a computor of this sort is employed to *simulate* the operation of some portion of a system or process under development, and then *its speed is auto*matically determined by that of the part being simulated. When there are no criteria of this sort present, there is a very important advantage *in choosing* a speed of operation such that a solution is completed faster than we can change the conditions and characteristics of the problem. This is true whether the equations are algebraic or differential. In the latter case, where time is the independent variable, a fast computor enables repetition of the solution with a period so short that a sustained oscillographic display of the unknown variables is possible, showing the solutions continually to the operator as apparently stationary functions on the screen. The screen pattern, of course, alters as the initial conditions or the parameters are manually changed. No waiting period is required to study the solutions for a wide variety of cases.

However, the cathode ray oscilloscope as customarily used suffers from several calibration defects. These objections are at once removed by the *electronic graph paper* technique recently developed, which presents results as indicated in the accompanying figure.

CALIBRATED ANALOG DISPLAY



Horiz. Scale 0.03 SEC   Vert. Scale 100V

Broadly speaking, the method is similar to the familiar TV raster presentation but rotated 90°. In one embodiment, a 20-cycle sawtooth wave is employed for the horizontal time sweep, so that a vertical scan of 2500 cycles produces 125 lines across the screen. This vertical sweep in voltage form is compared in passing with a set of constant equally-spaced reference voltages (comprising the horizontal grid) as well as with all voltage variables chosen for display. At each coincidence between scan voltage and these "plotted" points a unique method of velocity modulation is employed for producing a brightened spot on the screen.

If the vertical lines are then counted by fives and tens, with corresponding intensification, a complete calibration is made possible for both axes.

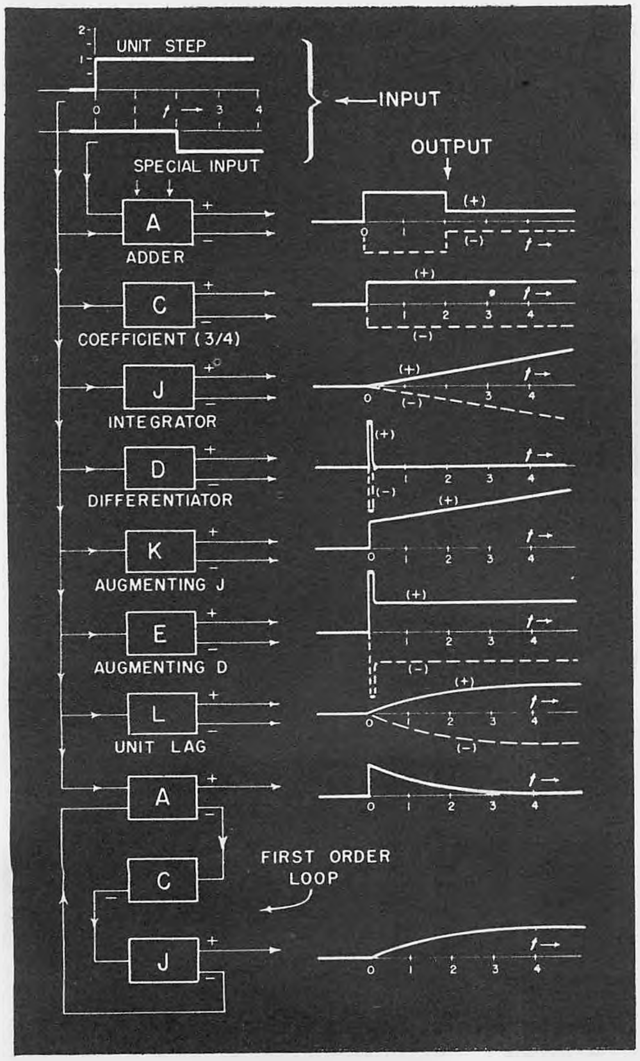
Using this method, repetitive displays may be read at a glance to better than 1%, with a presentation accuracy rendered *independent of distortion or bias* in the oscilloscope amplifiers or in the geometry of the cathode ray tube, since all signals and calibration lines are similarly transformed.

Fortunately the cost of an electric analog computor is minimum for this reasonably high-speed version. Electronic techniques and manufactured parts are plentiful for use in the audio-video band which is involved in devices of such a time scale. In terms of frequency, it is possible to employ a band extending from the repetition rate as fundamental up to frequency a thousand or more times higher. The usual procedure, however, is to operate all the way down to zero frequency (DC). While it may be difficult to speed up a DC Computor, it is relatively easy to make it operate more slowly merely by increasing the reactive impedances.

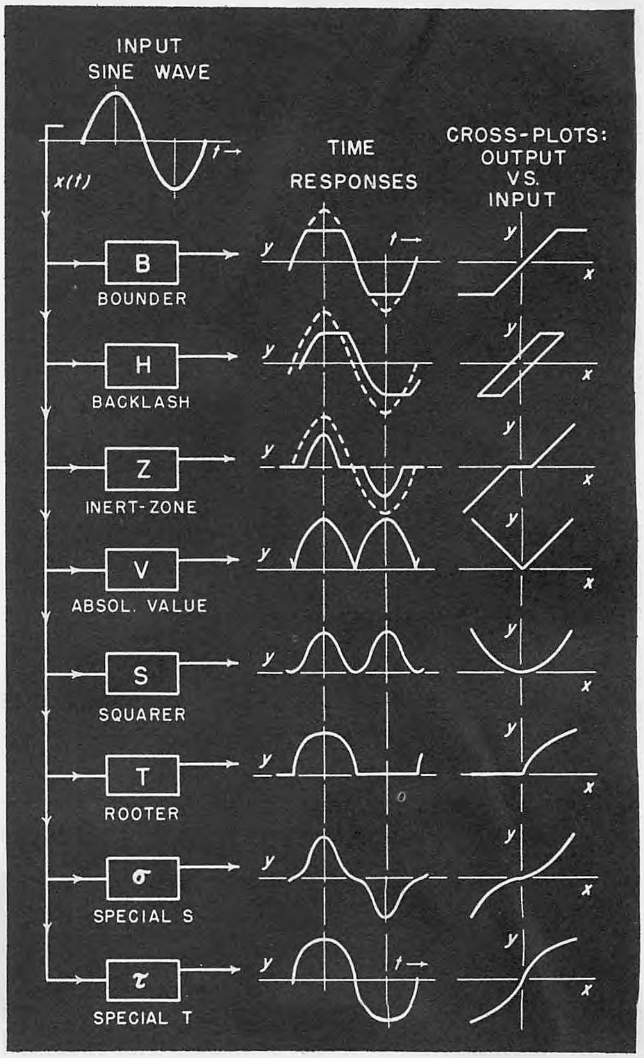
Computing Components

Electric analog components which add, set coefficients, integrate and differentiate have already been referred to. As a practical matter, these are designed with high-impedance inputs and low-impedance outputs, for foolproof interconnection in a computing system. Algebraic sign is allowed for by providing both positive and negative outputs for each component. With these and a few other practical features — for supplying power, adjusting, identifying, and so on—such components are engineered for convenient usage as versatile general-purpose analog *building blocks*.

While the above component operations are sufficient for all linear mathematics, or model-building, it is economical to add certain other components in the linear class for combined operations which keep arising. A large number of such combinations are current, some fairly complex; we mention here only three of the simpler examples. One is like the integrator, but transmits an additive signal equal to the input: this component is called the *augmenting integrator*. Similar in concept is the *augmenting differentiator*. An accompanying illustration shows responses of these and other components when subject to a test square wave. These augmenting variants are particularly useful in the study of control problems. Another such component is that for the simplest dynamic lag. Called the *unit-lag* component, it actually operates inversely to the augmenting differentiator. These two in sequence will leave a signal unchanged, if the adjustments are matched.

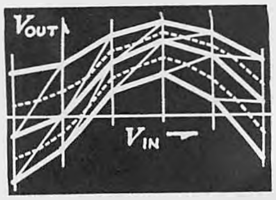


In this box of tricks are also components which embody basic nonlinear properties, notably *bounding, backlash, inert-zone, absolute-value, squaring,* and *square-rooting* components (variants of the last two are included which feature a complete inverse symmetry useful especially for hydraulic problems). All these characteristics may be demonstrated in terms of their response to a sine wave input, plotted either against time or versus the input itself. Components in this group are employed to represent real physical discontinuities and "distortions" which are often assumed absent in conventional analysis. In many cases the effects of such irregularities are important, often crucial.



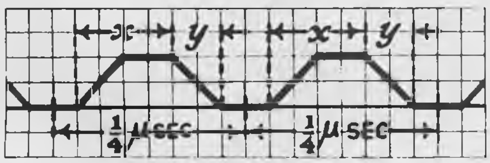
Besides the above elementary nonlinearities, it is often necessary to represent functional relationships between two variables which cover an unlimited variety of shapes and forms, and which perhaps are known only empirically. For this purpose, components are available which can fit such curves by linear-segmented polygons, with angles and lengths adjustable. In addition, modern forms of this device permit a manipulated degree of "rounding" at each vertex to permit greater articulation. By these means, arbitrary functional constraints can be represented in analog computors in a manner affording complete alteration during operation.

Furthermore, this particular method may be generalized in a direct fashion to implement functional relations involving more than one input variable. The behavior of one form of this device is indicated in an illustration.

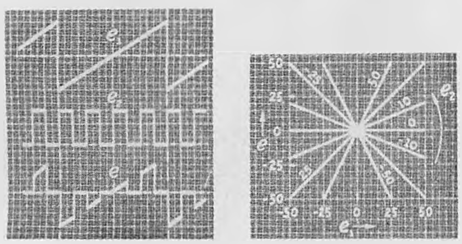


High-Speed Analog, Function of Two Variables

Perhaps the most important nonlinear operation in computing systems is that involving the multiplication of two variable voltage signals. One eminently successful method of all-electronic multiplication gives accurate voltage products from DC up to high repetitive speeds. This device employs a carrier signal of special form in the megacycle range combined with certain elementary nonlinear operations to yield a product waveform which, before final averaging, appears as the train of trapedzoidal pulses shown.



Multiplier Output Before Averaging



Multiplier Performance

The sides of the pulses have a constant slope and the two input variables, x and y, affect the pulse dimensions as indicated. Thus, when x and y are equal, a triangular pulse results, while, in general, the pulse height is determined by the smaller input. The system employed is thus completely symmetric, with the interchanging roles of the inputs assigned in a natural manner and with a similarly automatic conversion to negative pulses for inputs of opposite sign.

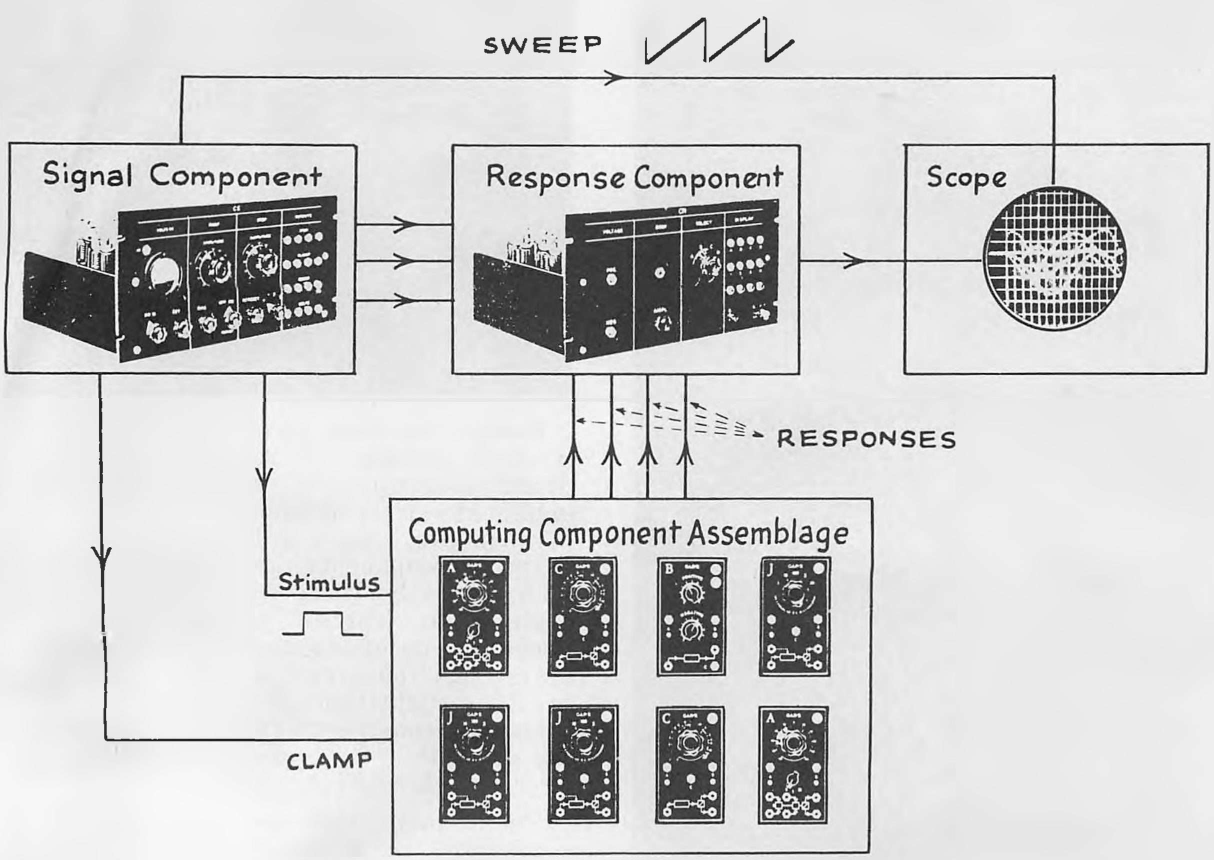
A multiplier so constructed will also perform division, made available simply through switching.

Applications, of course, abound for multiplication and division in computing structures. These include representation of variable parameters, computation of instantaneous powers of signals, and correlation analyses. In conjunction with dynamic components, multipliers permit integration with respect to variables other than time.

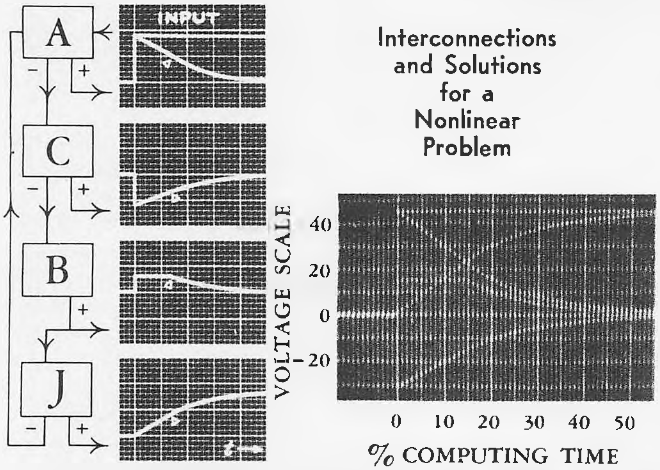
Computing Assemblages

A complete high-speed electronic computing installation requires in addition to the above computing and display components, auxiliary equipment to supply initial conditions, bias, clamping and timing signals, as well as the basic stimulae to be applied to the computing model itself. Thus a complete interconnected computing assemblage would appear as shown schematically in the illustration, indicative of all equipment (exclusive of essential power supplies) necessary to carry out the solution of complex engineering and scientific problems at lightning speed.

The arrangement of computing components into self-determining causal loops, together with



Interconnected Assembly Comprising Complete High-speed Computor consequent responses when subjected to input stimulation, are illustrated. The ready availability of time and magnitude information is apparent; this feature permits immediate quantitative determination of effects of parameter variations.



Interconnections and Solutions for a Nonlinear Problem

Only an idea may be given here of the scope of practical applications of the computing apparatus described. The broad mathematical usages should be evident from the text; an exhaustive summary along this line would depart from the matter of the present exposition, and might even be misleading. It is preferable to pick out a few familiar physical systems and to show how analog computor components may be interconnected to embody the laws and to represent the phenomena which operate therein: and thus to illustrate how dynamic performance may be "solved for."

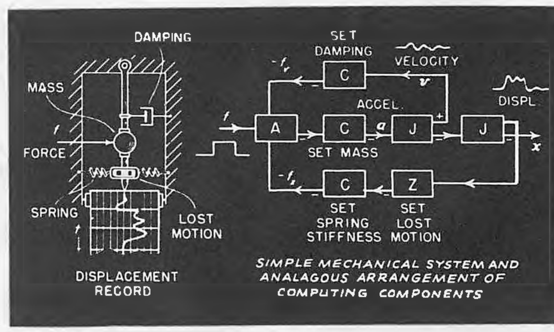
A first example is a mechanical structure exhibiting damped vibrations, involving a massive body, a spring, and a dashpot. Here the position of the body is developed as the integral of its velocity, which in turn is the integral of acceleration, which itself is related to the net force by the magnitude of the mass. The net force, finally, is the algebraic sum of an applied or external force, a spring force depending on position, and a damping force which depends on the velocity. Nonlinearities such as lost motion may be incorporated directly. All physical properties are quantitatively adjustable in the computing assembly.

Another example is offered by a thermal system under automatic control. The temperature in the inner of two concentric baths is to be regulated at an assignable value. Transfer lags are interposed between the application of heat and the location of the thermometer. Lagged responses are also present in the steam valve and in the bulb itself. One may represent, for example, all the classical control reactions, integral, proportional, derivative, and so on which might be applied in the regulatory installation. Limits of valve operation are directly imposed by the nonlinear bounding component. Solutions for optimum recovery, and consequently the most stable and effective controls, are obtained in short order by experimental variation of the design parameters under appropriate disturbances.

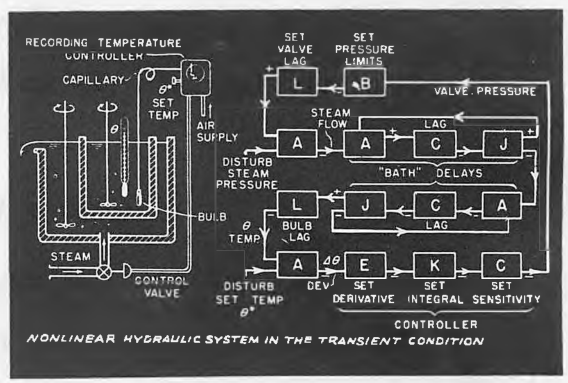
Seemingly paradoxical is the application of electronic computors to electrical networks, which are themselves used as models for phenomena in other media. Curiously, this is one of the most powerful and penetrating applications of the computor. The relations between voltage and current, for each inductor, capacitor, and resistor in the network, are merely set up in a consistent causal sequence by means of the simplest standard components. The result is a computing structure of wide flexibility whereby the desired network may be rapidly designed. Dual networks are implicit in the assembly: one needs only to reverse the roles of voltage and current. Negative values for each circuit element are as easily set in as are positive values.

Finally a hydraulic example may be selected, involving storage tanks, the inertia of fluid in a duct, nonlinear resistance to flow, and a check valve. The variables are levels (or hydraulic *heads*) and flows. The check valve may connect one tank to an infinite reservoir at fixed head.

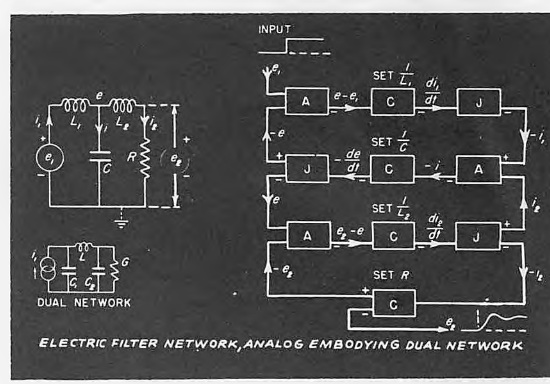
These examples demonstrate that a broad variety of useful systems and operating structures may be synthesized and analyzed to any desired detail with analog components of the type described.



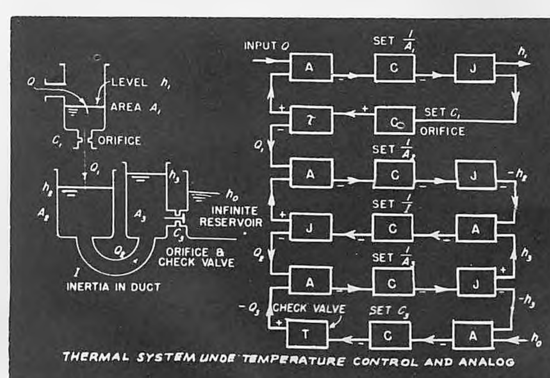
SIMPLE MECHANICAL SYSTEM AND ANALAGOUS ARRANGEMENT OF COMPUTING COMPONENTS



NONLINEAR HYDRAULIC SYSTEM IN THE TRANSIENT CONDITION



ELECTRIC FILTER NETWORK, ANALOG EMBODYING DUAL NETWORK



THERMAL SYSTEM UNDER TEMPERATURE CONTROL AND ANALOG

COMMENTARY

The foregoing article, which was brought up to date for inclusion in the "Palimpsest", covers in thumbnail fashion the history and art of the electronic analog field.

The next three papers cover in considerable detail significant developments in the evolution of electronic analog computors. That by Ragazzini et al demonstrates thinking and circuitry growing out of World War II efforts. The Mynall article outlines his inspired ideas conceived entirely independently of American accomplishments.

Lastly, the efforts of Alan Macnee at M.I.T. directed toward the simultaneous goals of high speed and accuracy in electronic differential analyzers have had a profound influence on subsequent developments.

May 19 PROCEEDINGS OF THE I.R.E.

Analysis of Problems in Dynamics by Electronic Circuits*

JOHN R. RAGAZZINI†, MEMBER, I.R.E., ROBERT H. RANDALL‡, AND FREDERICK A. RUSSELL§, MEMBER, I.R.E.

Summary—This paper describes a method for obtaining an engineering solution for integrodifferential equations of physical systems using an electronic system. The components consist of standard plug-in feed-back amplifier units. As the interconnections are wires, resistors, and capacitors, no complicated mechanical layout problem is involved and a generally flexible analyzer need not be set up, for it is a simple matter to assemble the particular circuit for any system of equations for which solutions are desired. The system should, therefore, be of interest to those involved in a study of the dynamics of physical systems.

I. Introduction

THE FORMULATION of electrical analogs of dynamic problems in fields other than electrical has long been used to obtain solutions for such problems.[1] Then, in most cases, a physically realizable network may be synthesized to fit the equations and a network used to obtain the electrical outputs representing the solution of the equations.[2] For complicated problems this method does not usually result in a network whose individual parameters correspond to the individual parameters of the original system, so that experimentation in the nature of varying the parameters is not simple. This objection is largely overcome through the generous use of isolating amplifiers within the electrical network. Until the modern methods of feed-back stabilization were developed, the use of amplifiers introduced complicating circuit elements which altered with variation in tube characteristics. The other method of attack on problems of this type has been through the use of the mechanical differential analyzer[3] having as its basic tool an ingenious mechanical integrator, recently improved through the use of a polarized-light servo-operated torque amplifier.[4,5]

The technique described herein employs as its basic tool a stabilized feed-back amplifier of standard design,[6] which by mere external changes in connection will serve as integrator, differentiator and sign changer. Professor J. B. Russell of Columbia University first brought these techniques to the attention of the authors in the circuits employed in the Western Electric M-IX antiaircraft gun director.[7] As an amplifier so connected can perform the mathematical operations of arithmetic and calculus on the voltages applied to its input, it is hereafter termed an "operational amplifier." The operations can be performed to any desired degree of precision, providing power supplies of excellent regulation and circuit components of high precision are used. For most engineering computations, ordinary circuit components are adequate.

II. Operational Amplifiers

The term "operational amplifier" is a generic term applied to amplifiers whose gain functions are such as to enable them to perform certain useful operations such as summation, integration, differentiation, or a combination of such operations. In view of the fact that

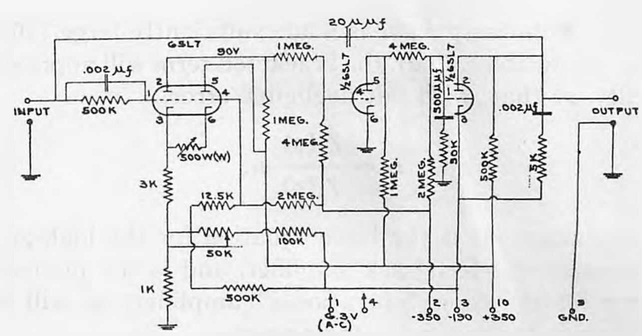


Fig. 1—Direct-current amplifier for use in electronic computers.

many operations involve steady or slowly changing inputs, the inherent frequency response of such amplifiers must extend down to zero frequency. The base unit in the operational amplifier is generally a direct-current amplifier having an odd number of stages. The unit shown in Fig. 1 was developed specifically for general laboratory use. However, any well-designed, stable, direct-current amplifier having an odd number of stages, or an equivalent phase shift, is adaptable to the uses which will be described.

* Decimal classification: 621.375.2. Original manuscript received by the Institute, April 30, 1946; revised manuscript received, September 25, 1946.
† Columbia University, New York, N. Y.
‡ City College of New York, N. Y.
§ Newark College of Engineering, Newark, N. J.

[1] M. F. Gardner and J. L. Barnes, "Transients in Linear System," John Wiley and Sons, New York, N. Y., 1942.

[2] H. W. Bode, "Network Analysis and Feedback Amplifier Design," D. Van Nostrand Co., Inc., New York, N. Y., 1945.

[3] V. Bush, "The differential analyser," *Jour. Frank. Inst.*, vol. 212, pp. 447–448; October, 1931.

[4] H. P. Kuehni and H. A. Peterson, "A new differential analyser," *Trans. A.I.E.E.*, (*Elec. Eng.*), vol. 63, pp. 221–228; May, 1944.

[5] T. M. Berry, "Polarized light servo-system," *Trans. A.I.E.E.* (*Elec. Eng.*), vol. 63, pp. 195–197; April, 1944.

[6] E. L. Ginzton, "DC amplifier techniques," *Electronics*, pp. 98–102; March, 1944.

[7] Instruction booklet prepared by the Bell Telephone Laboratories for the Western Electric M-IX antiaircraft gun director.

The basic equation of the operational amplifier may be derived by reference to Fig. 2. Here, if it is assumed that the box marked A is a direct-current amplifier and

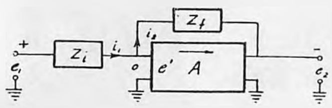


Fig. 2—Block diagram for basic feedback computation.

that the input at o is connected directly to the grid of the first tube, the currents at the junction o must add up to zero. Thus,

$$i_1 = i_2$$

or,

$$\frac{e_1 - e_1'}{Z_i(p)} = \frac{e_1' + e_2}{Z_f(p)} \tag{1}$$

where the voltages e_1, e_2, and e_1' are functions of time and $Z_i(p)$ and $Z_f(p)$ are used in the manner of the Heaviside impedance operators. It is noted that, if A is the voltage gain of the amplifier,

$$e_2 = Ae_1'. \tag{2}$$

Using this relation and simplifying (1),

$$e_2 = \frac{Z_f(p)}{Z_i(p)}\left[\frac{1}{1 + \frac{1}{A}\left(1 + \frac{Z_f(p)}{Z_i(p)}\right)}\right]e_1. \tag{3}$$

If the amplifier gain A is made sufficiently large (5000 is a practicable value), the bracketed term will approach unity, so that, with but negligible error,

$$e_2 = \frac{Z_f(p)}{Z_i(p)} e_1. \tag{4}$$

Equation (4) is the basic equation for the high-gain direct-current feed-back amplifier, and is the justification for the term "operational" amplifier, as will be shown in the succeeding paragraphs.

It will be noted in Fig. 2 that the output voltage e_2 is of opposite polarity from that of the input voltage e_1. Hence, if the impedances $Z_f(p)$ and $Z_i(p)$ are equal resistances, the amplifier will perform the simple operation of sign-changing. In fact, sign-changing will be included in all the operations which can be performed by the amplifier.

If the impedances of (4) are unequal resistances, a scale change will be accomplished, for

$$e_2 = \frac{R_f}{R_i} e_1. \tag{5}$$

It will be noted that the accuracy of the scale change depends only on the accuracy of the resistors, and does not depend on the amplifier components, so long as the amplifier gain remains large for all frequencies of interest.

Often it is desired to vary the scale of a given input, or to make the output adjustable. Three methods of accomplishing this operation are shown in Fig. 3. The circuits are self-explanatory and all of them may be used interchangeably. It is to be noted that circuits (*a*) and (*b*) fully realize the low output impedances which result in degenerative feed-back amplifiers. On the other hand, the output impedance of circuit (*c*) may reach a value as high as one quarter the resistance of the output potentiometer. This output impedance may be important in producing computation errors if the circuit following has a low impedance. In case (*a*) the gain varies linearly with the resistance of resistor R_f; in case (*c*) the output voltage will be linear with change in resistor R_c; but in case (*b*) the gain will vary as the reciprocal of the resistance R_i. These facts are of particular importance when the parameter is to be varied by a servomechanism whose output angle is most readily made linearly variable with the applied voltage.

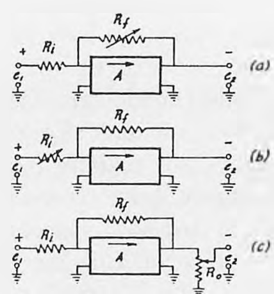


Fig. 3—Three methods for obtaining variable changes in scale or gain.

The more important operations which can be performed by virtue of (4) are those of the calculus, which make systems of these amplifiers capable of solving differential equations. Thus, if $Z_f(p)$ is a capacitor whose impedance expressed operationally is $1/Cp$, and $Z_i(p)$ is a resistance R, (4) becomes

$$e_2 = \left(\frac{1}{RC}\right)\frac{1}{p} e_1. \tag{6}$$

This amplifier is now an integrator which can accurately integrate input voltages with respect to time. It should be noted that the multiplying factor $1/RC$ may be made unity by choosing R as 1 megohm and C as 1 microfarad, and that the sign of the output voltage e_2 is negative as compared to the input voltage e_1. If the input voltage is constant, the output voltage will rise linearly with time up to the limit of the amplifier. If the input voltage is removed, the output voltage e_2 will remain constant at the integrated value. To remove this output voltage it is necessary to provide a switch which short-circuits the capacitor C, thus returning the output voltage to zero.

If $Z_f(p)$ is now made a resistance R, and $Z_i(p)$ is made a capacitor whose impedance is expressed operationally as $1/Cp$, (4) becomes

$$e_2 = (RC)pe_1. \quad (7)$$

The operational amplifier is now a differentiator whose output is the derivative of the input voltage. If a voltage of constantly increasing magnitude is applied to the input of this differentiator, the output will be a constant value. One of the disadvantages of the differentiator is its very good high-frequency response. For example, if the voltage fed to the differentiator is produced by a coarse potentiometer, the output voltage will contain high-magnitude pulses produced by each step of the contact arm.

Another basic operation which can be performed by the operational amplifier is the important one of summing the voltages obtained from any number of independent sources. A circuit to provide the summation of

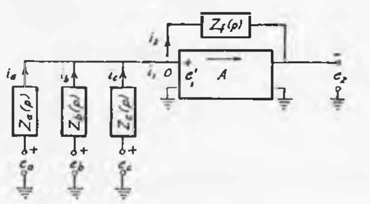


Fig. 4—Circuit for adding functions of three variables.

three input voltages is shown for illustration in Fig. 4. If it is assumed that the voltage e_1' is virtually zero, the current equation may be written in the same manner as was used to derive (4):

$$i_a + i_b + i_c = i_2. \quad (8)$$

So,

$$\frac{e_a}{Z_a(p)} + \frac{e_b}{Z_b(p)} + \frac{e_c}{Z_c(p)} = \frac{e_2}{Z_f(p)} \cdot \quad (9)$$

Rearranging,

$$e_2 = \frac{Z_f(p)}{Z_a(p)} e_a + \frac{Z_f(p)}{Z_b(p)} e_b + \frac{Z_f(p)}{Z_c(p)} e_c. \quad (10)$$

If all the impedances are equal resistances, (10) reduces to a summation of the three input voltages:

$$e_2 = e_a + e_b + e_c. \quad (11)$$

This summation may be accomplished for any number of input voltages.

It is evident that the summation may be made algebraic (that is, may include subtractions) if the input voltages are of proper sign, and they may be made so by the use of sign-changing amplifiers where required. It is easy to see that scale-changing may be made to accompany the summation if the resistances R_a, R_b, etc., are not selected to equal R_f, but ratios of R_f/R_a, R_f/R_b, etc., are chosen to give the desired individual multiplying factors. Furthermore, it is possible to use the general equation (10) and perform calculus operations on the input voltages at the same time they are summed and altered in magnitude. The only restriction is that the impedance $Z_f(p)$ is common to all the operations involved.

The operations described above may be summarized and the process involved becomes generalized and capable of further extension if (4) is written in the form

$$e_2(t) = \frac{F_1(p)}{F_2(p)} e_1(t) \quad (12)$$

or

$$F_1(p)e_1(t) = F_2(p)e_2(t). \quad (13)$$

In this form, the equation may be used for the setting up of complicated systems, examples of which will be described in the ensuing pages. The functions $F_1(p)$ and $F_2(p)$, as shown in Fig. 5, may be produced not only by

Fig. 5—Generalized circuit for operational computation.

passive circuits consisting of impedances, but also with additional operational amplifiers.

III. Synthesis of Computers for Linear First-Degree Equations

The techniques for setting up an electronic computer for an equation of a physical system can be divided into two classes, which will be referred to as the "in-line" and "current-junction" methods. One or a combination of these methods will apply to a particular problem. Usually there are many possible circuit arrangements, one of which will be more convenient to synthesize, more economical of equipment, or will permit easier adjustment of the desired parameters.

Before an equation is synthesized, it should be written in a form such that the independent variable is time, and the literal coefficients represent positive numerical values. A simple differential equation which fulfills the above conditions without rearrangement is that of the mechanical dynamics of a d'Arsonval meter or oscillograph element with constant rotational inertia J, viscous friction damping factor D, and stiffness factor K. The relationship between the angular displacement $\theta(t)$ and the applied torque $\tau(t)$ is given by the familiar second-order equation:[8]

$$J\frac{d^2\theta}{dt^2} + D\frac{d\theta}{dt} + K\theta = \tau(t) \quad (14)$$

or

$$(Jp^2 + Dp + K)\theta - \tau(t) = 0. \quad (15)$$

[8] R. E. Doherty and E. G. Keller, "Mathematics of Modern Engineering," vol. I, p. 17, John Wiley and Sons, New York, N. Y., 1936.

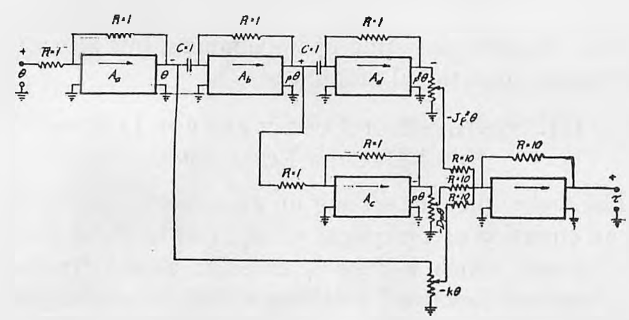
This equation, which can, of course, be solved by straightforward mathematical methods without recourse to a computer, will be used to illustrate the two techniques mentioned above.

(a) *In-Line Technique*

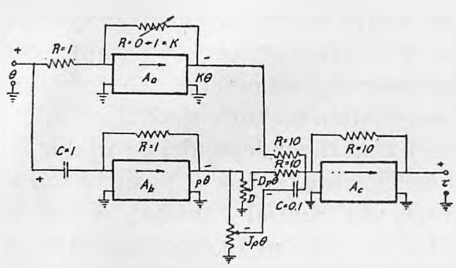
The in-line technique is adaptable to equations in which a number of functions of the independent variable are to be added to equal some function of the dependent variable. For example, if it is desired to examine the physical system of (15) in order to investigate the transient torque required to produce a given angular displacement for various values of inertia, damping, and stiffness, a computer would be synthesized to solve (15) for $\tau(t)$ in terms of $\theta(t)$.

Synthesis by means of the in-line technique consists simply of feeding the independent variable as an input voltage into a series of operational amplifiers, each of which computes one of the desired functions. These functions are then algebraically added in a final summation amplifier to give the desired output or dependent variable.

In the system of Fig. 6(a), an applied voltage proportional to the angular displacement from rest, $\theta(t)$, is fed into the left-hand terminals from a relatively low-impedance source. Amplifier A_a, having unit input and



(a)



(b)

Fig. 6—(a) Example illustrating "in-line" technique of synthesis. (b) Simplified computer for above example.

feed-back resistors (say, one megohm each) operates as an isolating unit and sign-changing amplifier, giving an output voltage proportional to $-\theta$. This voltage is differentiated and again changed in sign by amplifier A_f, having a unit input capacitor and a unit feed-back resistor (say, one microfarad and one megohm, respectively). Its output voltage is therefore proportional to $p\theta$. Amplifiers A_c and A_d receive $p\theta$ and produce voltages proportional to $-p\theta$ and $-p^2\theta$, respectively, the latter by means of another differentiation.

Assuming initially that the constant positive coefficients J, D, and K are all to be adjusted to values between zero and one, potentiometers of relatively low impedance (e.g., 50,000 ohms) can then be arranged as shown to deliver fractions of the voltages symbolized by $-p^2\theta$, $-p\theta$, and $-\theta$, respectively proportional to the coefficients J, D, and K. These voltages are then summed in amplifier A_e, using high-resistance input-circuit resistors (e.g., 10 megohms) so that the equivalent resistance from the potentiometer tap to ground will cause a negligible addition to the input resistance of amplifier A_e.

Suppose, however, that $1<J<10$. Then the feed-back resistance of amplifier A_d could be made equal to ten times the input resistance, so that the voltage output would be proportional to $-10p^2$.

Economy of amplifiers can be obtained with the circuit of Fig. 6(b). In this circuit the summing amplifier A_c also differentiates $-Jp\theta$, thus producing $Jp^2\theta$ without recourse to a separate differentiator and an additional sign-changing amplifier. In cases such as that described above, the potentiometer resistance to ground must be sufficiently low so that the real component of the capacitive input impedance will be negligible at the highest frequency for which accuracy of response is desired.

Many other circuit arrangements are possible, each having its advantages and disadvantages. It should be emphasized that all of these computer systems are unidirectional; a voltage proportional to $\tau(t)$ in the present example can not be applied to the terminals at the right to obtain the angular displacement θ as a voltage at the left-hand terminals. If it is desired to obtain $\theta(t)$ as a function of $\tau(t)$, the technique of synthesis which follows must be employed.

(b) *Current-Junction Technique*

The second method by means of which computer circuits may be synthesized, which is referred to as the "current-junction" technique, is perhaps the most basic method of attacking the problem. It was demonstrated in (13) that any amplifier, including the summing amplifier, solves the general equation of the form

$$F_1(p)x - F_2(p)y = 0 \tag{16}$$

where $F_1(p)$ is the operator for the independent variable x and $F_2(p)$ is the operator for the fed-back terms of the dependent variable y.

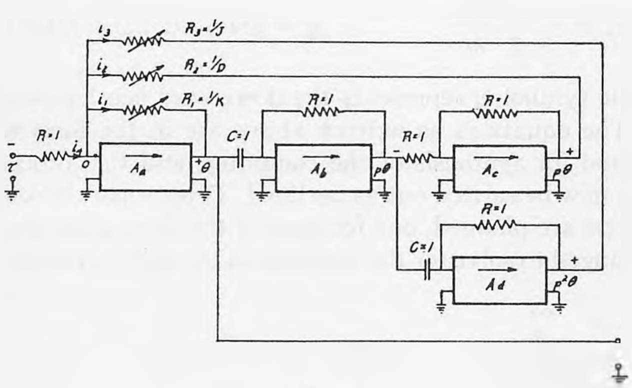
In the general case there may be several independent variables x, x', x'', etc., each with its own operator; and several different operators for the dependent variable y all of which are fed to the current junction o of Fig. 5. Thus the general equation would be of the form

$$F_1(p)x + F_1'(p)x' + F_1''(p)x'' + \cdots - F_2(p)y - F_2'(p)y - F_2''(p)y - \cdots = 0. \quad (17)$$

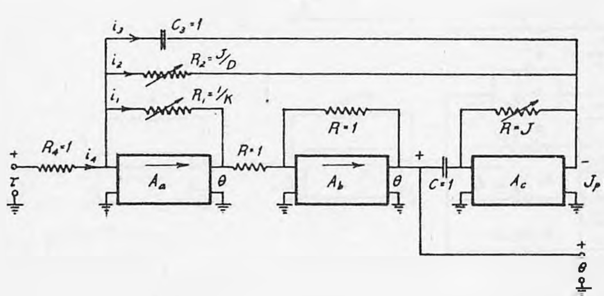
If the variables x, x', x'', $\cdots$, y are voltages such as $e_1(t)$ and $e_2(t)$, then the various expressions $F(p)$ are admittances, and the synthesizing process consists of summing algebraically the currents flowing into the junction o, of an amplifier such as that shown in Fig. 5, and setting the result equal to zero. The various currents are mutually independent since the grid input terminal o is virtually at zero potential (since e_1' is negligibly small) and hence effectively grounded.

The technique is illustrated in Fig. 7(a) for (15), which is now to be solved for $\theta(t)$ in terms of $\tau(t)$. Amplifier A_a is first assumed to produce an output voltage equal to $\theta(t)$. In order that $\theta(t)$ may exist at this point, it is necessary to feed four currents (one for each term of the equation) to its grid input terminal, junction o, thus solving the equation

$$-\tau(t) + K\theta + Dp\theta + Jp^2\theta = 0. \quad (18)$$



(a)



(b)

Fig. 7—(a) Example illustrating "current-junction" technique of synthesis.
(b) Simplified computer for above example.

The current i_4, representing the term of the independent variable $\tau(t)$, will exist if we let the input resistor be a unit value and apply a voltage proportional to $-\tau(t)$ at the left-hand terminals. The second term is $K\theta$, obtained by feeding back the voltage proportional to θ through a conductance proportional to K, or a resistance, $R_1 = 1/K$. Thus a current $i_1(t) = K\theta$ will also flow at junction o. Voltages proportional to $p\theta$ and $p^2\theta$, necessary to produce currents $i_2(t)$ and $i_3(t)$, are produced by the in-line technique described previously. When the currents i_1, i_2, i_3, i_4 are applied to junction o, the output of amplifier A_a must then be proportional to $\theta(t)$, as assumed at the start. The computer of the above example may be somewhat simplified, at the expense of having resistor R_2 dependent on two of the coefficients, J and D, as shown in Fig. 7(b).

The choice of the computer system is sometimes determined by the variations of the coefficients of the equations. If the coefficients such as J, D, and K are merely adjustable but remain constant as inputs are applied, practically any system is workable. However, occasionally a coefficient is not constant but a function of either time or one of the variables of the system. In that case a circuit must be chosen so that the value of the coefficient is proportional to the angle of rotation of a potentiometer shaft, so that a simple servo-driven unit may be used.

IV. Problems Involving Simultaneous Equations

Simultaneous integrodifferential equations arise when problems of coupled physical systems are encountered. Frequently it is inconvenient or impracticable to solve these equations mathematically for the desired unknown, and then set up a computer for the resulting equation. In addition, a solution may be desired for each of the unknown quantities. In such cases it is convenient to synthesize a computer for each of the equations of the set, and then to interconnect the circuits in a manner such that the equations are satisfied simultaneously.

The steps required for synthesis of a computer for simultaneous linear equations may be tabulated as follows:

(a) Each equation is written in a form such that one of the dependent variables, with its coefficients, stands alone on the right-hand side of the equation, this dependent variable being different for each equation of the set.

(b) By one of the methods previously described a separate computer is set up for each equation, assuming that all the variables save the one on the right are independent variables, so that the computer will produce an output voltage proportional to the dependent variable on the right-hand side of the equation.

(c) Functions of the output of each computer required to provide the proper input functions for the other computers are noted. Circuits are added to the output of each to provide these additional output functions.

(d) The heretofore-separate systems are cross-connected so that each input terminal receives its proper voltage from one of the outputs of the system. Only the input terminals for the true independent variable or variables will then be left free, and to these external driving functions will be applied, as in the cases discussed previously.

(e) Scales of computation are adjusted so that each amplifier will have a voltage output of reasonable magnitude.

As an example consider the airplane shown in Fig. 8, for which it is desired to make a study of the pitching characteristics when the elevators are manipulated. A pair of axes fixed in the airplane will be used as references, with the customary positive directions as shown.

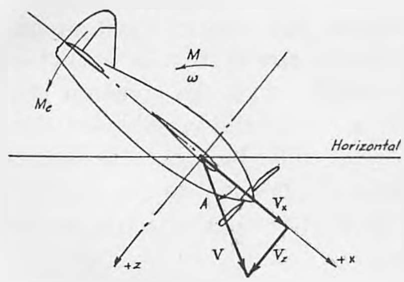


Fig. 8—Diagram illustrating notation for equations (19), (20), (21), and (22).

At the instant illustrated the airplane is pointing at a negative angle θ from the horizontal, and is moving with a resultant linear velocity V in a direction at an attack angle A from the longitudinal axis. The velocity V is considered to have two rectilinear components, V_x and V_z.

It is desired to synthesize a computer which will solve for the disturbances or changes in angular velocity ω and the disturbances or changes in the linear velocities v_x and v_z caused by small moments M_e applied to the airplane by means of the elevators. The airplane is assumed to have a mass m and moment of inertia J about the y axis, perpendicular to the plane of the paper.

Let it be assumed that initially (at $t=0$) the airplane is flying with velocities and angular positions indicated by upper-case symbols with the subscript o. Then equations for the changes or disturbances in linear and angular velocities (indicated by lower-case symbols) can be written for the symmetrical motions of the airplane. Three equations of motion are required, which are:[9]

(1) The sum of the x-directed forces equals zero; $\sum f_x = 0$.

(2) The sum of the z-directed forces equals zero; $\sum f_z = 0$.

(3) The sum of the angular moments about the center of gravity and the y axis equals zero; $\sum M = 0$.

With certain approximations,[9] these three equations can be written in the following form:

$$(p + k_{xx})v_x + V_o(\sin A_o)\omega + g(\cos\theta_o)\frac{\omega}{p} = k_{xz}v_z \qquad (19)$$

$$-(p + k_{zz})v_z + V_o(\cos A_o)\omega - k_{z\omega}\omega + g(-\sin\theta_o)\frac{\omega}{p} = k_{zx}v_x \qquad (20)$$

$$-(k_{M\dot z}p + k_{Mz})v_z + k_{Mx}v_x + \frac{M_e}{J} = (p + k_{M\omega})\omega \qquad (21)$$

where V_o, A_o and θ_o are respectively the initial resultant velocity, attack angle, and angle from the horizontal (chosen negative); v_z, v_x, and ω are the increments in downward, forward, and angular velocities, respectively, for which solutions are desired; and the coefficients (all arranged so as to be numerically positive) are:

$$\begin{aligned}
k_{xx} &= -\frac{1}{m}\frac{\partial f_x}{\partial v_x} & k_{zz} &= -\frac{1}{m}\frac{\partial f_z}{\partial v_z}\\
k_{xz} &= \frac{1}{m}\frac{\partial f_x}{\partial v_z} & k_{zx} &= -\frac{1}{m}\frac{\partial f_z}{\partial v_x}\\
k_{z\omega} &= -\frac{1}{m}\frac{\partial f_z}{\partial \omega} & k_{M\dot z} &= -\frac{1}{J}\frac{\partial M}{\partial \dot v_z}\\
k_{Mz} &= -\frac{1}{J}\frac{\partial M}{\partial v_z} & k_{Mx} &= \frac{1}{J}\frac{\partial M}{\partial v_x}\\
k_{M\omega} &= -\frac{1}{J}\frac{\partial M}{\partial \omega} & g &= \text{gravitational constant.}
\end{aligned} \qquad (22)$$

(The symbol $\dot v_z$ represents the downward acceleration.)

The equations as written above are in the form required for synthesis of the computer, and this process can now be carried out as outlined. Three separate computers are planned, one for each of the three equations, arranged to solve for the variable on the right-hand side.

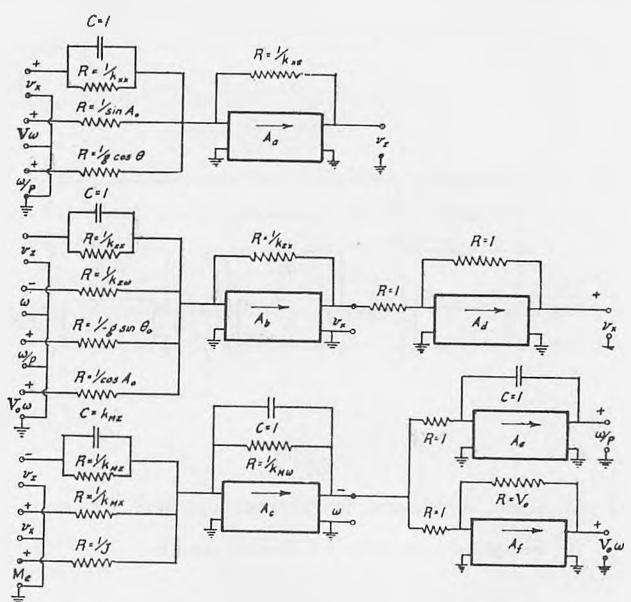


Fig. 9—Electronic computer for symmetric motions of airplane.

In Fig. 9, the output of amplifier A_a is chosen to be $-y_z$; the feed-back network is chosen so as to multiply by the coefficient k_{zx}. Similarly, amplifiers A_b and A_c produce $-v_x$ and $-\omega$, respectively; the feed-back networks provide for multiplication by k_{xz} and $(p + k_{M\omega})$, respectively. The required inputs are then assumed to be available, and input impedances are selected so that these inputs will be multiplied by the proper coefficients.

[9] W. F. Durand, "Aeronautic Theory," vol. 5, division N, by B. Melvill Jones, chap. 5, J. Springer, Berlin, 1934–1936.

It will be noted that, with the chosen feed-back impedances, it is possible to use capacitors to differentiate but not to integrate input terms. For this reason, it is necessary to assume as inputs the integrated quantities, such as ω/p, where such are required. It is also convenient to assume that the multiplication by V_o is to be taken care of elsewhere, so that $V_o\omega$ is available for an input. It is now evident that the inputs required include, in addition to the three outputs $-v_x$, $-v_z$, and $-\omega$, the quantities v_x, ω/p, and $V_o\omega$. Amplifiers A_d, A_e, and A_f are added to the circuits in order to produce these terms.

The next step consists of interconnecting the three computers so that correspondingly labeled input and output terminals are connected; these wires are not shown on the diagram. When these connections are complete, the only input terminal left free is that of the independent variable M_e, the applied elevator moment. This moment may be assumed proportional to rudder angle, and may therefore be an input obtained from a potentiometer supplied with a direct voltage and attached to the elevator control operated by the "pilot" of the simulated airplane, or by an automatic pilot which it is desired to study.

There are, of course, further practical considerations, which have not been mentioned. Stabilization of each loop of the computer circuit must be accomplished, a problem to be discussed later. Scales must be adjusted so that the output voltages of the v_z and ω computers will be of the same order of magnitude as that of the v_x computer.

V. Equations with Nonconstant Coefficients and of Higher Degree

In the analysis of some systems, the equations may be of higher degree and, in addition, some of the coefficients may be functions either of time or of some of the variables. In view of the fact that the quantities involved are usually voltages, simple servomultiplier and divider circuits will be described.

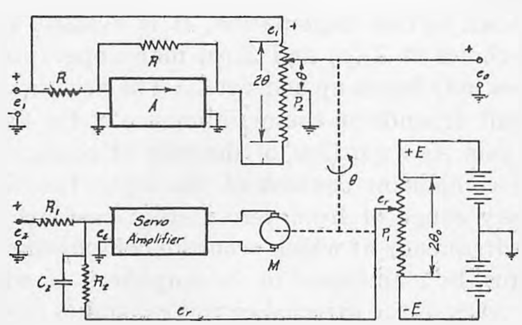


Fig. 10—Servomultiplier circuit.

As shown in Fig. 10, the servomechanism, being a degenerative device, causes the error voltage e_ϵ to be brought to zero. The angle θ at which the servomechanism just closes its error is given by the relation

$$\frac{e_r}{R_2} = \frac{e_2}{R_1} \tag{23}$$

and

$$e_r = \left(\frac{\theta}{2\theta_o}\right)2E = \left(\frac{\theta}{\theta_o}\right)E \tag{24}$$

where θ is the shaft rotation and $2\theta_o$ is the total angle subtended by the follow-up potentiometer P_1. Solving for the shaft rotation,

$$\theta = \left(\frac{R_2}{R_1}\right)\left(\frac{\theta_o}{E}\right)e_2. \tag{25}$$

To produce multiplication with the second voltage e_1, a second potentiometer P_2, is supplied on one side with the voltage e_1 directly, and on the other side with the voltage e_1 through a sign changer. The shaft of this potentiometer is connected to that of the servomechanism, so that its output is

$$e_o = \left(\frac{e_1}{\theta_o}\right)\theta, \tag{26}$$

substituting the value of θ from (25),

$$e_o = \left(\frac{R_2}{R_1E}\right)e_1e_2. \tag{27}$$

By proper choice of R_2, R_1, and E, a scale factor for the multiplication may be produced to specifications. The capacitor C_2 is merely for stabilization of the servosystem and does not enter in the computation, except during transient conditions.

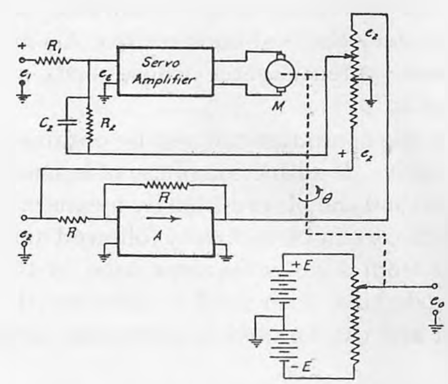


Fig. 11—Servodivider circuit.

In a similar manner, it is possible to divide one voltage by another using a servodriven potentiometer. A schematic circuit of this system is shown in Fig. 11. The servosystem closes the error voltage e_ϵ to zero, so that

$$\frac{e_1}{R_1} = \frac{2\theta e_2}{2\theta_o R_2}. \tag{28}$$

Solving for the shaft rotation θ,

$$\theta = \left(\frac{R_2\theta_o}{R_1}\right)\frac{e_1}{e_2}. \tag{29}$$

The output voltage e_o is obtained from a potentiometer on the same shaft supplied with $\pm E$ volts to ground. Thus,

$$e_o = \left(\frac{2E}{2\theta_o}\right)\theta = \left(\frac{ER_2}{R_1}\right)\frac{e_1}{e_2}. \qquad (30)$$

It is observed that the output voltage e_o is the quotient of e_1 and e_2 multiplied by a scale factor. Thus it is possible to multiply or divide a term in any equation by another term with any suitable scale constant by use of these small servodriven potentiometers.

Frequently it is desired to insert a function of a given variable in the computer system. For instance, if a study of aerodynamic systems is made in which a coefficient is a function of wind velocity, it is possible to adapt one of the servodriven systems to perform this operation. One technique is to construct a servomechanism similar to the multiplier of Fig. 10, and to drive a cam through an angle θ proportional to an independent variable e_2. The desired function may be cut on a cam which drives another linear potentiometer whose output voltage is the function of the independent variable. The same result may be obtained without a cam by winding a potentiometer whose voltage output is the desired function of shaft angle.

For instance, referring to the electronic computer for the symmetric motions of an airplane in Fig. 9, the quantity V_o is the initial airspeed which is constant. The term $V_o\omega$ produced by amplifier A_f is used in the computer for inputs which, for more accurate computations, should use the magnitude of the instantaneous velocity V, the vector value of which is given by the equation

$$\overline{V} = \overline{V}_o + \bar{v} \qquad (31)$$

where $\bar{v}$ is the change in velocity from the initial value $\bar{v}_o$. To make this correction in the computer, it would be necessary to vary the feed-back resistor R_{19} by means of a servodriven potentiometer whose shaft rotation is proportional to $|V|$.

Inputs to the computer can also be obtained by manual curve tracing. If a function of the independent variable is plotted and the plot rotated by means of the servomotor, the curve can be manually followed and the voltage output from a potentiometer used in the system. Such methods have been used in differential analyzers in the past and can be used in electronic computers as well.

Practical physical problems offer numerous examples of system variables which have fixed upper limits. For example, an angle specifying the position of a boat rudder may vary between zero and a fixed maximum. Similar limits apply to airplane controls. A mechanical system involving spring forces will often involve maximum spring extensions.

The purely mathematical solution of the dynamics of such systems where such limiting action occurs is often difficult. The electronic representation of such limits is comparatively simple and can be introduced into the system wherever the limits occur. Fig. 12 shows one such circuit. Whenever the input voltage e_1 exceeds the bias voltage E, one or the other diode will conduct, depending on the polarity of e_1. When this occurs the output voltage will be limited closely equal to the bias voltage, provided the resistance R is high compared to the diode resistance. A limiter of this type may be inserted in the computers previously described wherever needed.

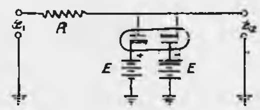


Fig. 12—Circuit for introducing nonlinearity in form of limits.

VI. Operating Considerations

The frequency response of any operational amplifier may be obtained by replacing the operator p by the quantity $j\omega$. Thus the frequency response of an integrator is given by

$$\frac{E_2}{E_1} = \left(\frac{1}{RC}\right)\frac{1}{j\omega}. \qquad (32)$$

This expression indicates a gain characteristic which drops 6 decibels per octave. On the other hand, the frequency response of a differentiator is given by

$$\frac{E_2}{E_1} = (RC)j\omega, \qquad (33)$$

which indicates a gain characteristic that rises 6 decibels per octave.

These frequency characteristics may be used as a convenient test of amplifier accuracy in producing a given operation. For instance, if the gain curve of a differentiator indicates a linearly rising gain of 6 decibels per octave up to a given frequency, but beyond this frequency shows considerable deviation from linearity, it may be concluded that as a differentiator it is correct up to this particular frequency. If voltages containing frequency components of higher value are applied at the input, error in differentiation will result. Extending the frequency response of the base amplifier will correct the situation.

Without further explanation, it is evident that by proper choice of $Z_f(p)$ and $Z_i(p)$ many operational expressions may be set up accurately. The accuracy which will result depends on the maintenance of the high values of gain A, regardless of the rate of change or frequency component content of the input functions. A frequency range of from zero (direct current) to the highest frequency at which accurate computation is desired must be maintained in the amplifiers. In addition, the accuracy of the external circuit constants used must be assured.

For the impedance functions $Z_f(p)$ and $Z_i(p)$, capacitors and resistors are generally used. It is evident that, while theoretically usable, inductances are avoided, since at the relatively low frequencies involved it is difficult to obtain as pure or accurate an impedance function as can be obtained with a capacitor of low loss and low retentivity.

Whever the computer involves the use of capacitors other than small ones inserted to achieve stability, considerable difficulty may be found due to retentivity of charge. It has been recommended by the Bell Telephone Laboratories that capacitors with polystyrene dielectric be used, as they are relatively free of this effect.

A perfect representation of the equation for a stable mechanical system should, of course, be stable electrically. Unavoidable progressive phase shifts, both within individual amplifier units and appearing accumulatively when several units are connected in a loop, may lead to oscillation at some upper frequency. A careful redesign of the complete system will often minimize the number of separate amplifiers used in any single loop so that oscillation does not occur. The nature of the system itself which is being represented will often automatically stabilize the electronic equivalent, as, for instance, when a mechanical system having large mass will act as a low-pass filter, eliminating from the feed-back loop those higher frequencies for which phase shift may occur.

The individual amplifiers may exhibit instability when the over-all gain of the amplifier is too high. This condition may be overcome by inserting a small capacitor across the feed-back impedance of differentiators and scale changers, or by inserting a resistor in series with the capacitor of an integrator, thereby limiting the frequency response. The higher harmonics in the system response are, of course, attenuated as a result. In some cases it may be necessary to choose a new time scale[10] in which these attenuated high frequencies are well above the upper frequencies of response of the dynamical system being simulated.

While the amplifier units described earlier are reasonably drift free, residual drifts may often be annoying, particularly where small input voltages and long time intervals are involved. These effects can be readily minimized by changing the voltage scale and the time scale so that the useful voltages are large compared to the drift voltages and so that the experimental "run" is completed before errors have had time to accumulate. Drift and stability for these systems are functions of the direct-current power supply, which should be extremely well regulated and of low dynamic output impedance.

A typical laboratory electronic-computer arrangement is shown in Fig. 13.

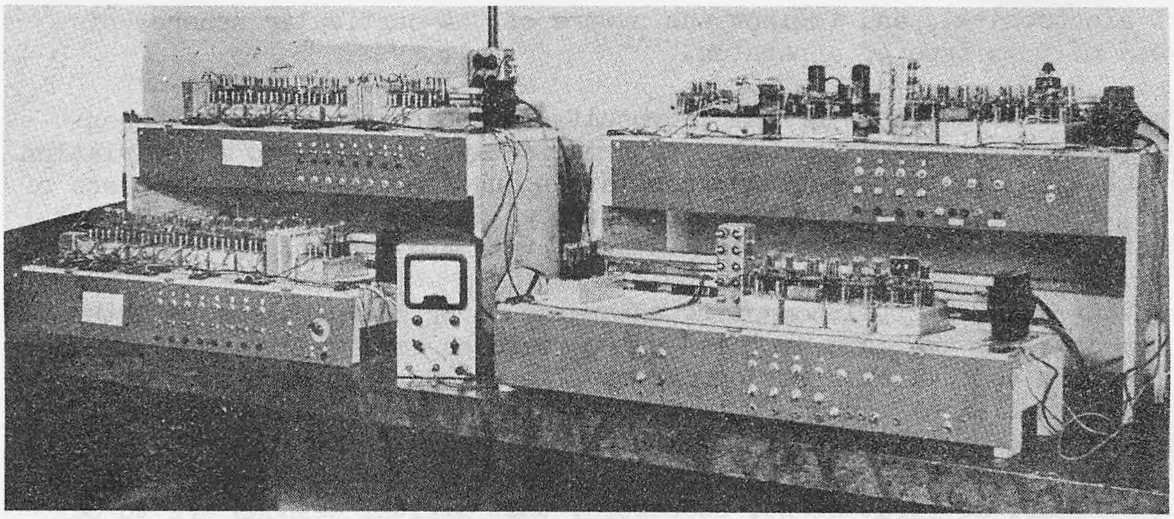


Fig. 13—Typical laboratory electronic-computer arrangement. One set of amplifiers in the lower bay on the left is the computer described in Fig. 9 for aircraft symmetrical motions.

VII. Acknowledgment

The authors wish to express their appreciation to Section 7.2 of the National Defense Research Council, its chief, S. H. Caldwell, and the technical aides of that Section, G. A. Philbrick, J. B. Russell, and H. C. Wolfe, for their assistance, co-operation, and guidance in carrying out this work at Columbia University under contract number OEMsr-237. In addition, the authors wish to acknowledge the contributions of L. Julie and M. Hestenes to some phases of this development.

[10] J. L. Gardner and M. F. Barnes, "Transients in Linear System," vol. 1, p. 226; John Wiley and Sons, New York, N. Y., 1942.

The following passages represent significant excerpts from the pioneering article by D. J. MYNALL entitled "Electrical Analog Computing" appearing in Electronic Engineering (Br.), June - September, 1947.

"In a study of the means for carrying out mathematical computations automatically, it is essential to be clear as to the distinction between the analogue and the digital principles.

"This article sets out to describe characteristic methods employed in electrical analogue computing, to indicate the direction in which development is likely to tend (in an attempt to realise the potentialities offered by the full exploitation of electronics), and to show the nature of the particular spheres of usefulness of this type of apparatus.

"Employment of the analogue principle results in apparatus in which chosen physical quantities vary in a manner mathematically analogous to the variation of the numbers in the problem under consideration. In any particular piece of apparatus, there exists, of course, an accuracy limit beyond which it is not possible to set up the input quantities or to read the result.

"On the other hand, digital computing apparatus works out problems by methods which are basically the same as those which would be employed if an attempt were made to perform the calculations on paper. The elements in such apparatus have purely positional significance, after the fashion of beads on a counting-frame, and there is thus no fundamental limit to the accuracy obtainable.

"It would appear, at first sight, therefore, that digital methods are to be preferred. However, in general, the analogue principle gives apparatus which is smaller and lighter than the equivalent digital apparatus, and it thus has fields of application in which it is particularly advantageous. The two principles are complementary rather than competitive."

. . . .

"The mathematical processes dealt with in this article are the basic operations such as addition, multiplication, differentiation, integration, and a few others which do not come directly under these heads.

"The examples which are given later have been chosen for the generality of their application and for their usefulness in demonstrating principles.

"The art is extensive, so that, of necessity, some interesting examples of technique have been omitted or given but brief mention. It is hoped that the references quoted will in some measure compensate for this."

. . . .

"ALGEBRAIC ADDITION

"To move from the general to the particular, the operation of algebraic addition may be considered first, since it is obviously a prime requirement, and is one which linear electrical circuits are well adapted to deal with, by a variety of methods."

. . . .

"Another important method of adding voltages may be called "star mixing". It is particularly well suited to electronic systems, and is able to deal with a wide frequency range, from D.C. upwards.

"Suppose, for example, that it is desired to sum the voltages from N sources by this method. It is sufficient to join N equal admittances in a star-point and to connect each of the free ends of the star to one of the sources to obtain a voltage at the star-point which is $\frac{1}{N}$ th of the desired sum.

"The Star admittances are generally (and for D.C., necessarily) conductances.

"Precisely the same remarks with regard to alternating voltages apply as before, and general types of linear functions may be set up by making the star admittances unequal.

"Loading from the star-point to "earth" alters the detail design but not the principle, and may readily be taken into account.

"Fig. 4 is an illustration of the most general form of star mixing when $N = 3$. It is useful to work in terms of admittance rather than impedance, as the expressions for the weighting coefficients in the output function assume a simpler mathematical form, which makes it clear that in order to adjust one of the coefficients, without affecting the others, an equal and opposite variation should be imposed on the associated star admittance and the loading admittance.

"Apart from their use in more elaborate systems, these principles suffice in themselves to form the basis for simultaneous linear equation solvers. This subject may be followed up for example in Reference 4, which deals with a commercial application of the star mixing method and the general theory of the classical iterative method of using such apparatus.

"So far, in the consideration of addition, electronics has not appeared specifically, though it will be understood that circuits such as ...[Fig.] 4 may well employ electronic excitation and indication.

"In particular, an electronic amplifier is well adapted to restore the loss of level which is sustained in every star mixing process.

"THE THERMIONIC VALVE AS A COMPUTING CIRCUIT ELEMENT

"Before going further, it will be as well to consider some general properties of a vital element in electronic computing systems — the thermionic valve.

"The thermionic valve has the advantage that it can operate at speeds far in excess of any mechanical device. It also offers the features of readily providing power gain from point to point of the circuit and of acting as a more-or-less unidirectional buffer, thus simplifying interaction problems when compounding units to form a complete system.

"But for these advantages the valve would not find a place in precision systems, because it also offers the disadvantage that one cannot rely upon its characteristics remaining unchanged during warming-up, under changes of ambient temperature, or with normal ageing. Nor is there likely to be close similarity of characteristics between valves of one and the same type. This means that the direct use of the valve characteristic is ruled out.

"Fortunately, various artifices are available to make the overall performance of valve circuits depend almost entirely on circuit elements which are more uniform and consistent in performance. The principle involved, which is known as "negative feedback", was first explicitly investigated and used on an extensive scale in connexion with communication networks,[56] and the technique was thus ready for application to computing circuits. This principle is so important that a simple but characteristic application will be described before proceeding with the description of units for performing other mathematical operations.

"NEGATIVE FEEDBACK

"In Fig. 5, the block represents a high-gain, direct-coupled, voltage amplifier. The numerical value of the gain, A, is an "instantaneous" value which is not necessarily known with any exactitude: it will vary due to factors which have already been enumerated and will also be a function of the input voltage, owing to non-linearity of the valve characteristics. For simplicity, it is assumed that the input resistance of the amplifier is infinite, the output resistance zero, and that zero output voltage corresponds to zero input voltage. Departure from these conditions can readily be taken into account, but would merely obscure the main point. A resistor, R, connected between the input and output terminals, provides the feedback link.

"With these assumptions, it is easy to show that the input resistance and the transresistance of the whole circuit have the values given on the diagram. These are instantaneous values, which will vary as A varies. The transresistance has the dimensions of resistance since it is defined as to the ratio of the output voltage to the input current.

"The important point to note is that the transresistance approaches a limit --R, as the value of A is raised, and that this limit is independent of A. Thus, provided A be sufficiently great, large variations in A result in negligible variations in the transresistance. In other words, this property of the circuit has been made to depend on the characteristics of a resistor rather than on the characteristics of the valve or valves, which are merely the means for obtaining a high value of A.

"Furthermore, the input resistance, though directly subject to the variations of A, approaches zero as the value of A is raised, and can be made extremely small.

"It is a short step to the amplifier of Fig. 6, which has been formed by the addition of a resistor, r, on the input side. The input resistance and voltage gain are as shown on the diagram, and both approach independence of A as A is raised in value.

"An amplifier of this type is commonly used for addition by star mixing, the resistor r being replaced by the whole star of mixing resistors, as shown in Fig. 7. The advantage over the arrangement of Fig. 4 is that adjustment of the star admittance associated with one of the input variables affects only the associated constant in the output function (no compensating adjustments elsewhere being called for) and the output level is independently adjustable by variation of the beedback admittance. Where the change of sign is significant it can be corrected by a unity-gain amplifier of the same general type.

"It may be noted in passing, that when direct voltage is being handled by such a system the stability of the zero is not ensured by the means described. One way of overcoming this difficulty is to convert the input to A.C., by high-speed mechanical interruption or some form of modulation process, reverting to D.C. at the output terminal by means of synchronous demodulation at high level. This technique is an art in itself, too extensive to be dealt with in detail in this article. A useful explanation of the purpose of chopping the input quantity is contained in Reference 7. Several useful embodiments of the principle have not yet appeared in the literature of the subject.

"Another method is to employ an auxiliary circuit which examines the zero setting and corrects it when necessary.[8] This avoids the need for chopping the input to the main amplifier, where it may impose an artificial upper limit on speed of response.

"Feedback and input elements are not necessarily resistive, as will be seen in examples of electronic differentiators and integrators which are dealt with later."

. . . .

"ULTRA HIGH-SPEED OPERATION

"In any automatic computing system the speed of operation is ultimately limited by the acceleration which can be imposed on the active elements. In this respect electronics is unique. Nothing is known to Science which can be more readily accelerated than the electron.

"In electro-mechanical systems the overall speed of operation is ultimately limited by the inertia of the mechanical elements. Even with the lightest of servo-mechanisms, rates of variation of the data cannot be dealt with at levels even distantly approaching the speed at which electronic devices, such as the thermionic valve, the cathode-ray tube and the photo-electric cell, can operate.

"The introduction of electricity into computing systems offers two main advantages. The first of these is realised in electro-mechanical systems, and is the feature of flexibility of design and layout. The second of the advantages is the possibility of operating the whole system at "electronic" speeds."

. . . .

"The possibility of dealing with fast-changing data, outside the speed capabilities of existing systems, is an obvious feature of ultra-high-speed technique. A less obvious, but nevertheless valuable, advantage could result from its application to devices such as algebraic equation solvers and differential analysers, where the substantially instantaneous presentation of the result of changing any of the conditions in the problem would greatly facilitate searching for desired types of solutions.

"The means which have already been described for dealing with addition do not involve mechanical elements, so that it is unnecessary to consider this process anew.

"Since the principles on which electronic time differentiators and integrators may be constructed are well established. these are described first."

. . . .

"SPECIAL FEATURE OF ELECTRONIC APPROACH

"Since the whole system is electronic, it is easy to arrange that the solution for any particular set-up is traced out in a time of the order of a few milliseconds. It is practicable, for example, to repeat the whole process (including the time for restoring the circuit to the "initial" state) at 50 cycles per second.

"This rate of repetition is above optical flicker frequency and an apparently steady trace is seen on the screen of the cathode-ray tube.

"A feature of considerable practical importance arising from this method of working is that the effect of altering any of the boundary conditions or constants in the equation, or of varying the form of $f(t)$, can be followed instantly and continuously. Families of solutions can be run through in the course of a few minutes."

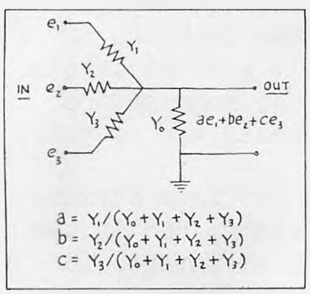


FIG. 4, STAR MIXING, ANOTHER METHOD OF FORMING A LINEAR FUNCTION OF THE INPUT VARIABLES.

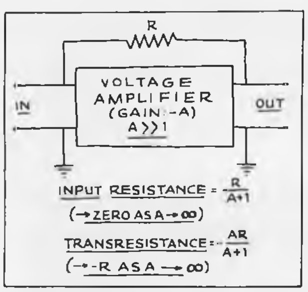


FIG. 5, THE APPLICATION OF NEGATIVE FEED- TO A VALVE AMPLIFIER, SHOWING EFFECT OF R ON THE INPUT RESISTANCE AND TRANSRESISTANCE.

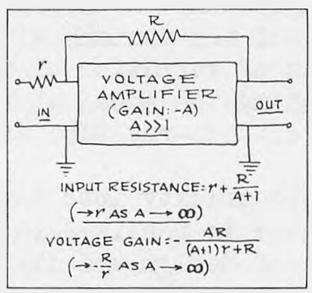


FIG. 6, ADDITION OF INPUT RESISTANCE r TO FIG. 5 TO FORM FEEDBACK AMPLIFIER.

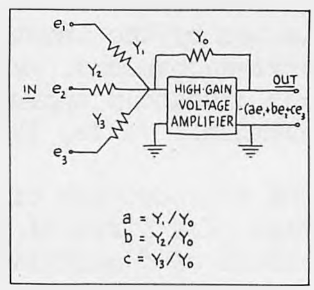


FIG. 7, FEEDBACK CIRCUIT APPLIED TO STAR MIXING.

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We offer in the several paragraphs below some direct quotations from the clairvoyant article by another justly famous pioneer, ALAN B. MACNEE, appearing in the IRE Proceedings of November, 1949, under the title "An Electronic Differential Analyzer."

"Summary—An electronic differential analyzer is described, capable of solving ordinary differential equations of orders through the sixth, both linear and nonlinear, and with coefficients that are either constant or variable. This analyzer has a high speed of operation and is extremely flexible with regard to equation parameters and initial conditions. The flexibility permits rapid investigation of wide ranges of equation solutions with regard to periodicity, instability, and discontinuities. It renders practicable the solution of end-point boundary-value problems; that is, problems in which the final rather than initial values are specified.

"Two new computing elements, an electronic function generator and an electronic multiplier, are employed in this differential analyzer. A diode clamping circuit permits the use of ac coupled amplifiers.

"A number of representative differential equations of the linear and nonlinear types have been solved. Comparison of observed and calculated solutions reveals an accuracy of from 1 to 5 per cent, while the precision (or repeatability) of the solutions ranges from 0.002 to 0.1 per cent. An analysis of the errors introduced into the differential- equation solutions by the frequency limitations of the computing elements, such as the integrators and adders, has been made, and the results of this analysis have been verified experimentally."

" 1. INTRODUCTION

"The present trend in mathematical calculators is toward large machines capable of solving more and more complicated problems with the greatest accuracies. These large machines are extremely expensive to build and operate; the use of the MIT Differential Analyzer for an eight-hour day, for example, costs about $400. This high cost, together with the limited number availabke, has prevented many investigators from realizing the advantages of these machines.

"Early in the fall of 1945 it was felt that there was considerable need for a differential analyzer of somewhat different characteristics from any then in existence or under development. There appeared to be the need for a machine having the following characteristics: (a) moderate accuracy, of perhaps 1 to 10 per cent; (b) much lower cost than existing differential analyzers; (c) the ability to handle every type of ordinary differential equation; (d) high speed of operation; (e) above all, extreme flexibility, in order to permit the rapid investigation of wide ranges of equation parameters and initial conditions.

"A differential analyzer of this type bears the same relation to the larger differential analyzers which a slide rule bears to a desk calculating machine. Its uses are numerous: (a) It can be used, as a slide rule is used, to give rapid solutions of moderate accuracy to the differential equations encountered by the engineer, physicist, and mathematician. In this role it is valuable both in solving higher-order ordinary differential equations with constant coefficients, which are very tedious to handle analytically, and also nonlinear equations and equations with variable coefficients which often can not be solved at all analytically. (b) Such a differential analyzer can also be used as an adjunct to one of the larger differential analyzers. It is used then to carry out the time-consuming exploratory solutions necessary to determine the ranges of equation parameters and initial conditions of interest. This preliminary work can be done at a great saving in time and money, and then, if warrented, the larger and more accurate machine can be used to obtain the final desired solution. (c) Such a differential analyzer, by virtue of its moderate cost and great flexibility, is very useful as a teaching tool in the fields of mathematics, engineering, and physics.

"In the author's opinion, the objectives of flexibility, moderate cost, and high-speed operation are most easily achieved with an all-electronic machine.

"It is important to realize that the main advantage of the high speed of this analyzer is that it permits rapid exploration of a wide range of solutions. In a fundamental way this is perhaps the most significant advantage of the electronic differential analyzer over mechanical differential analyzers.

"There are many important problems involving the solution of a differential equation in which the crux of the matter is to find the initial conditions for which the solutions (a) are stable, or periodic, or continuous, or (b) satisfy certain specified final conditions. Such problems frequently require exploration of 1,000 different solutions (e.g., 10 values for each of three parameters) and might well represent a prohibitive investment of time on a slow differential analyzer which requires several minutes for each solution; on the present electronic analyzer such an exploration can be carried out in an hour or two. An example of this sort (the solution of the equation $d^3y/dt^3 - y(t-t_0)/4 = 0$ over the range $0 \le t \le t_0$ for specified final conditions) is given in Section V-F.

"For problems of this type, the usefulness of the electronic analyzer actually surpasses that of the more elaborate and accurate, but slower, mechanical differential analyzers."

. . . .

[1] V. Bush and S. H. Caldwell, "A new type of differential analyzer," Jour. Frank. Inst., vol. 240, pp. 255-325; October, 1945.

[2] F. Rockett, "Selective sequence digital computer for science," Electronics, vol. 21, p. 138; April, 1948.

[3] A. W. Burks, "Electronic computing circuits of the ENIAC," Proc. I.R.E., vol. 35, pp. 756-767; August, 1947.

[4] This need was first pointed out to the author by Henry Wallman, professor in the Department of Mathematics, Massachusetts Institute of Technology, who suggested such a machine. The author wishes to express to Prof. Wallman his deepest appreciation for his guidance of this work.

HISTORY AND NATURE OF ANALOG COMPUTORS

The following articles by Jurgen Roedel discuss in far greater detail than any of the foregoing material the history and nature of electronic analog computors.

In particular, the second article terminates with a table of transfer impedances of useful computing networks which should be particularly helpful to the student and practitioner.

AN INTRODUCTION TO ANALOG COMPUTORS

ABSTRACT

As an introduction, to the newcomer, into the field of computing devices the history and classification of these devices is duscussed with particular emphasis on analog computors. A discussion of the precision of computing devices follows to acquaint the reader with this important problem. The paper is concluded with a consideration of basic elements in the operational amplifier type of analog computor, and is illustrated with photographs of typical installations of this type.

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INTRODUCTION

Although this paper contributes nothing new to the art of computing devices, the need for a very general introduction to this subject is well recognized. It is for the benefit of the neophyte that this paper will discuss the history and classification of computing devices. To introduce more detailed concepts of analog computation, a discussion will follow of the basic elements of the operational amplifier analog computor. It is with this equipment that the author is most familiar.

HISTORY

The history of computing devices may well extend to the very beginning of civilization. The early forms of digital computation probably manifested themselves when early man first started to keep tab on numbers by counting on his fingers or with pebbles. The earliest known record of analog computation is the surveying and map making in Babylonia in 3800 B. C. for the purpose of taxation.[1] By 1300 B. C. surveying and map making were common in Egypt.[2]

The earliest digital machine is probably the abacus which evolved from the counting with pebbles. In its early form, which was used in the Tigress Euphrates Valley as early as 5000 years ago[3] and in Egypt as early as 460 B. C., it consisted of a clay board with groves in which pebbles were placed. It later appeared in Rome, China and Japan in the form of a wire frame with beads. A picture of the Japanese abacus (Soroban), which is still used by Japanese and Chinese tradesmen, is shown in Figure 1.

In 960 A. D., Gerbert brought back from the Universities of the Moors the concept of Arabic numerals, and tried for years to make practical a calculating machine, which the Moors had conceived. John Napier described his invention of logarithms in 1614[4] and in 1615 John Briggs,[5] in collaboration with Napier, converted them to the base 10. In 1617[6] John Napier devised a method of multiplication utilizing numbering rods. These were known as "Napier's Bones." Edmund Gunter utilized Napier's logarithms in 1620[7] to create a slide rule with no moving parts. This was subsequently improved on by William Oughtred's conception in 1632 of the astrolabe,[8] which was the forerunner of the slide-rule and nomogram with a sliding scale.

Blaise Pascal in 1642 invented the first desk calculator.[9] This device, which is shown in Figure 2, using toothed wheels was limited in its operations to addition and subtraction.

G. W. Liebnitz made an important contribution by his improvements to the Pascal machine in 1671,[10] to facilitate multiplication by repeated addition. His first machine, Figure 3, was completed in 1694, but it was never practical because of mechanical imperfections.

The planimeter, an analog device first appeared in 1814 after its invention by J. A. Hermann, a Bavarian engineer.[11] Between 1814 and 1854 when Amsler invented the popular modern polar planimeter,[12] Figure 4, many new and improved types of planimeters appeared.

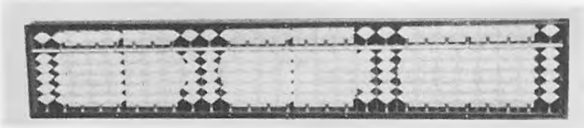


FIG. 1 A JAPANESE ABACUS -
of the type used today by Japanese Tradesmen

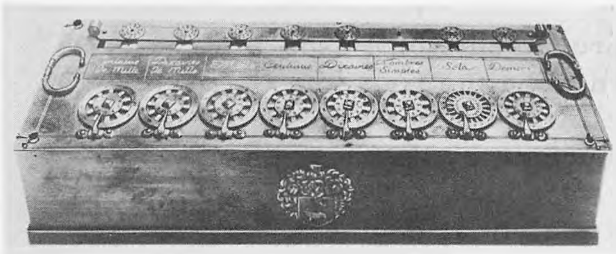


FIG. 2 ONE OF PASCAL'S CALCULATING MACHINES
Courtesy of Felt & Tarrant Mfg. Co.

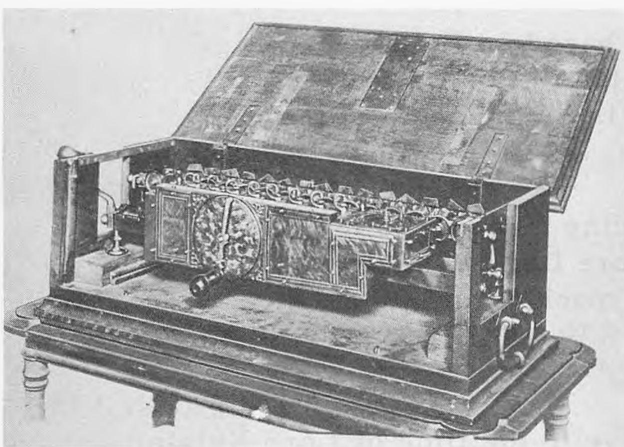


FIG. 3 LEIBNITZ CALCULATOR - The first two-motion machine designed to compute multiplication by repeated addition.
Courtesy Felt & Tarrant Mfg. Co.

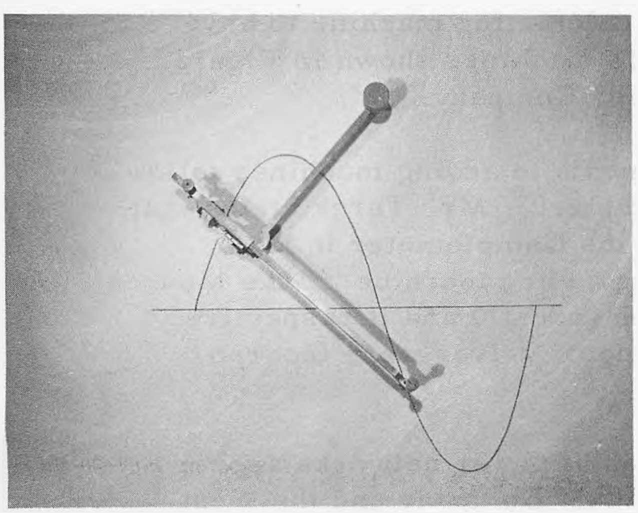


FIG. 4 POLAR PLANIMETER -
Based on Amsler's invention in 1854

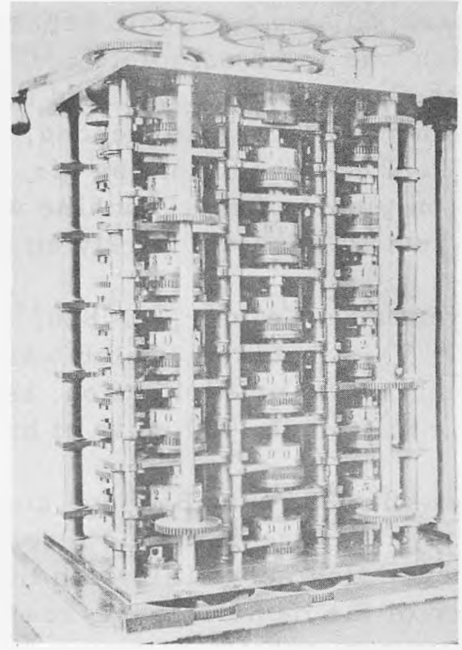


FIG. 5 BABBAGE'S DIFFERENCE ENGINE - Mr. Turck states that this part consists merely of an accumulator mechanism
Courtesy of Science Museum,
South Kensington, England

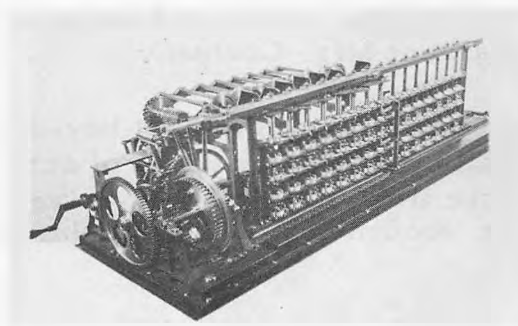


FIG. 6 SCHEUTZ DIFFERENTIAL CALCULATOR
Courtesy Felt & Tarrant Mfg. Co.

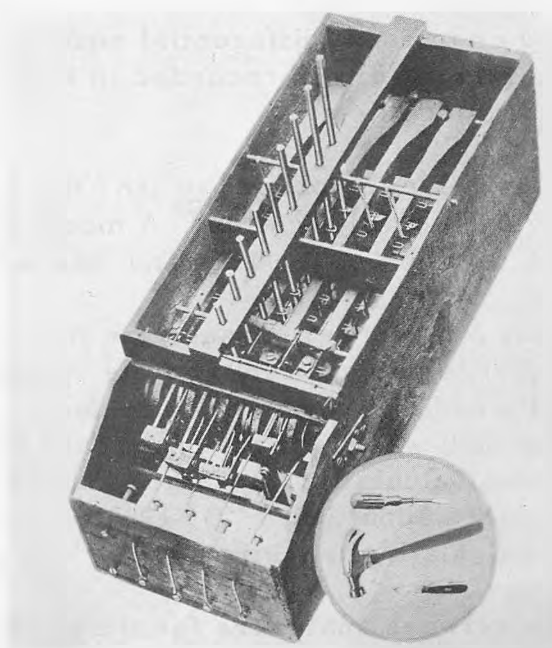


FIG. 7 FELT'S "MACARONI BOX"- This is the first model of the comptometer which Felt built using the tools shown in the insert.
Courtesy Felt & Tarrant Mfg. Co.

Charles Babbage in 1812 conceived a digital difference-equation solver which was to be complete with means to print the answers.[13] The English Government supported the construction of this machine until 1833 when work was suspended, and they finally abandoned the project in 1842. I have been told by Mr. J. A. V. Turck[14] that his inspection of the only part of this machine which was ever built, shown in Figure 5, revealed it was merely an accumulator mechanism.

Thomas De Colmar, in 1820,[15] made improvements on Pascal's calculator. This machine, which was considered to be the first successful machine for multiplication, is still made in Paris by Darras. Thomas himself made 1500 units of his 6 place machine by 1878.

It was in 1833 that Babbage conceived his analytical engine.[16] This was to be a more versatile machine than his earlier unsuccessful "Difference Engine". It is said that this machine was the true fore-runner of the modern large-scale digital computers. The memory and programming were to be in the form of Jacquard-Hollerith principle cards while the arithmetic section used tooth wheels.

Scheutz in Sweden completed a machine based on Babbage's idea of the "Difference Engine" which he exhibited in London in 1855.[17] The original model, shown in Figure 6, is now in the museum of the Felt and Tarrant Mfg. Company.

The patenting of the first key-driven adding machine in 1850[18] was followed by thirty-six years of activity before Dorr E. Felt developed the first practical key-driven adding machine in 1886.[19] A photo of the wooden box model and the tools used to make it are shown in Figure 7.

The early art of analog devices was active in the time of Lord Kelvin. Kelvin's brother, James Thomson, had invented an integrating mechanism, shown in Figure 8.[20] Lord Kelvin conceived the idea of connecting these devices together to solve differential equations in 1876, and an early use of Thomson's integrators is recorded in Kelvin's "Harmonic Synthesizer" built in 1878 to predict tides.[21]

Leon Bolée introduced in 1887 the first calculating machine to have single operation multiplication.[22] A model of this machine, shown in Figure 9, is in the museum of the Felt and Tarrant Mfg. Company.

Many patents were issued for improvements to adding machines following D. E. Felt's patent in 1887 of the "Comptometer". Mr. Turck states that D. E. Felt's addition of the printing feature to the Comptometer in 1889 ("comptograph") was the first practical printing adding machine.[23] The Monroe and Marchant Calculating Machines, shown in Figures 10 and 11 respectively, were introduced about 1911.[24] By 1920 electric motor drives were incorporated into calculating machines.

The network analyzers for the simulation of power networks appear principally as developments of the General Electric Company and the Westinghouse Company. The D. C. Network Analyzer shown in Figure 12 was the first form of these devices to appear in 1925.[25] Since this is a resistive analog,

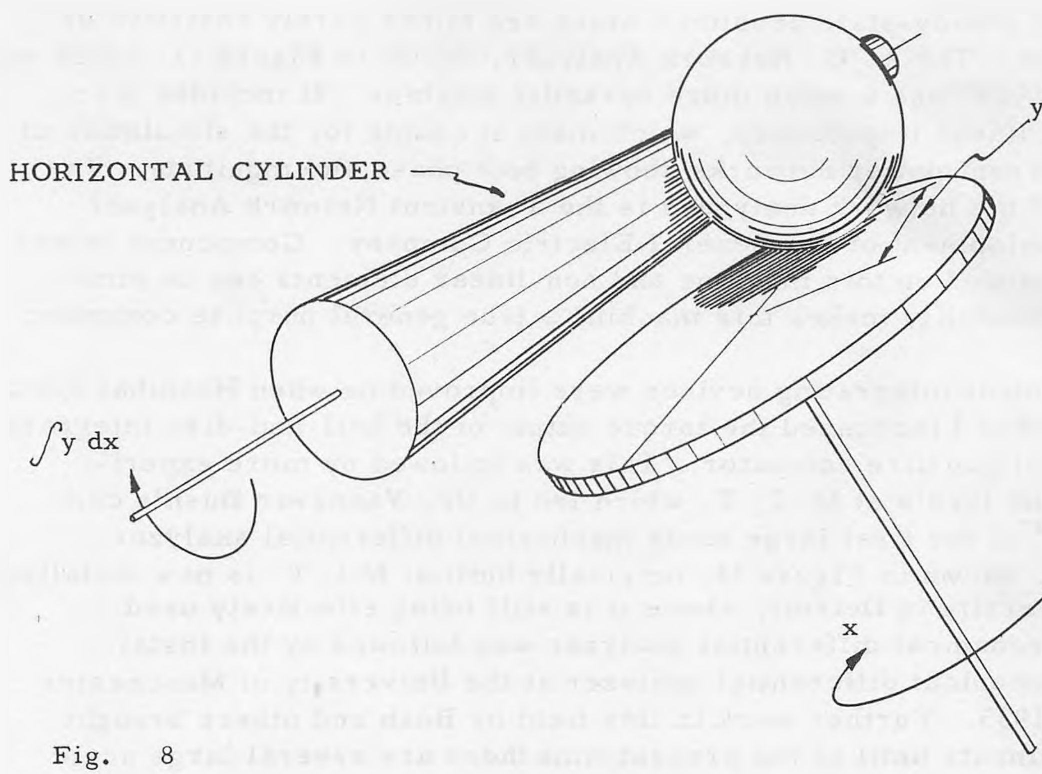


Fig. 8

ARTISTS SKETCH OF THOMSON'S INTEGRATING MECHANISM

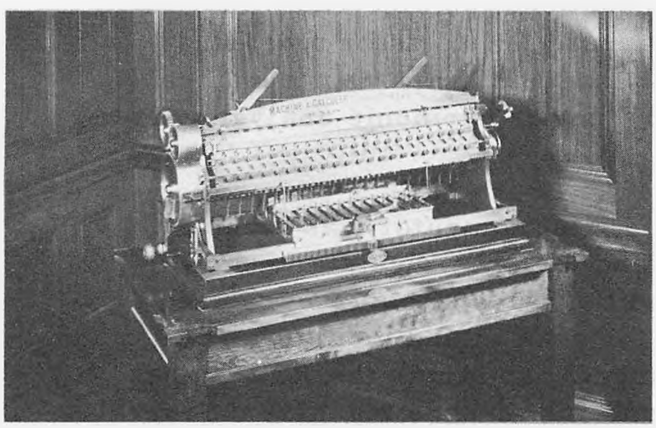


FIG. 9 BOLLEE CALCULATING MACHINE
Courtesy Felt & Tarrant Mfg. Co.

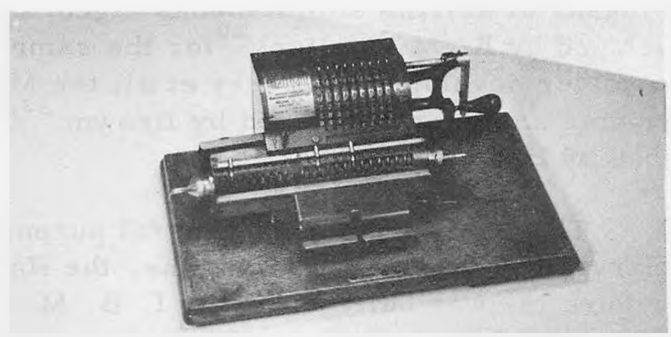


FIG. 11 EARLY MARCHANT CALCULATING MACHINE

FIG. 10 EARLY MONROE CALCULATING MACHINE

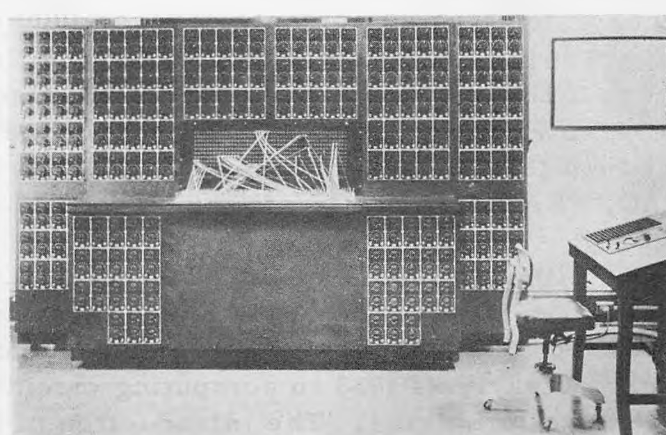


FIG. 12 D.C. NETWORK ANALYZER
Courtesy American Gas and Electric Service Co.

it is limited to steady-state problems which are either purely resistive or purely reactive. The A. C. Network Analyzer, shown in Figure 13, which was introduced in 1929[26] was a much more versatile machine. It included three types types of linear impedances, which made it usable for the simulation of alternating current power networks showing both phase and mgnitude. The most recent of the network analyzers is the Transient Network Analyzer which is a development of the General Electric Company. Component values are readily changed on this machine and non-linear elements can be simulated. This flexibility makes this machine a true general purpose computor.

Mechanical integrating devices were improved on when Hannibal Ford during World War I increased the torque output of the ball-and-disc integrator to make a naval gun-fire computor. This was followed by more experimentation in the 1920's at M. I. T. which led to Dr. Vannevar Bush's completion in 1931[27] of the first large scale mechanical differential analyzer. This machine, shown in Figure 14, originally built at M.I. T. is now installed at Wayne University in Detroit, where it is still being effectively used. Dr. Bush's mechanical differential analyzer was followed by the installation of a mechanical differential analyzer at the University of Manchester in England in 1935. Further work in this field by Bush and others brought more improvements until at the present time there are several large scale mechanical machines in operation in this country.

Simultaneous equation solvers and harmonic analyzers of many types appeared in the 1930's. Among these are Wilbur's Mechanism,[28] a mechanical means of solving simultaneous algebraic equations; an electrical machine developed by R. M. Mallock[29] for the same purpose; an adjuster type equation solver developed by Berry et al; the Multi-harmonigraph, a mechanical harmonic analyzer developed by Brown;[30] and a newer harmonic analyzer described by Hagg and Laurent.[31]

The first large scale general purpose digital computor was completed at Harvard in 1944.[32] This machine, the Harvard Mark I Calculator, shown in Figure 15, was built jointly by I. B. M. and Harvard. Relays were used for the arithmetic section and punched cards for read in and memory. Bell Telephone Laboratories also built a relay computor known as the BTL model I which was completed in 1940.[33] This was a special purpose machine. Several other special purpose relay computors were built at the laboratory and in 1944 work was started on a general purpose relay computor which was to be designated as the BTL model V.[34] This machine contained 9000 relays, 50 pieces of teletype equipment, covered 1000 square feet of floor space and weighed twice the 5 tons of the Mark I. At about the same time the Moore School of Engineering completed its all-electronic digital computor for the Aberdeen Proving Grounds.[35] This machine, the ENIAC, shown in Figure 16, contained 18,000 vacuum tubes. It now has many direct descendants such as SWAC, SEAC, MANIAC, etc.

Although Lovell of Bell Telephone Laboratories is generally credited with the introduction of the operational amplifier during the second World War,[36] such a device was independently discovered and used by Philbrick[37] as early as 1938 in computing circuits for the solution of servomechanism problems. The introduction of the operational amplifier has made possible the newest class of general purpose analog computor. This type is commercially known by such names as BOEING, EASE, GAP/R. GEDA, IDA, and REAC. It is also used in many special purpose computors such as those used in gun directors.

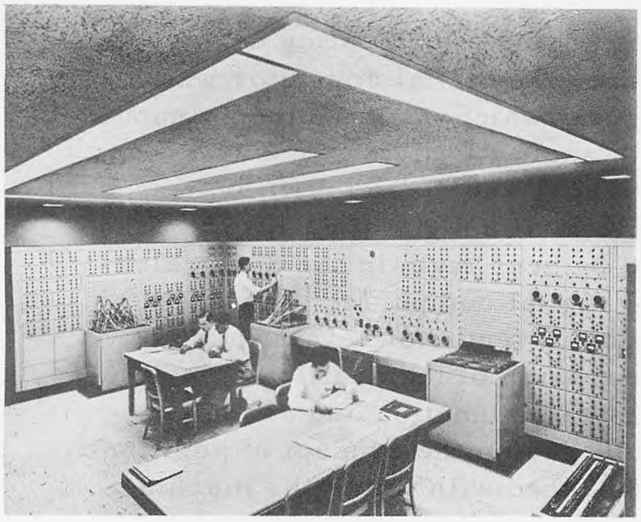


FIG. 13 A.C. NETWORK ANALYZER
Courtesy American Gas and Electric Service Co.

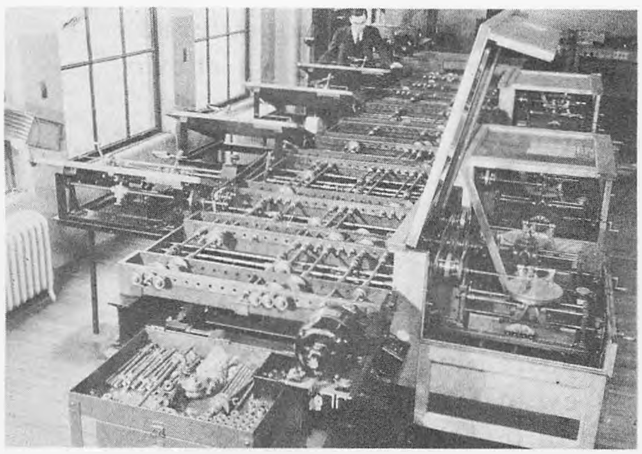


FIG. 14 MECHANICAL DIFFERENTIAL ANALYZER
Courtesy Massachusetts Institute of Technology

FIG. 15 THE HARVARD MARK I CALCULATOR
Courtesy of Harvard University

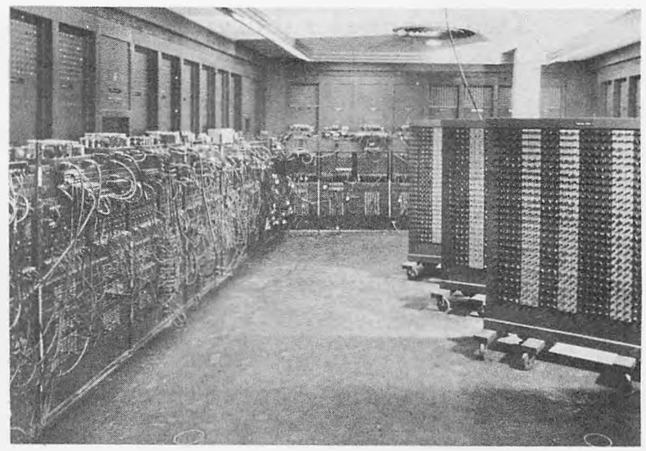
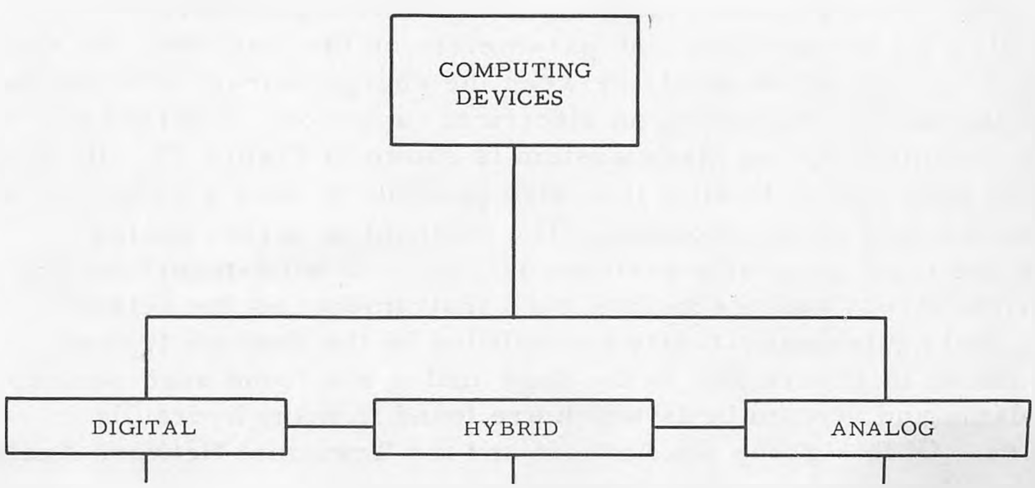


FIG. 16 ENIAC - The first all electronic digital computor
Courtesy Aberdeen Proving Grounds



CLASSIFICATION OF COMPUTING DEVICES

There are two major classes or categories of computing devices as is seen by the chart in Figure 17. These are, the digital computor which deals in numbers only and the analog computor which deals with continuous physical variables. With the perfection of "analog to digital" converters there should probably be a considerable swing in the direction of hybrids which are a combination of analog and digital. The digital computor works directly with integers which are expressed by gear teeth in the desk calculator or electrical pulses in the electronic digital computor. The desk calculator adds by the addition of revolutions or tenths of revolutions, while multiplication is carried out by an extension of the addition principle. The digital machine then has addition as its basic function and all other complex arithmetic is described in a logic based on the concept of addition. The power of the digital machine lies in the speed with which the machine will add. To solve the more complex problems such as the solution of differential equations, the machine computes by repeated refining of an approximation. By sacrificing speed the precision of digitally computed results can be greatly increased.

Two general subdivisions of digital computors will be considered. They are the general purpose and the special purpose machines as shown in Figure 18. In each of these subdivisions we find machines which are either mechanical, electrical or electromechanical. The only true all-mechanical general purpose machine is probably Babbage's "Analytical Engine" which was never developed beyond the stage of Babbage's original idea as to how it would work and what it would do. The ENIAC represents that group of machines in the general purpose class which is all electrical except possibly for some read-in and read-out equipment. Finally, the relay computors which are exemplified by the Harvard Mark I are predominantly electro-mechanical. A few of the devices in the special purpose class of digital computors are shown in the skeletal diagram in Figure 18.

The analog devices as has been stated operate with physical variables such as shaft rotations or electrical voltages. From the chart in Figure 19 we find two types of analogy. The direct analogy is characterized by those cases where problem variables and problem parameters are represented directly by variables and parameters on the machine. An example is the direct analogy which exists between the energy storage in a mechanical spring and the energy storage in an electrical capacitor. A direct electrical analogy for a simple spring mass system is shown in Figure 20. By application of the principle of Duality it is also possible to have a computor which operates as the dual of the problem. The mechanical direct analog computors are most generally scale models such as wind-tunnel models. The electrical direct analogs include such instruments as the network analyzers, and equivalent circuits exemplified by the Anacom type of computor shown in Figure 21. In the fluid analog are found such devices as model dams and stream beds which are found in many hydraulic laboratories. Of this group the Anacom and the Transient Network Analyzer are probably the only general purpose computors.

The indirect analog computors are of a type which can carry out or assist in the solution of algebraic or differential equations. The most

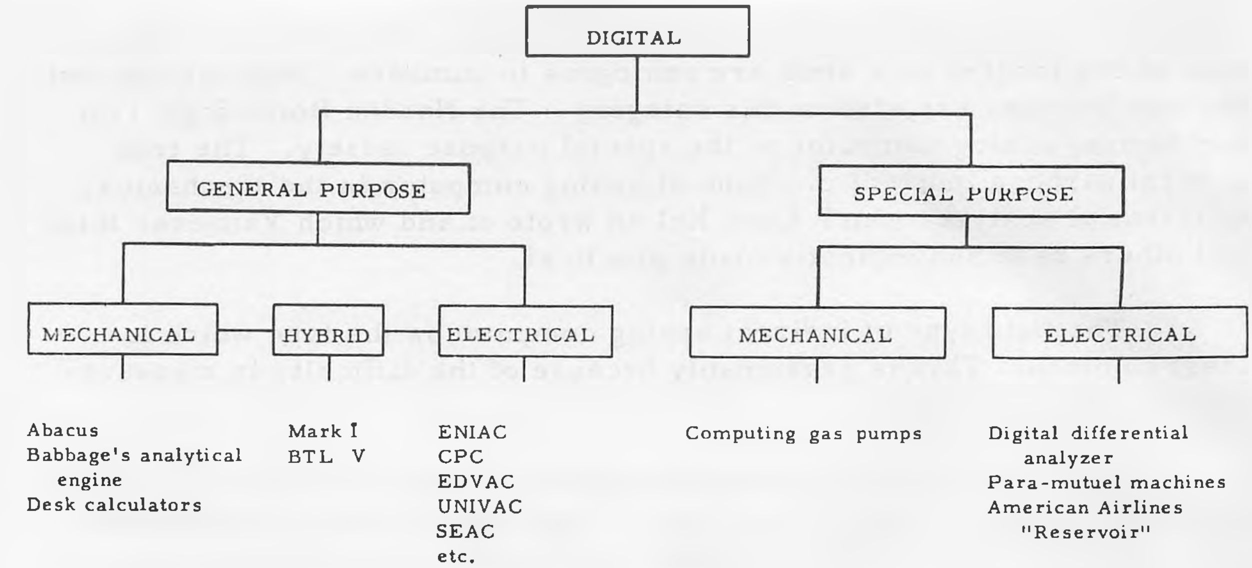


Fig. 18 CLASSIFICATION OF DIGITAL DEVICES

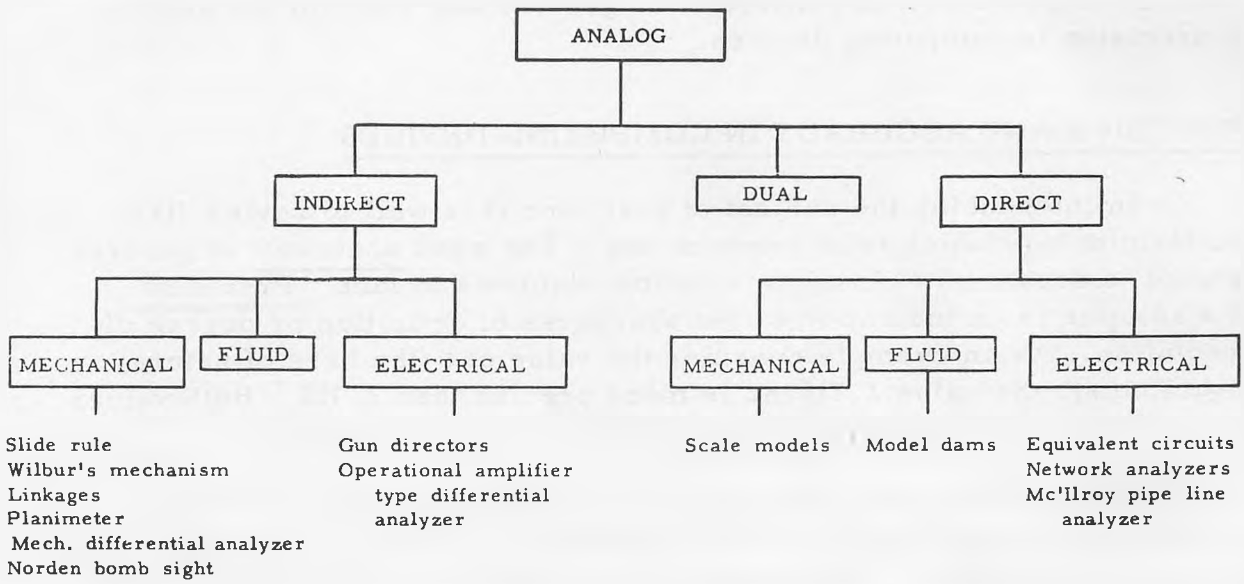
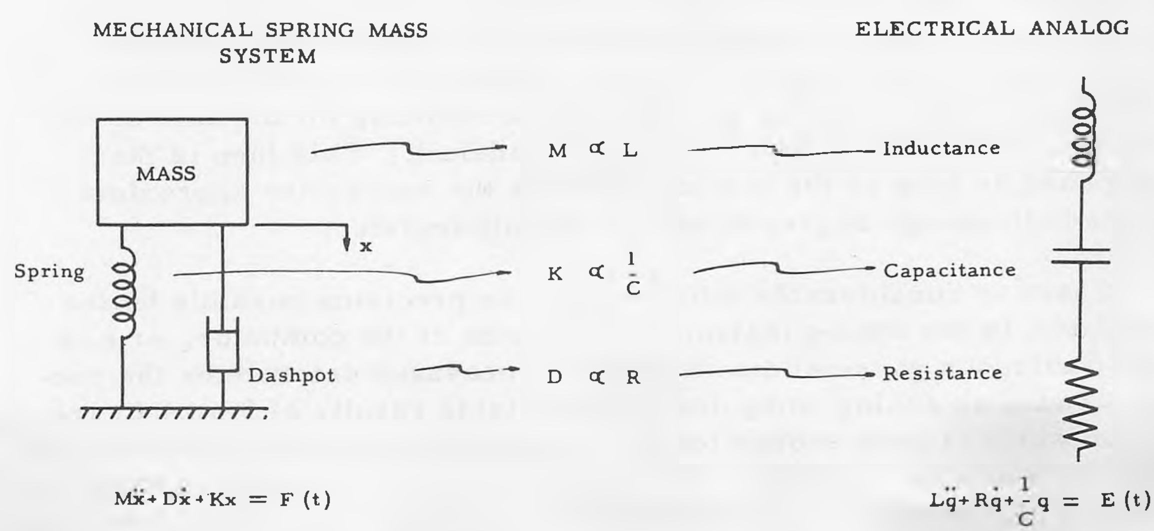


Fig. 19 CLASSIFICATION OF ANALOG DEVICES



ELECTRICAL - MECHANICAL ANALOGY

Fig. 20

common example of a mechanical indirect analog computor is the slide rule where lengths on a stick are analogous to numbers. Nomograms and various linkages are also in this category. The Norden Bomb Sight is a mechanical analog computor of the special purpose variety. The true general purpose indirect mechanical analog computor is the mechanical differential analyzer which Lord Kelvin wrote of, and which Vannevar Bush and others have subsequently made practical.

The fluid type of indirect analog computor is the type which is least common. This is presumably because of the difficulty in measurement in the fluid system.

The electrical (or electronic) indirect analog computor is probably the most common of the indirect type. This type normally employs high gain amplifiers which when applied in appropriate feedback loops will perform many mathematical operations. It is the computors in this class which use the operational amplifier discovered by both Philbrick and Lovell. Before the components of this type of computor are considered in detail it would be well to consider in a general way some of the aspects of precision in computing devices.

PRECISION AND ACCURACY IN COMPUTING DEVICES

In introducing the subject of precision it is well to review first the terminology which is in common use. The word accuracy in general is used to denote how closely a solution conforms to fact. Precision of a solution is an indication of the sharpness of definition or degree of resolution. As an example consider the value of ε (the base of natural logarithms), the value 2.718282 is more precise than 2.718. Both values are accurate statements describing the value of the constant ε.

In expressing the precision of an analogically computed result, the sharpness of resolution in the components and the measuring devices is the limiting factor. The accuracy of a result depends upon the validity of the analog which is set up. The digital computor is accurate to the degree with which it will perform operations required of it without making mistakes. The UNIVAC with its self checking system would by this definition then exhibit a high degree of accuracy. The precision of the digital machine is determined by the number of digits in the register and the freedom from round-off errors. (Round-off errors are those errors which accumulate in a problem due to rounding off the answer to the number of significant figures being considered.) This then is the limiting case so long as the machine carries the successive approximations to a high enough degree to utilize the full register.

There is considerable difference in the precision possible by the two methods; in the analog instruments the size of the computor, or how well the analogous physical quantity can be measured determines the precision. Thus, an analog computor usually yields results of 3 or 4 figure precision which is good enough for many engineering purposes because the original data may be no better. The digital machine on the other hand can give nearly any desired precision without an increase in machine size, by exchanging time for more significant figures. In digital machines it is usual to use 10 to 20 significant figures.

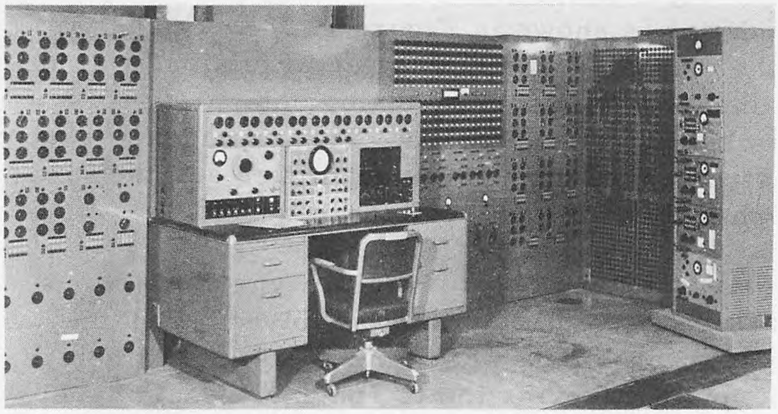


FIG. 21 DIRECT ANALOG TYPE COMPUTOR
at California Institute of Technology
Courtesy of California Institute of Technology

Courtesy:
Electronic Analog Computors
By Korn & Korn
Published by McGraw Hill
Book Company

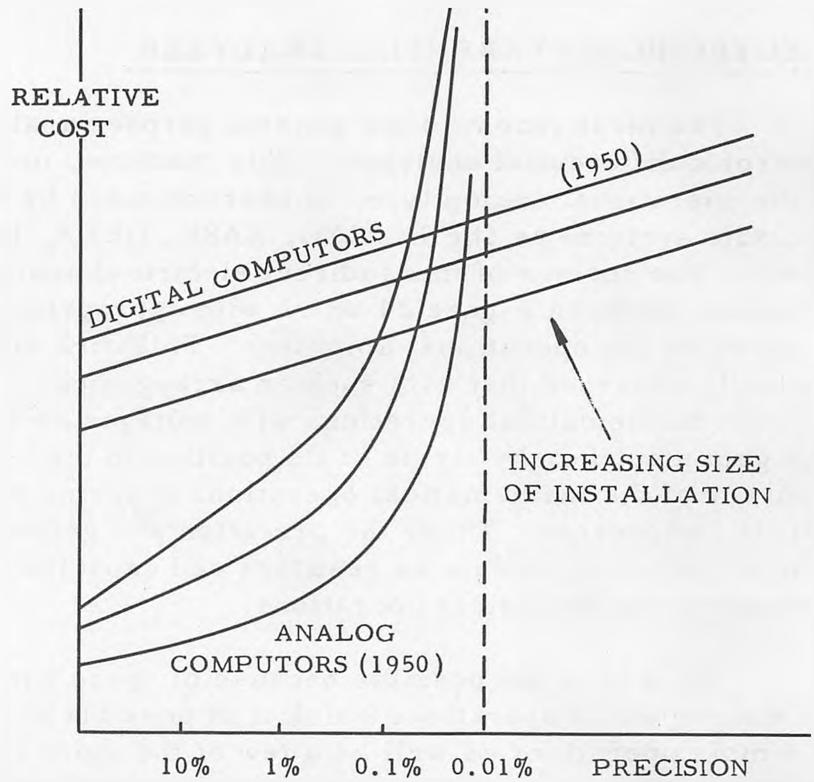
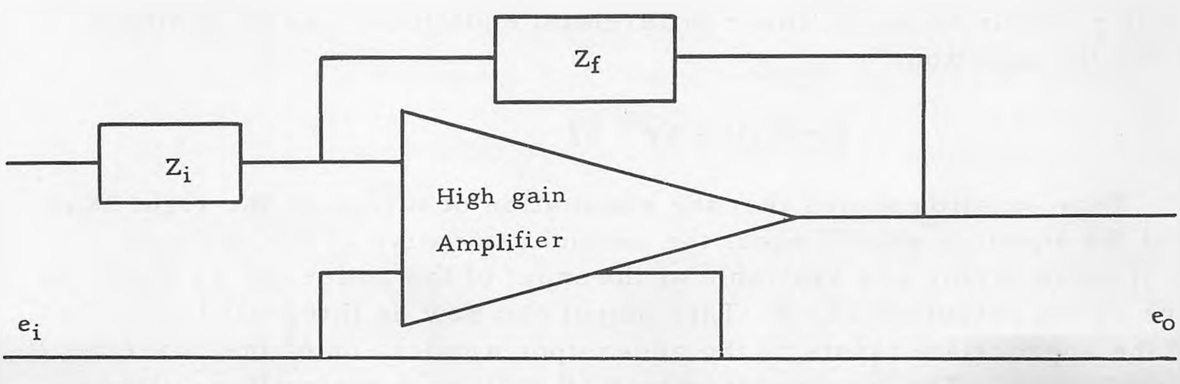


Fig. 22 Relative cost vs precision for digital and analog computors (1950) for different size installations



$$e_o = \frac{Z_f}{Z_i} \; e_i$$

SCHEMATIC DIAGRAM OF OPERATIONAL AMPLIFIER COMPUTING ELEMENT

Fig. 23

An interesting curve showing relative cost of attaining greater precision in analog and digital machines is shown in Figure 22. As is seen in the curve by Korn and Korn[38] the cost of attaining greater precision in a digital machine is essentially linear. Precision in analog computation on the other hand becomes increasingly more expensive. This is understandable since the cost of measuring instruments for physical quantities and the cost of componenets both rise sharply with the precision required.

How much precision to buy in an analog or a digital type computor is a problem governed mainly by the problem to be solved. For the solution of some structural dynamic problems where the system parameters are no more than 100% precise, and the fabrication is rough, it would hardly be conceivable to purchase digital or analog equipment with a precision of .01%.

ELECTRONIC DIFFERENTIAL ANALYZER

The most recent of the general purpose analog computors is the electronic differential analyzer. This machine, more generally known as the operational analog type, is characterized by such commercially available systems as the BOEING, EASE, GEDA, IDA, GAP/R and REAC. The nucleus of this indirect electrical analog is a high gain d-c amplifier shown in Figure 24 which with appropriate feed-back networks is known as the operational amplifier. Philbrick and Lovell each independently observed that with such an arrangement it would be possible to perform mathematical operations with voltages as the variables. The high gain amplifier by virtue of its position in the circuit assures the accuracy of the mathematical operations in terms of a few well known circuit components. Thus, the precision and general availability in pure form of such components as resistors and capacitors is exploited to perform strict mathematical operations.

While it is not possible because of space limitations to list all of the mathematical operations which it is possible to perform, the basic or simple operations as well as a few of the more common non-linearities would be of interest. Diagrams of the four basic mathematical operations of addition, scale change or proportioning, integrations and differentiation are shown in Figure 25. With these basic operations available, which I shall now designate by boxes with the appropriate symbol as shown in Figure 25, it is possible to solve linear differential equations. As an example consider the equation.

$$\ddot{y} = F(t) - a\dot{y} - by$$

This equation states that the summation of terms on the right hand side of the equation should equal the second derivative of the variable y. thus, if these terms are available at the input of the adder (A) as shown in Figure 26 the output will be $\ddot{y}$. This output can now be integrated twice to yield the appropriate inputs to the adder upon application of the correct scaling factors. The forcing function F (t) may be a perfectly arbitrary function of time so long as it is possible to generate it as an electrical voltage. The usefulness of the computor does not stop at the linear realm. It is also possible to consider non-linear systems such as those involving

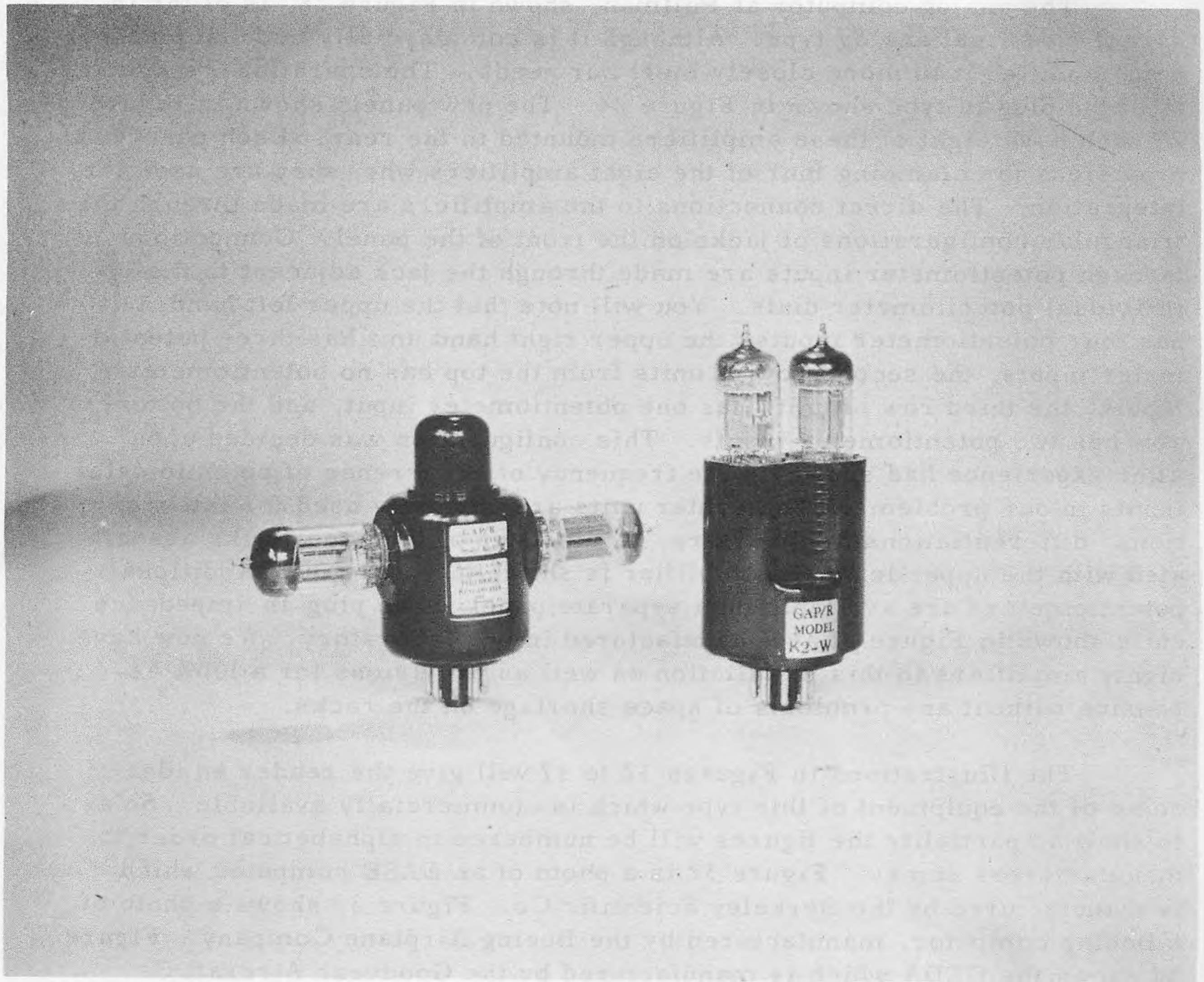


FIG. 24 PLUG-IN TYPE OPERATIONAL AMPLIFIERS

products of two variables, limits, deadzones, or dry friction. These operations are shown schematically in Figure 27. With these concepts available it is now possible to perform an infinite number of combinations of operations to solve linear and non-linear differential equations, algebraic equations, or to perform specific mathematical operations in the process of data reduction.

The analog computor at Pullman, shown in Figure 28, is of the indirect electrical analog type. Although it is commercially manufactured we have modified it to more closely meet our needs. The operational amplifier is of the plug in type shown in Figure 24. The new panels shown in Figure 29 each have eight of these amplifiers mounted in the rear. Each panel has provisions for clamping four of the eight amplifiers when they are used for integration. The direct connections to the amplifiers are made through the triangular configurations of jacks on the front of the panel. Connections through potentiometer inputs are made through the jack adjacent to the individual potentiometer dials. You will note that the upper left hand unit has four potentiometer inputs, the upper right hand unit has three potentiometer inputs, the second row of units from the top has no potentiometer inputs, the third row of units has one potentiometer input, and the bottom row has two potentiometer inputs. This configuration was decided upon after experience had shown us the frequency of occurrence of potentiometer inputs in our problems. The center units are normally used for integrations, differentiations or inverters. A close-up of the panel jacks associated with the upper left hand amplifier is shown in Figure 30. Additional potentiometers are available on a separate panel. The plug-in impedance units shown in Figure 31 are manufactured in our laboratory. We now have eighty amplifiers in this installation as well as provisions for a 100% expansion without any problems of space shortage on the racks.

The illustrations in Figures 32 to 37 will give the reader an idea of some of the equipment of this type which is commercially available. So as to show no partiality the figures will be numbered in alphabetical order to manufacturers names. Figure 32 is a photo of an EASE computor which is manufactured by the Berkeley Scientific Co. Figure 33 shows a photo of a Boeing computor, manufactured by the Boeing Airplane Company. Figure 34 shows the GEDA which is manufactured by the Goodyear Aircraft Co. Figures 35 and 36 show GAP/R computors. The first at Woodward Governor Company is used primarily on "real time" while the second (Figure 36) at Bendix is used principally as a fast time or repetitive computor. Both of these units are manufactured by G. A. Philbrick Research Inc. Finally a REEVES installation is shown in Figure 37. This unit is manufactured by the Reeves Instrument Company.

ACKNOWLEDGEMENTS

The author wishes to express his sincere appreciation to the many people and organizations who have assisted in securing illustrations for this paper. He is especially indebted to Mr. Dick Drake of the Felt and Tarrant Mfg. Company for his encouragement and cooperation during the writing of this paper, and to both Professor V. C. Rideout of the University of Wisconsin and Dr. Rufus Oldenburger of the Woodward Governor Company who edited early copies of this text.

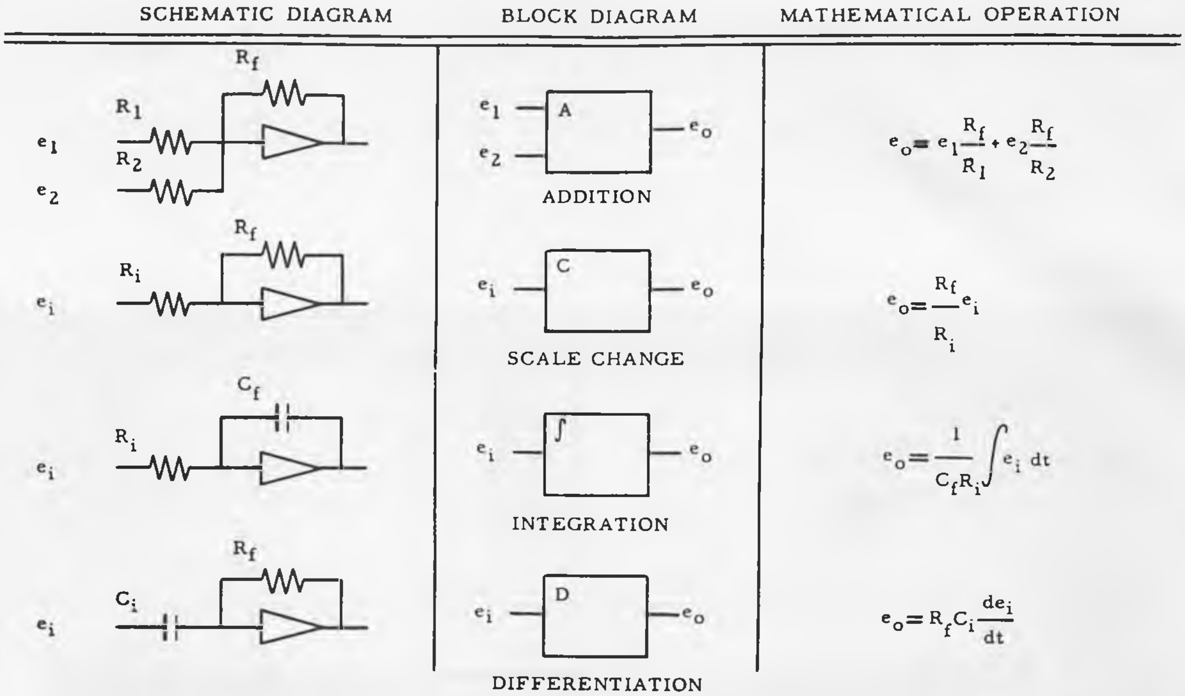


Fig. 25 BASIC LINEAR COMPUTING ELEMENTS

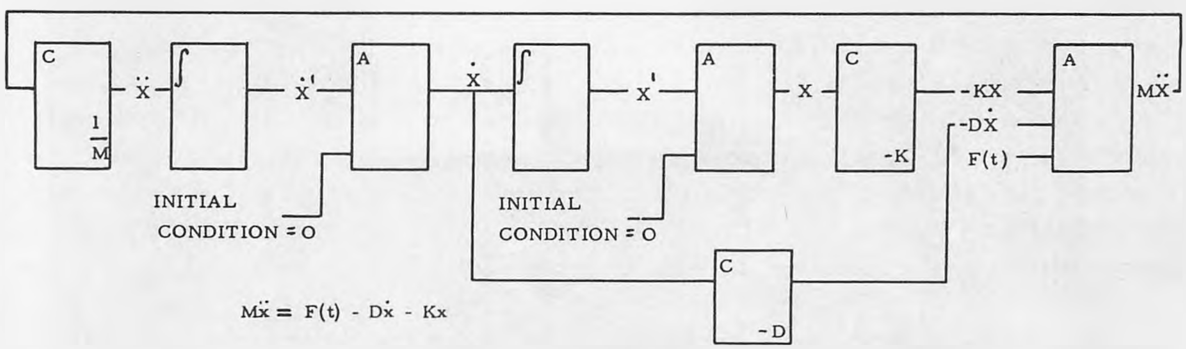


Fig. 26 COMPUTOR BLOCK DIAGRAM FOR THE SOLUTION OF A SINGLE DEGREE OF FREEDOM SYSTEM

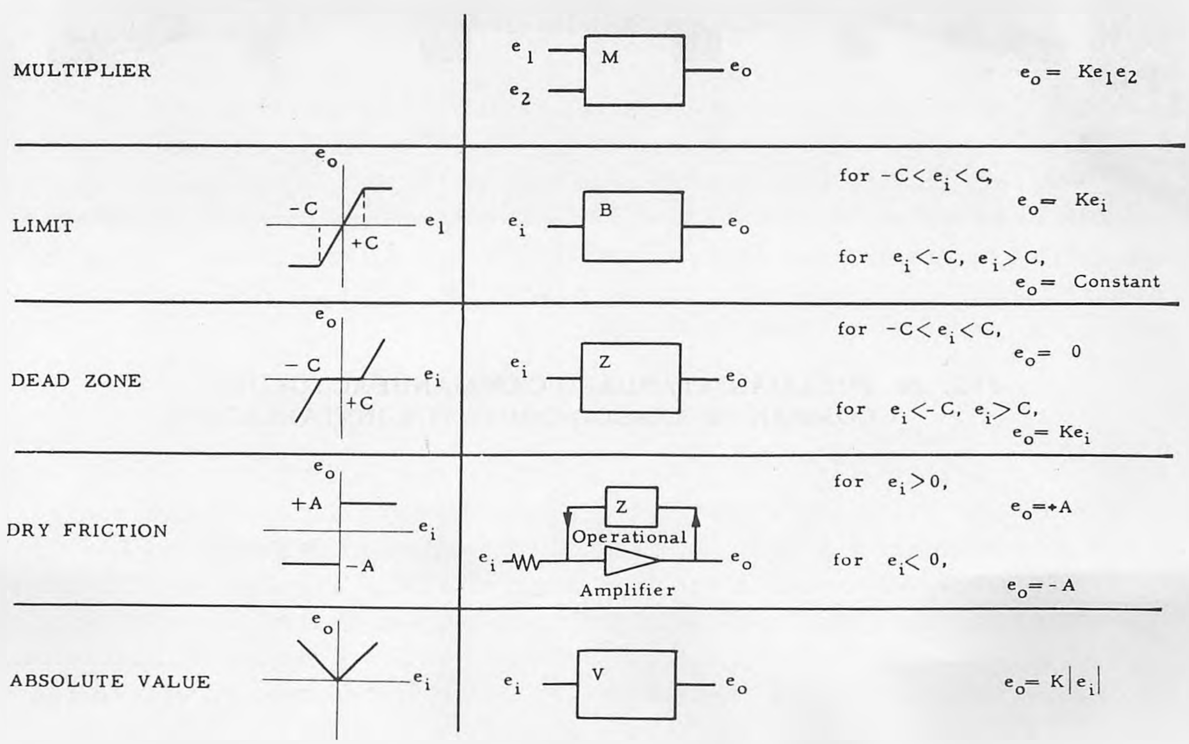


Fig. 27 NON - LINEAR OPERATIONS

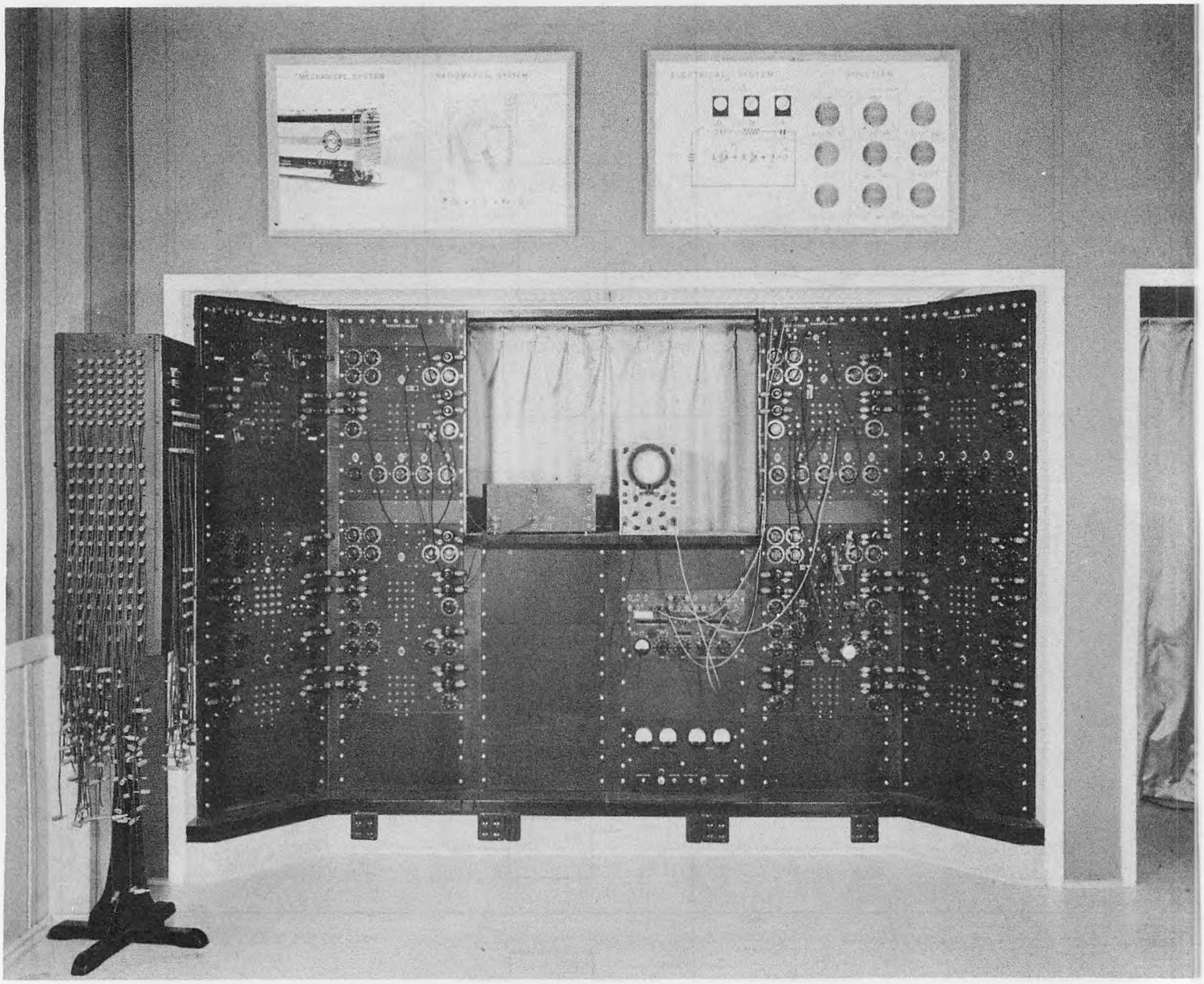


FIG. 28 PULLMAN-STANDARD CAR MANUFACTURING COMPANY'S ANALOG COMPUTOR INSTALLATION

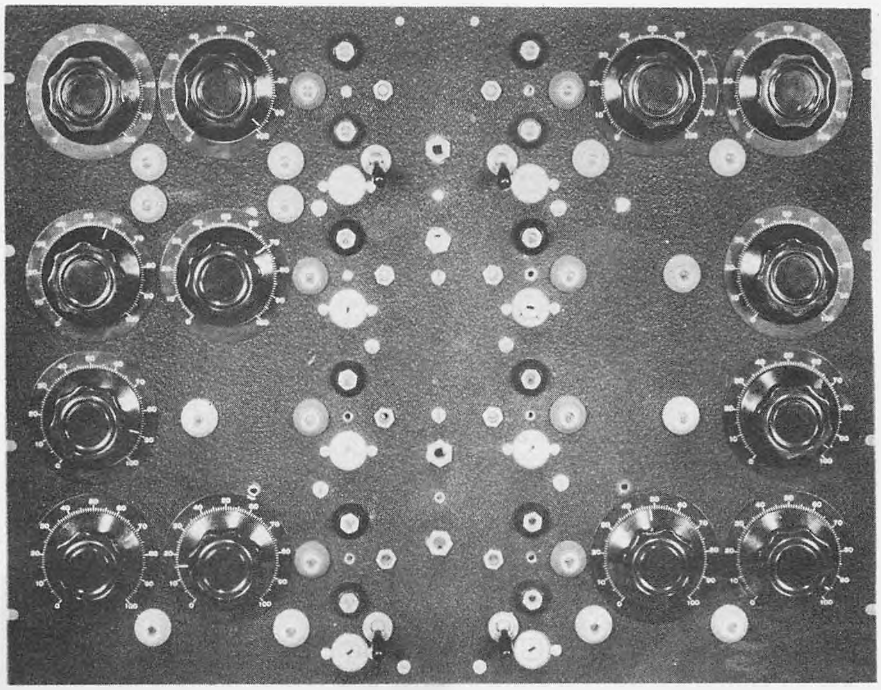


FIG. 29 INDIVIDUAL COMPUTING PANEL - of the Pullman-Standard Analog Computor

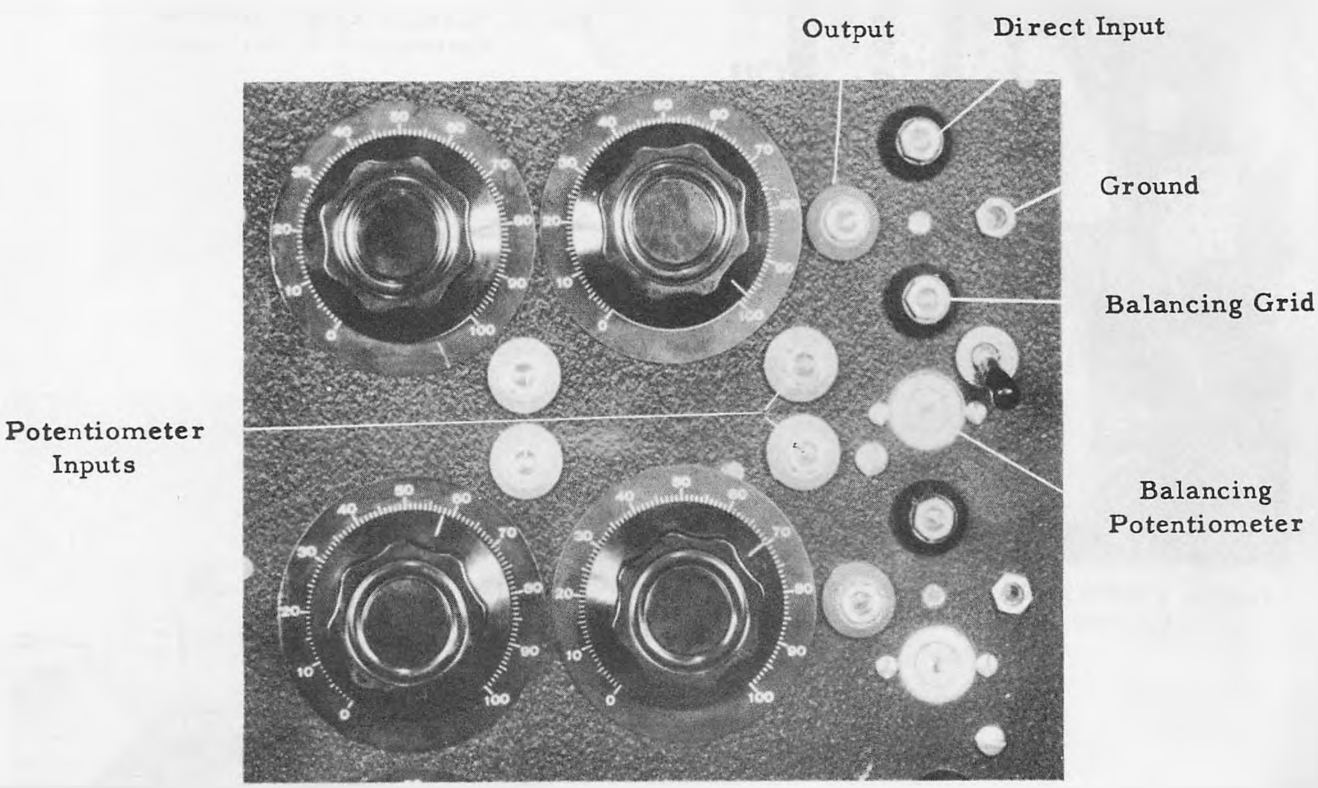


FIG. 30 DETAIL OF UPPER LEFT SECTION OF COMPUTING PANEL

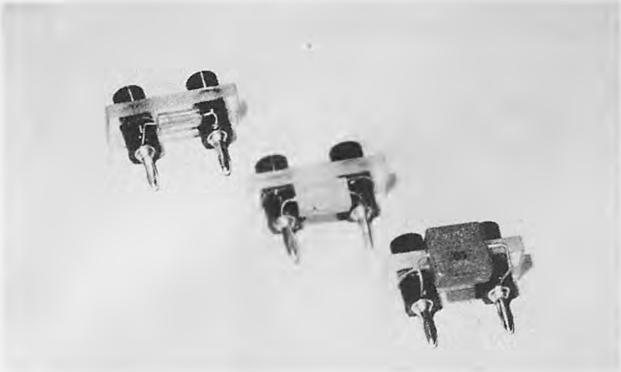


FIG. 31 PLUG-IN IMPEDANCE ELEMENTS - For use with the Pullman-Standard Computor.

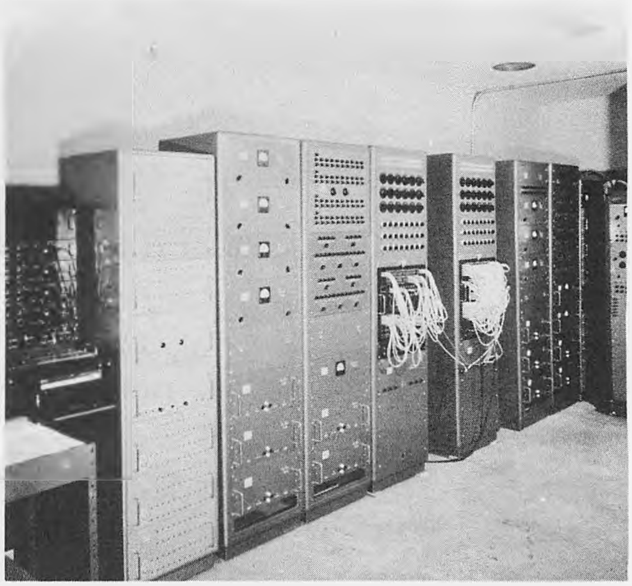


FIG. 32 BERKLEY EASE COMPUTOR
Courtesy of J. B. Rae, Los Angeles

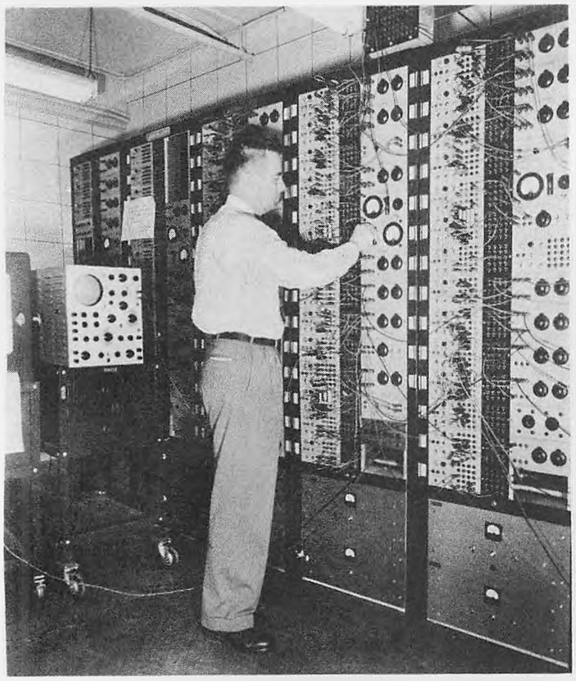


FIG. 33 BOEING ANALOG COMPUTOR
Courtesy of Boeing Airplane Company

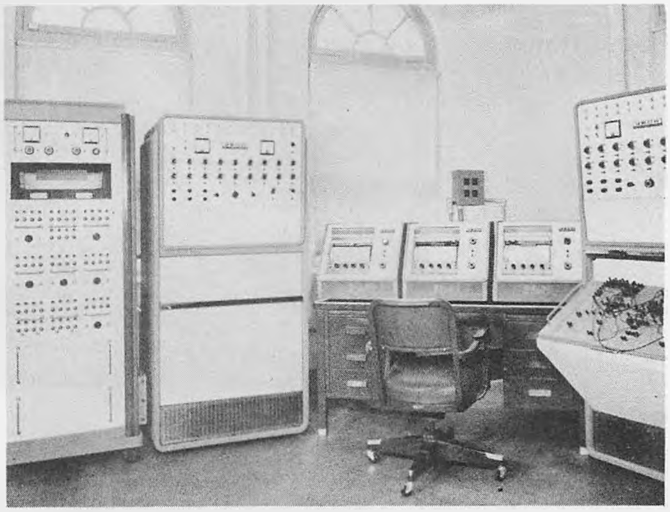


FIG. 34 GOODYEAR GEDA COMPUTOR INSTALLATION
Courtesy of Goodyear Aircraft Company, Akron, Ohio

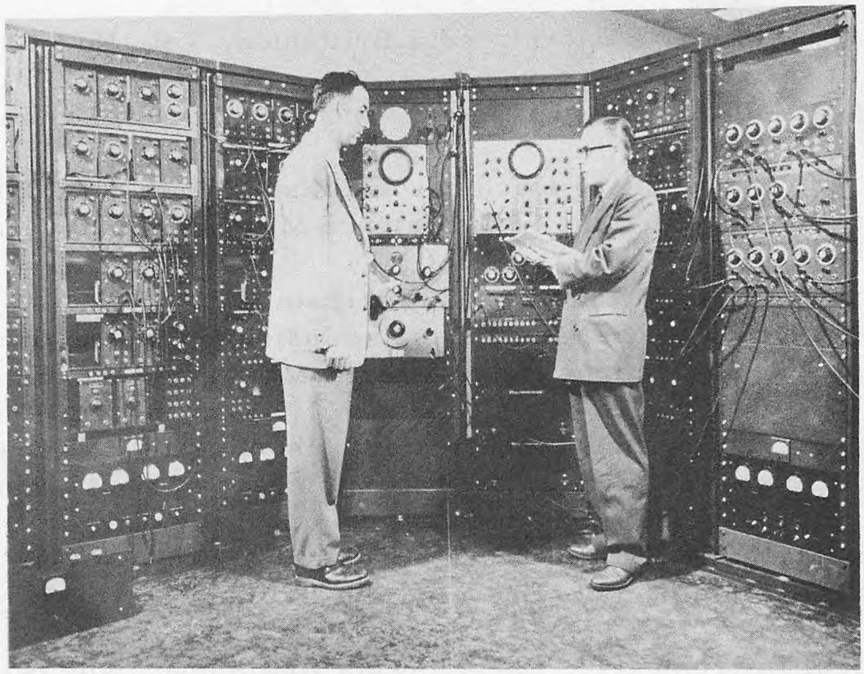


FIG. 35 PHILBRICK ANALOG COMPUTOR INSTALLATION
at Woodward Governor Company
Courtesy Woodward Governor Company

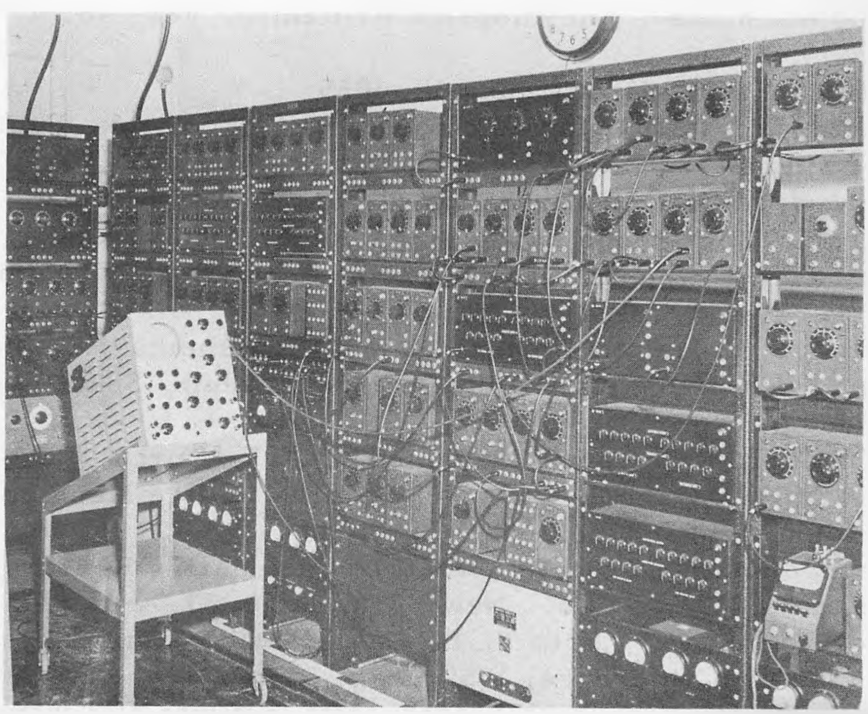


FIG. 36 PHILBRICK COMPUTOR INSTALLATION
at Bendix Products Division
Courtesy Bendix Aviation Co.

FOOTNOTES

[1] "Maps", Chambers Encyclopedia, Vol. No. 15, 1950, Pages 1-144.
[2] "Maps", Encyclopedia Brittanica, Vol. No. 21, 1949, Page 837.
[3] "Abacus", Chambers Encyclopedia, Vol. No. 1, 1950.
[4] "Calculating Machines", Encyclopedia Brittanica, Vol. No. 4, 1949, Page 548.
[5] A History of Mathematics, F. Cajori, 2nd Edition, MacMillan Co., 1919 Page 163.
[6] "Mathematics I, Calculating Machines and Instruments", D. Baxandall Catalogue of the Collections in the Science Museum, South Kensington, England, 1926, Page 7.
[7] "Calculating Machines", Encyclopedia Brittanica, Vol. No. 4, 1949, Pg. 553.
[8] "Mathematics I, Calculating Machines and Instruments", D. Baxandall Catalogue of the Collections in the Science Museum, South Kensington, England, 1926, Page 37.
[9] "Mathematics I, Calculating Machines and Instruments", D. Baxandall Catalogue of the Collections in the Science Museum, South Kensington, England, 1926, Page 37.
[10] "Calculating Machines", Encyclopedia Brittanica, Vol. No. 4, 1949, Pg. 549.
[11] "Mathematical Instruments", Encyclopedia Brittanica, Vol. No. 15, 1949 Page 70.
[12] "Mathematical Instruments", Encyclopedia Brittanica, Vol. No. 15, 1949, Page 70.
[13] "Calculating Machines", Encyclopedia Brittanica, Vol. No. 4, 1949, Pg. 552.
[14] Personal letter from Mr. J.A.V. Turck, 2371 No. Bay Road, Miami Beach, Florida, May 18, 1953.
[15] "Mathematics I, Calculating Machines and Instruments", D. Baxandall, Catalogue of the Collections in the Science Museum, South Kensington, England, 1926, Page 8.
[16] "Calculating Machines", Encyclopedia Brittanica, Vol. No. 4, 1949, Page 552.
[17] The Illustrated London News, June 30, 1855.
[18] "Origin of Modern Calculating Machines", J.A.V. Turck, Pub. by Western Society of Engineers, 1921, Pages 17-19.
[19] Origin of Modern Calculating Machines, Pages 50-70.
[20] J. Thomson, Proceedings of the Royal Society, Vol. No. 24, 1876, Pg. 262.
[21] "Mathematical Instruments", Encyclopedia Brittanica, Vol. No. 15, 1949, Page 71.
[22] "Mathematics I, Calculating Machines and Instruments", D. Baxandall, Catalogue of the Collections in the Science Museum, South Kensington, England, 1926, Page 9.
[23] Origin of Modern Calculating Machines, J.A.V. Turck, Pages 111-120.
[24] American Office Machines Research Service, Office Machines Research, Inc., New York, 1940, Sec. 4.3.
[25] "High Speed Computing Devices", Staff of Engineering Research Assoc. Inc., 1950, McGraw Hill Book Co., New York, Page 187.
[26] "*Introductory* Remarks", Eric T. B. Gross, Symposium on Network Analyzers, American Power Conference, 1952, Page 383.
[27] Letter from F. M. Verzuh, Massachusetts Institute of Technology, June 4, 1953.
[28] "The Mechanical Solution of Simultaneous Equations", J. B. Wilbur, Journal of Franklin Institute, Vol. No. 222, 1936, Pages 715-724.

[29] "An Electrical Calculating Machine", R. R. M. Mallock, Proceedings of The Royal Society, Vol. No. 140, 1933, Page 457.

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[31] "A Machine for the Summation of Fourier Series," G. Hagg and T. Laurent, Journal of Scientific Instruments, Vol. No. 23, 1946, Pages 155-158.

[32] "The Automatic Sequence Controlled Calculator", H. H. Aiken and G. M. Hopper, Electrical Engineering, Vol. 65, Nos. 8 and 9, August and Sept. 1946, Pages 384-391.

[33] "High Speed Computing Devices", Staff of Engineering Research Assoc., Inc., McGraw Hill Book Co., New York, 1950, Page 187.

[34] "A Bell Telephone Laboratories Computing Machine-I, F. L. Alt, Mathematical Tables and Other Aids to Computation, Vol. III, No. 21, 1948, Pages 1-13.

[35] "The Electronic Numerical Integrator and Computor, H. H. Goldstine and Adele Goldstine, Mathematical Tables and Other Aids to Computation, Vol. II, No. 15, 1946, Pages 97-110.

[36] Lecture Notes: Analog Computor Course by Professor V. C. Rideout, University of Wisconsin, Spring 1951, Page 5.

[37] Personal Unpublished papers of George A. Philbrick, Boston, Mass.

[38] "Electronic Analog Computers," Korn & Korn, McGraw-Hill Book Co. 1952, Page 3.

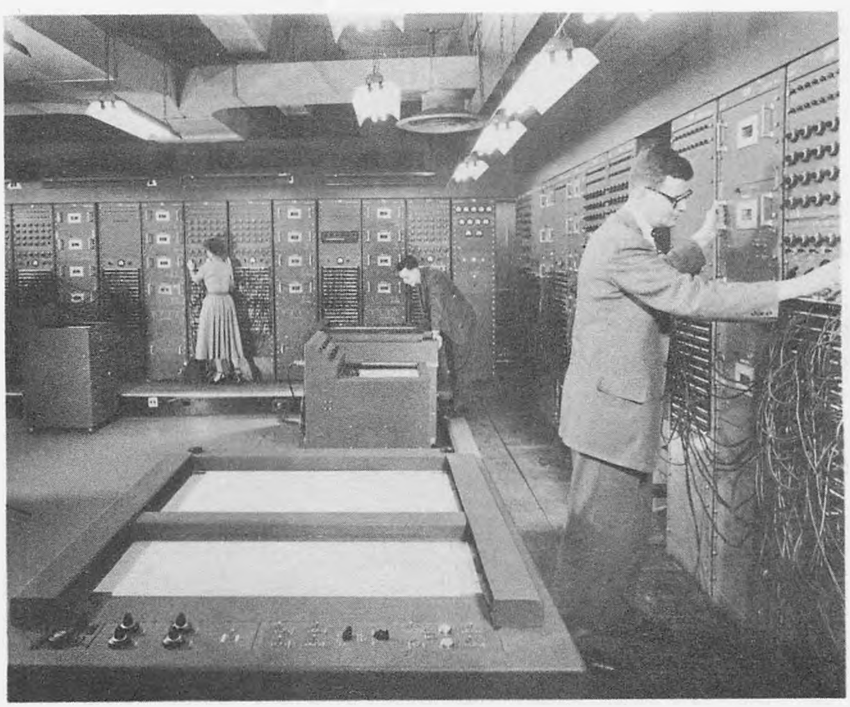

FIG. 37 REEVES COMPUTOR INSTALLATION
Courtesy Reeves Instrument Co.

THE ELECTRONIC DIFFERENTIAL ANALYZER

ABSTRACT

This text is intended as a foundation upon which the research engineer may find usefulness for these analog computors through the understanding and application of the principles of operation. Consideration is also given to computor time scales and precision.

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FOR PRESENTATION AT:

Case-Industry Conference on
Applications of Computing Machines
Case Institute of Technology
Cleveland, Ohio
April 13, 1954

PRINCIPLE OF OPERATION

The operational amplifier shown schematically in Figure 7 consists of a high-gain d. c. amplifier with appropriate generalized input and feedback impedance. If the input impedance of this amplifier is sufficiently high so that grid current is negligible, the output voltage (e_o) will be a complicated function of the gain of the amplifier, the input and feedback impedances and the input voltage. This is shown in equation (1).

$$\frac{e_o}{e_i} = \frac{-Z_f}{Z_i}\left[\frac{1}{1+\frac{1}{A}\left(\frac{Z_f}{Z_i}+1\right)}\right] \qquad (1)$$

For the case where the gain A of the amplifier is very large with respect to unity, this equation reduces to the form shown in equation (2),

$$e_o \doteq -\frac{Z_f}{Z_i}\, e_i \qquad (2)$$

where the output voltage is approximately proportional to the input modified by the ratio of the feedback impedance to the input impedance. The real significance of this phenomena becomes evident immediately when it is noted that Z_i and Z_f are very general impedances which may be complex. With this as background information, it would now be well to consider some of the simple single mathematical operations which can be simulated.

LINEAR OPERATIONS

If the feedback and the input impedances are purely resistive, equation (2) takes the form shown in Figure 8. In this figure is shown the diagram and equation for the operation of scale change or coefficient change. The operation is illustrated by the inclusion of photographs of the input and output relationship for an actual scale change operation. The block representation with which I shall denote this operation is shown as a block with the letter C in the upper lefthand corner. In a computor diagram the magnitude of the scale change will be shown as a number in the lower right-hand corner with the arrows indicating the direction of flow.

If the feedback impedance is a pure capacitance (of impedance value $\frac{1}{pC_f}$ in operational notation) and the input is purely resistive, the output voltage becomes proportional to a constant times the operator $\frac{1}{p}$ as shown in equation (3).

$$e_o = -\frac{e_i}{R_i C_f}\,\frac{1}{p} = \frac{1}{R_i C_f}\int e_i\,dt \qquad (3)$$

This operator designates integration of the input voltage with respect to time. The schematic diagram for this operation is shown in Figure 9. As in the previous illustration the input and output relationship are shown for a step input, as well as the block diagram representation for this operation.

The third operation, and one which it is generally well to avoid in analog computation is that of differentiation. I shall indicate the problems of using this device later in the article. This device has the operational impedance $\frac{1}{pC_i}$ as an input element and a pure resistance as a feedback, as shown in Figure 10. The output voltage now becomes proportional to a constant times the operator p as shown in equation (4).

$$e_o \doteq -R_f C_i p e_i = -R_f C_i \frac{de_i}{dt} \qquad (4)$$

This operator denotes a time derivative of the applied voltage e_i. The photos show the restoration of the integrated wave in Figure 9 by differentiation to the original step input. The block diagram for this unit is designated by the letter D.

As a final step the operation of addition is illustrated. It can be shown that if two or more inputs are connected to an operational amplifier through appropriate input resistances as shown in Figure 11, that the output voltage will be proportional to the sum of the applied voltages. This relationship is indicated in equation (5).

$$e_o \doteq -\left[\frac{R_f}{R_1} e_1 + \frac{R_f}{R_2} e_2 + \frac{R_f}{R_3} e_3\right] \qquad (5)$$

As an extension of simple addition, it is also possible to add integrals, add derivatives, or add combinations of these with direct addition. Three such combinations are illustrated in Figure 12.

To illustrate the usefulness of simple impedance elements in combination to form complex impedances, eight operations are shown in Figures 13 and 14. In Figure 13 the use of a parallel resistor-capacitor combination in the feedback element generates a simple lag function of time constant $R_f C_f$. This is followed by the parallel resistor-capacitor combination in the input element, which gives an operation of proportion plus derivative. The third case in this figure is the series combination of resistance and capacitance in the feedback. This gives rise to the operation of scale change plus integration. In the last case the series combination is moved to the input element which sets up an operation of derivative times lag which is commonly called a lead network in control system terminology. In Figure 14 four cases are illustrated with complex impedances in both the input and the output. The output relations for each of these cases may be easily derived by substituting the appropriate values for Z_i and Z_f.

In addition to these active computing networks there are many passive networks which will perform mathematical operations. These are shown in Figures 15, 16 and 17.

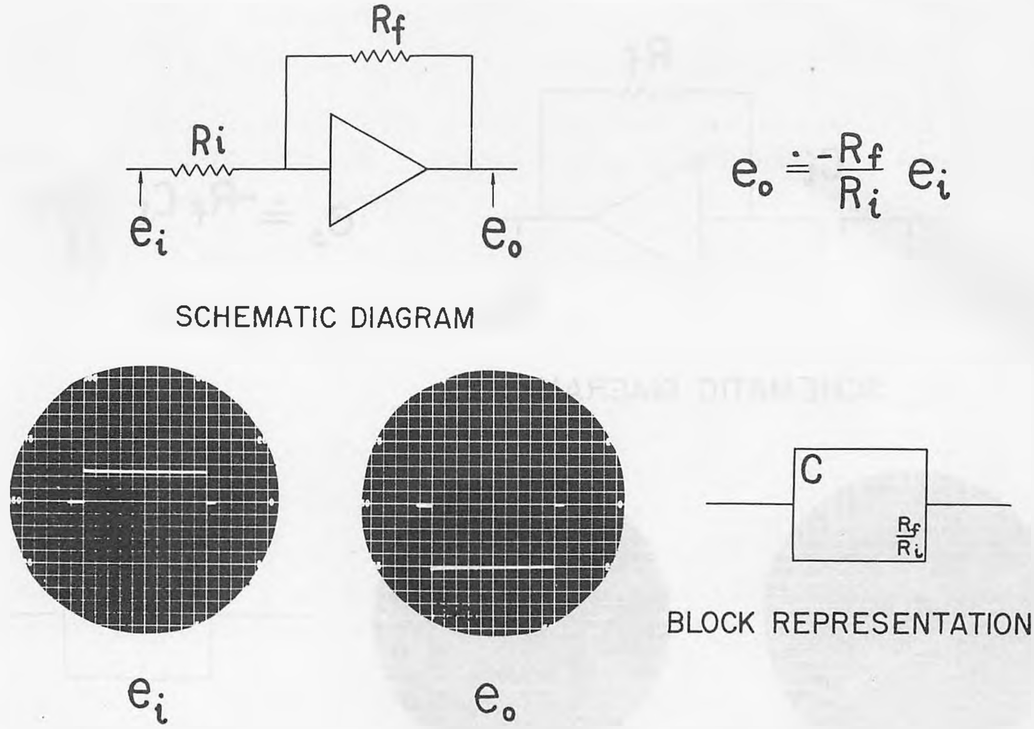


FIG. 8

SCALE CHANGER

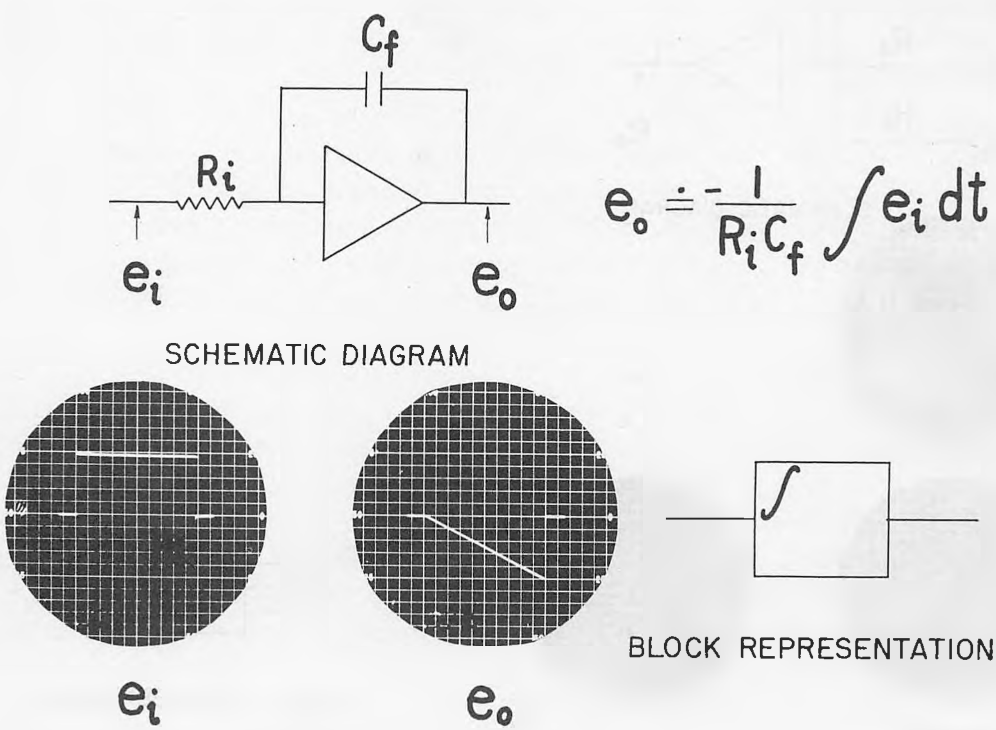
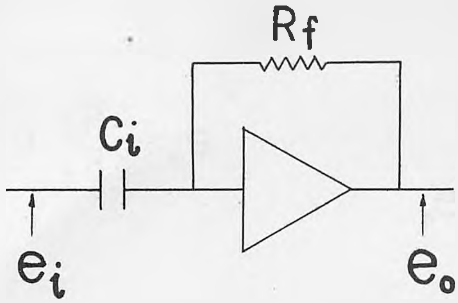


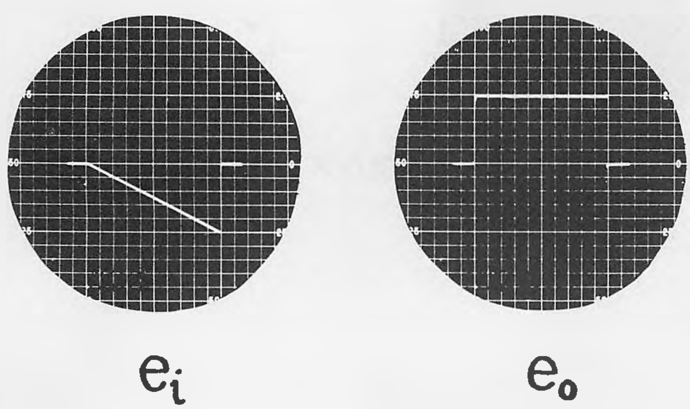
FIG. 9

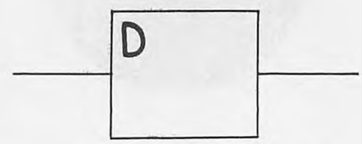
INTEGRATION



$$e_o \doteq -R_f C_i \frac{de_i}{dt}$$

SCHEMATIC DIAGRAM

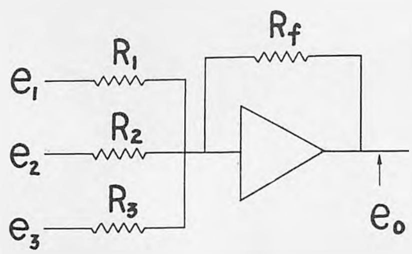




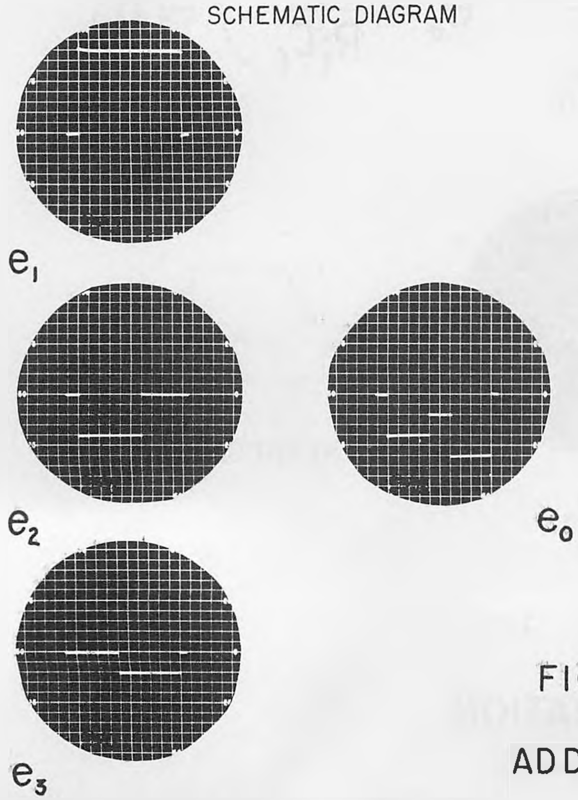
BLOCK REPRESENTATION

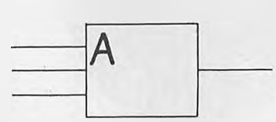
FIG. 10

DIFFERENTIATION



SCHEMATIC DIAGRAM

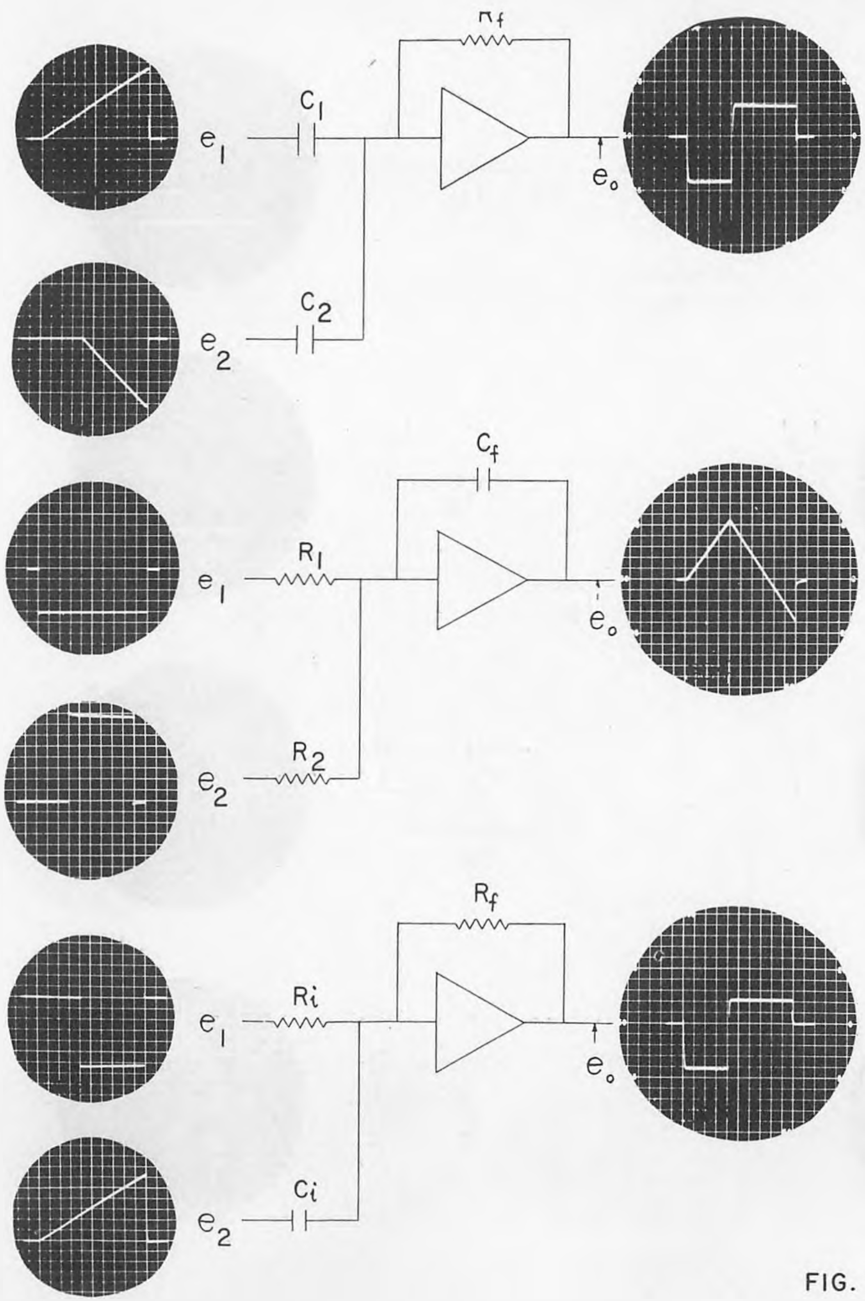




BLOCK REPRESENTATION

FIG. 11

ADDITION



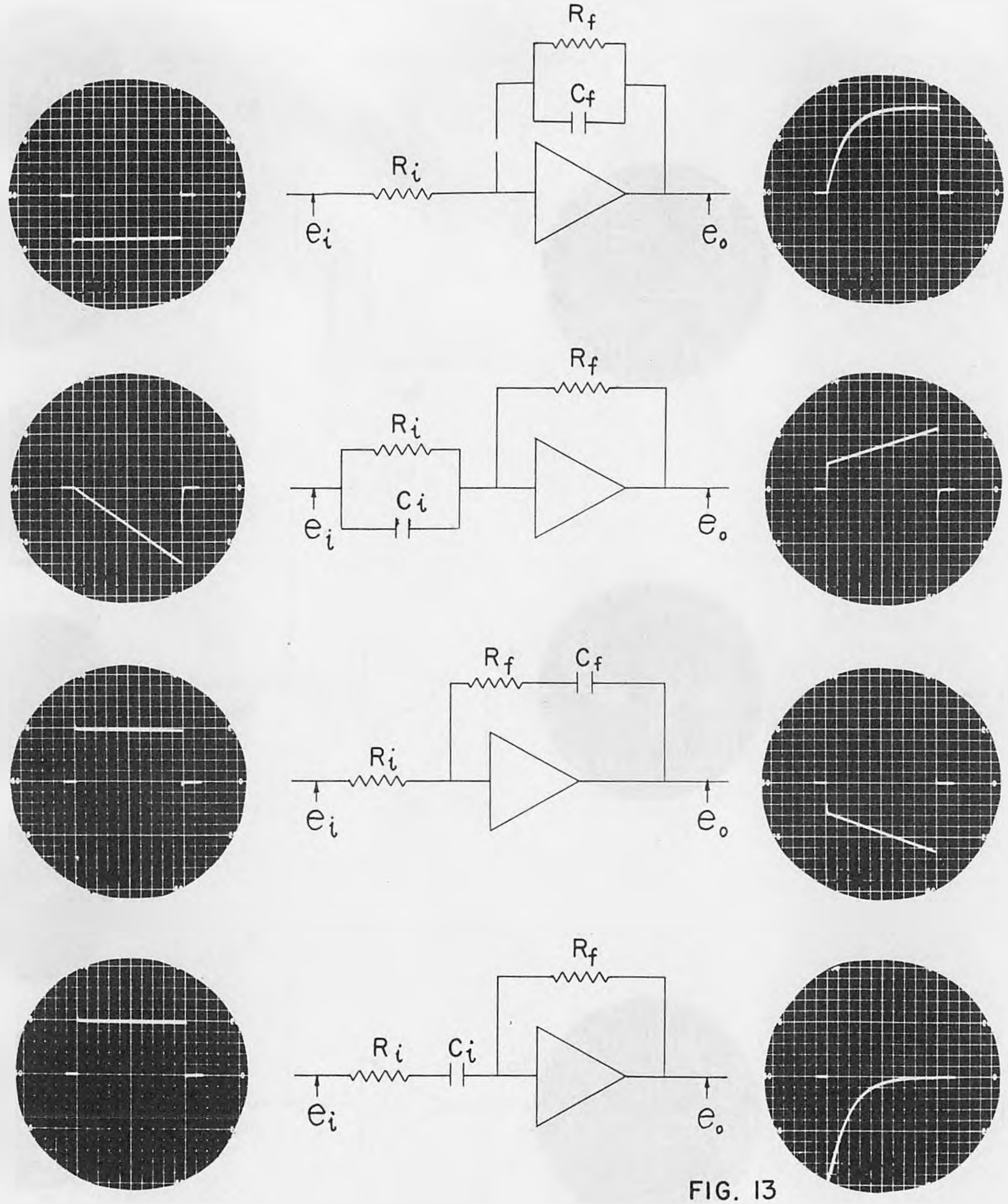
$$e_o \doteq -\left(R_f C_1 \frac{de_1}{dt} + R_f C_2 \frac{de_2}{dt}\right)$$

$$e_o \doteq -\left(\frac{1}{R_1 C_f}\int e_1\,dt + \frac{1}{R_2 C_f}\int e_2\,dt\right)$$

$$e_o \doteq -\left(\frac{R_f}{R_i} e_1 + R_f C_i \frac{de_2}{dt}\right)$$

FIG. 12

COMBINATIONS OF BASIC OPERATIONS



$$e_o \doteq -\frac{R_f}{R_i}\,\frac{1}{1+pR_fC_f}\,e_i$$

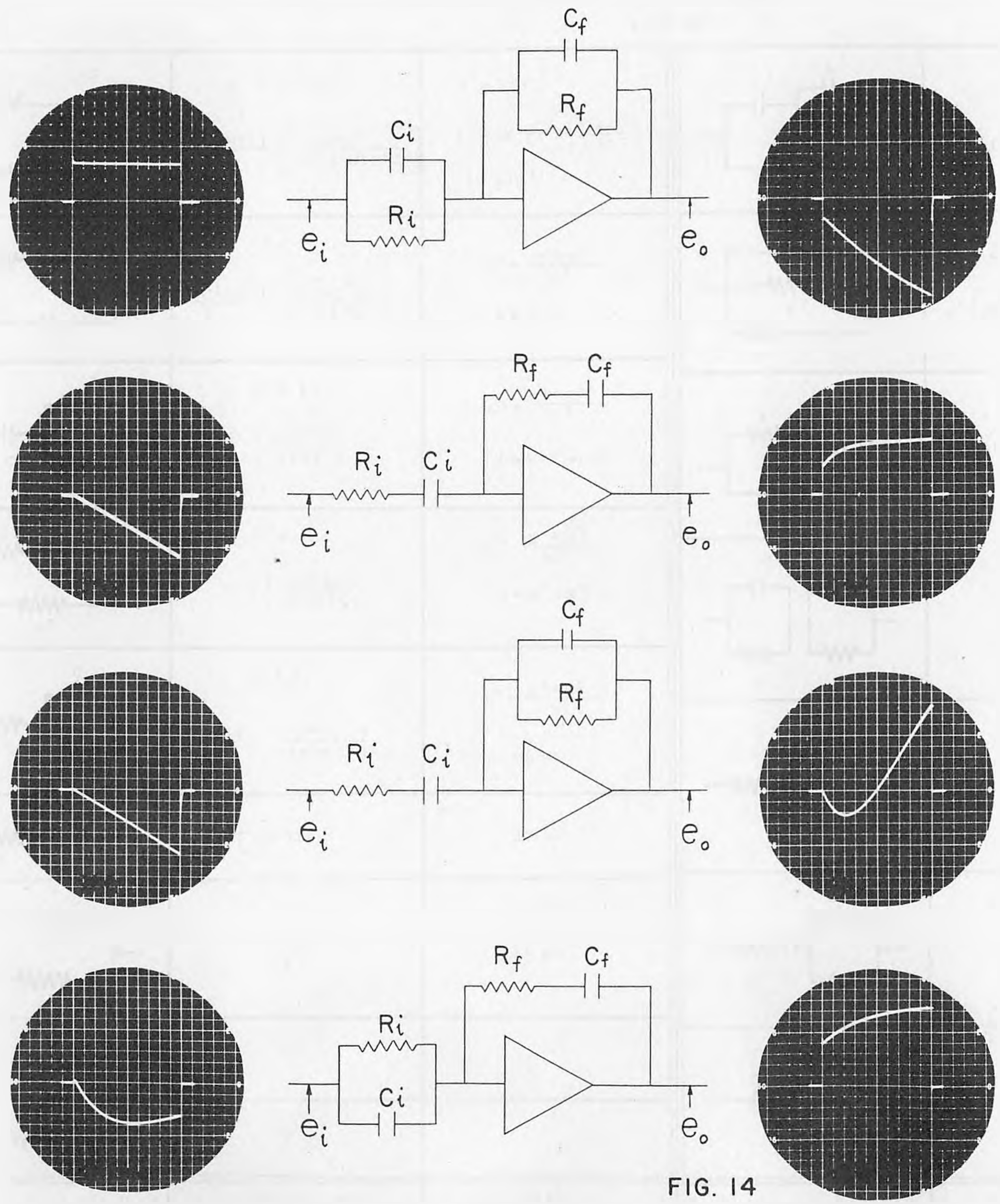
$$e_o \doteq -\frac{R_f}{R_i}(1+pR_iC_i)\,e_i$$

$$e_o \doteq -\frac{1+pR_fC_f}{pR_i\,C_f}\,e_i$$

$$e_o \doteq -\frac{pR_fC_i}{1+pR_iC_i}\,e_i$$

FIG. 13

OPERATIONS INVOLVING COMPLEX IMPEDANCES



$$e_o \doteq -\frac{R_f}{R_i}\,\frac{1+pR_iC_i}{1+pR_fC_f}\,e_i$$

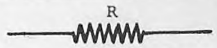
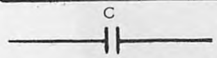
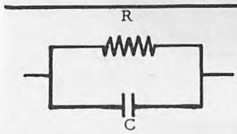
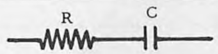
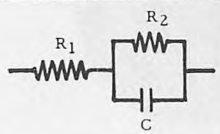
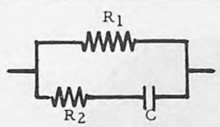
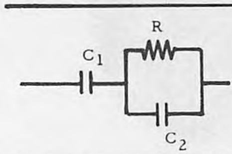
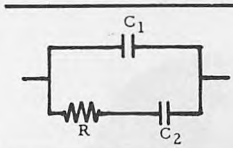
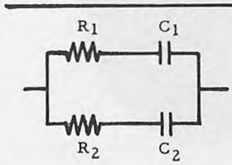
$$e_o \doteq -\frac{C_i}{C_f}\,\frac{1+pR_fC_f}{1+pR_iC_i}\,e_i$$

$$e_o \doteq -\frac{pR_fC_i}{(1+pR_iC_i)(1+pR_fC_f)}\,e_i$$

$$e_o \doteq -\frac{1}{pR_iC_f}(1+pR_iC_i)(1+pR_fC_f)\,e_i$$

FIG. 14

OPERATIONS INVOLVING COMPLEX IMPEDANCES

NETWORK	TRANSFER IMPEDANCE	SPECIAL RELATIONS
	R	
	C	
	$\frac{R}{1+pT}$	$T = RC$
	$\frac{1}{pC}(1+pT)$	$T = RC$
	$(R_1+R_2)\ \frac{(1+pR_eT)}{(1+pT)}$ $R_e < 1$	$T = R_2C$ $R_e = \frac{R_1}{R_1+R_2}$
	$R_1\ \left(\frac{1+pR_eT}{1+pT}\right)$ $R_e < 1$	$T = (R_1+R_2)\ C$ $R_e\ \frac{R_2}{R_1+R_2}$
	$\frac{1}{pC_1}\ \left(\frac{1+pT}{1+pC_eT}\right)$ $C_e < 1$	$T = R(C_1+C_2)$ $C_e = \frac{C_2}{C_1+C_2}$
	$\frac{1}{p(C_1+C_2)}\ \left(\frac{1+pT}{1+pC_eT}\right)$ $C_e < 1$	$T = RC_2$ $C_e = \frac{C_1}{C_1+C_2}$
	$\frac{1}{p(C_1+C_2)}\left[\frac{(1+pT_1)(1+pT_3)}{1+pT_2}\right]$ $T_1 < T_2 < T_3$	$T_1 = R_1C_1$ $T_2 = (R_1+R_2)\frac{(C_1C_2)}{(C_1+C_2)}$ $T_3 = R_2\ C_2$

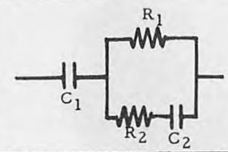
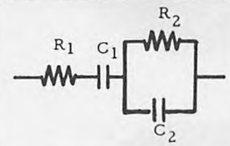
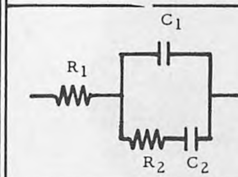
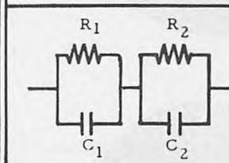
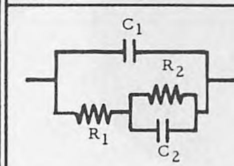
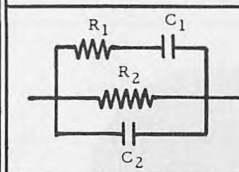
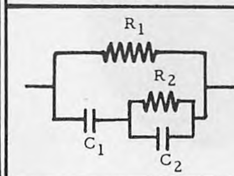
NETWORK	TRANSFER IMPEDANCE	SPECIAL RELATIONS
	$\frac{1}{pC_1}\left[\frac{(1+pT_1)(1+pT_3)}{1+pT_2}\right]$ $T_1 < T_2 < T_3$	$T_2 = (R_1+R_2)\ C_2$ $T_1T_3 = R_1R_2C_1C_2$ $T_1+T_3 = R_1C_1+R_2C_2+R_1C_2$
		$T_2 = R_2C_2$ $T_1T_3 = R_1R_2C_1C_2$ $T_1+T_3 = R_1C_1+R_2C_2+R_2C_1$
	$\frac{1}{p(C_1+C_2)}\left[\frac{(1+pT_1)(1+pT_3)}{1+pT_2}\right]$ $T_1 < T_2 < T_3$	$T_2 = R_2\ \frac{(C_1C_2)}{(C_1+C_2)}$ $T_1T_3 = R_1R_2C_1C_2$ $T_1+T_3 = R_1C_1+R_2C_2+R_1C_2$
	$(R_1+R_2)\left[\frac{1+pT_2}{(1+pT_1)(1+pT_3)}\right]$ $T_1 < T_2 < T_3$	$T_1 = R_1C_1$ $T_2 = \frac{(R_1R_2)}{(R_1+R_2)}(C_1+C_2)$ $T_3 = R_2C_2$
		$T_2 = \frac{(R_1R_2)}{(R_1+R_2)}\ C_2$ $T_1T_3 = R_1R_2C_1C_2$ $T_1+T_3 = R_1C_1+R_2C_2+R_2C_1$
	$R_2\left[\frac{1+pT_2}{(1+pT_1)(1+pT_3)}\right]$ $T_1 < T_2 < T_3$	$T_2 = R_1C_1$ $T_1T_3 = R_1R_2C_1C_2$ $T_1+T_3 = R_1C_1+R_2C_2+R_2C_1$
	$R_1\left[\frac{1+pT_2}{(1+pT_1)(1+pT_3)}\right]$ $T_1 < T_2 < T_3$	$T_2 = R_2(C_1+C_2)$ $T_1T_3 = R_1R_2C_1C_2$ $T_1+T_3 = R_1C_1+R_2C_2+R_2C_1$

FIG. 15

PASSIVE NETWORKS

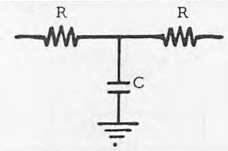
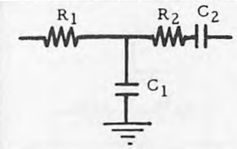
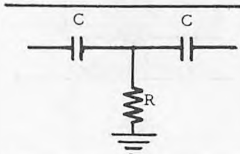
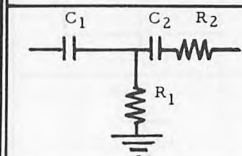
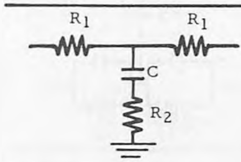
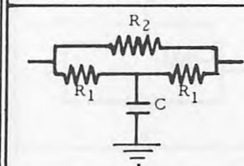
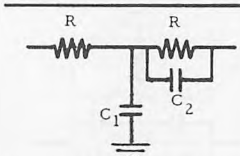
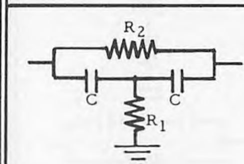
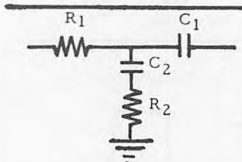
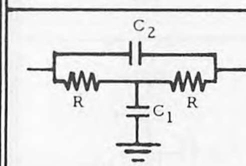
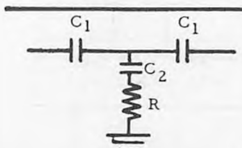
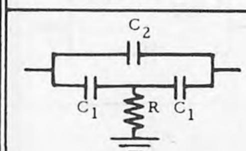
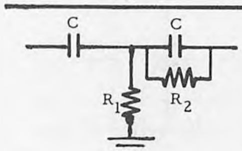
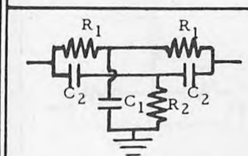
NETWORK	TRANSFER IMPEDANCE	SPECIAL RELATIONS	NETWORK	TRANSFER IMPEDANCE	SPECIAL RELATIONS
	$2R\,(1+pT)$	$T=\frac{RC}{2}$		$\frac{1}{pC_2}\,(1+pT_1)(1+pT_2)$ $T_1 \neq T_2$	$T_1T_2=R_1R_2C_1C_2$ $T_1+T_2=R_1C_1+R_2C_2+R_1C_2$
	$\frac{2}{pC}\,\frac{(1+pT)}{(pT)}$	$T=2RC$		$\frac{1}{pC_2}\,\frac{(1+pT_1)(1+pT_2)}{(p\sqrt{T_1T_2})}$ $T_1 \neq T_2$	$T_1T_2=R_1R_2C_1C_2$ $T_1+T_2=R_1C_1+R_2C_2+R_1C_2$
	$2R_1\,\frac{(1+pT)}{(1+pR_eT)}$ $R_e<1$	$T=(R_2+\frac{R_1}{2})\,C$ $R_e=\frac{2R_2}{2R_2+R_1}$		$\frac{2R_1R_2}{2R_1+R_2}\,\frac{(1+pT)}{(1+pR_eT)}$ $R_e<1$	$T=\frac{R_1C}{2}$ $R_e=\frac{2R_1}{2R_1+R_2}$
	$2R\,\frac{(1+pT)}{(1+pC_eT)}$ $C_e<1$	$T=\frac{R}{2}\,(C_1+C_2)$ $C_e=\frac{2C_2}{C_1+C_2}$		$R_2\,\frac{(1+pT_1)}{(1+pT_1+p^2T_1T_2)}$	$T_1=2R_1C$ $T_2=\frac{R_2C}{2}$
	$\frac{1}{pC_1}\left[\frac{(1+pT_1)(1+pT_3)}{1+pT_2}\right]$ $T_1<T_2<T_3$	$T_2=R_2C_2$ $T_1\,T_3=R_1R_2C_1C_2$ $T_1+T_3=R_1C_1+R_2C_2+R_2C_1$		$2R\,\frac{(1+pT_2)}{(1+pT_1+p^2T_1T_2)}$	$T_1=2RC_2$ $T_2=\frac{RC_1}{2}$
	$\frac{(2C_1+C_2)}{(pC_1^2)}\,\frac{(1+pC_eT)}{(1+pT)}$ $C_e<1$	$T=RC_2$ $C_e=\frac{2C_1}{2C_1+C_2}$		$\frac{1}{pC_2}\,\frac{(1+pC_eT)}{(1+pT)}$ $C_e<1$	$T=RC_1\,\frac{(2C_2+C_1)}{C_2}$ $C_e=\frac{2C_2}{2C_2+C_1}$
	$\frac{(R_1+R_2)}{(pR_1C)}\,\frac{(1+pR_eT)}{(1+pT)}$ $R_e<1$	$T=R_2C$ $R_e=\frac{2R_1}{R_1+R_2}$		$2R_1\,\frac{(1+pT_1)}{(1+p^2T_1T_2)}$	$T_1=\frac{R_1C_1}{2}$ $T_2=R_1C_2$ $R_1C_1=4R_2C_2$

FIG. 16

PASSIVE NETWORKS

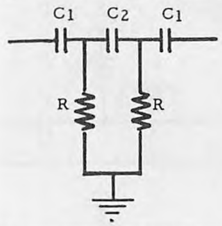
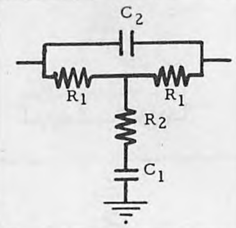
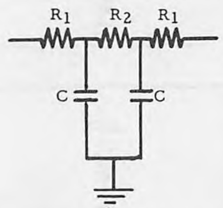
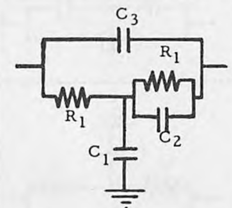
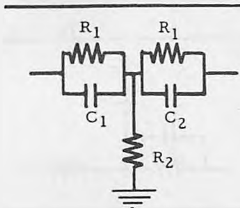
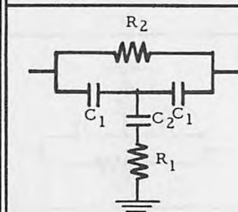
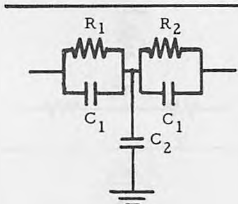
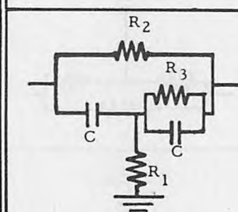
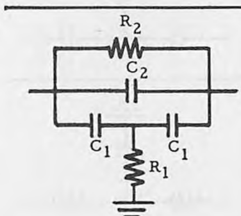
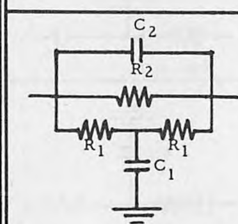
NETWORK	TRANSFER IMPEDANCE	SPECIAL RELATIONS	NETWORK	TRANSFER IMPEDANCE	SPECIAL RELATIONS
	$\frac{C_1+2C_2}{pC_1C_2}\left[\frac{(1+pT_1)(1+pT_2)}{p^2T_1T_2}\right]$ $T_1 < T_2$	$T_1 = RC_1$ $T_2 = R(C_1+2C_2)$		$2R_1\left[\frac{1+pT_3}{1+pT_1+p^2T_1T_2}\right]$ $T_2 > \frac{T_1}{4}$ (Complex) (Roots) $T_3 > T_2$	$T_1 = R_2C_1+2R_1C_2$ $T_2 = \frac{R_1(R_1+2R_2)C_1C_2}{R_2C_1+2R_1C_2}$ $T_3 = (R_2+\frac{R_1}{2})C_1$
	$(2R_1+R_2)(1+pT_1)(1+pT_2)$ $T_1 < T_2$	$T_1 = \left(\frac{R_1R_2}{2R_1+R_2}\right)C$ $T_2 = R_1C$			$T_1 = R_1(C_2+2C_3)$ $T_2 = \frac{R_1C_3(C_1+C_2)}{C_2+2C_3}$ $T_3 = \frac{R_1}{2}(C_1+C_2)$
	$(2R_1+\frac{R_1^2}{2})\left[\frac{1+pT_2}{(1+pT_1)(1+pT_3)}\right]$ $T_2 \leq T_1 \leq T_3$	$T_1 = R_1C_1$ $T_2 = \left(\frac{R_1R_2}{R_1+2R_2}\right)(C_1+C_2)$ $T_3 = R_1C_2$		$R_2\left[\frac{1+pT_3}{1+pT_1+p^2T_1T_2}\right]$ $T_2 > \frac{T_1}{4}$ (Complex) (Roots) $T_3 < T_1$	$T_1 = \frac{C_1(2R_1C_2+R_2C_1)}{2C_1+C_2}$ $T_2 = \frac{R_1R_2C_1C_2}{2R_1C_2+R_2C_1}$ $T_3 = \frac{2R_1C_1C_2}{(2C_1+C_2)}$
	$(R_1+R_2)\left[\frac{1+pT_2}{(1+pT_1)(1+pT_3)}\right]$ $T_1 \leq T_3 \leq T_2$	$T_1 = R_1C_1$ $T_2 = \frac{R_1R_2}{R_1+R_2}(2C_1+C_2)$ $T_3 = R_2C_1$			$T_1 = \frac{R_1(R_2+2R_3)C}{R_1+R_3}$ $T_2 = \frac{R_2R_3C}{(R_2+2R_3)}$ $T_3 = \frac{2R_1R_3C}{(R_1+R_3)}$
	$R_2\left[\frac{1+pT_3}{1+pT_1+p^2T_1T_2}\right]$ $T_2 > \frac{T_1}{4}$ (Complex) (Roots) $T_3 < T_1$	$T_1 = 2R_1C_1+R_2C_2$ $T_2 = \frac{R_1R_2C_1(C_1+2C_2)}{2R_1C_1+R_2C_2}$ $T_3 = 2R_1C_1$		$\frac{(2R_1+R_2)}{2R_1R_2}\left[\frac{1+pT_3}{1+pT_1+p^2T_1T_2}\right]$ $T_2 > \frac{T_1}{4}$ (Complex) (Roots) $T_3 > T_2$	$T_1 = \frac{R_1(R_1C_1+2R_2C_2)}{2R_1+R_2}$ $T_2 = \frac{R_1R_2C_1C_2}{R_1C_1+2R_2C_2}$ $T_3 = \frac{R_1C_1}{2}$

FIG. 17
PASSIVE NETWORKS

The High-speed Analog as Applied in Industry

George A. Philbrick

(Presented et the Spring Meeting of the ASME,New London,Conn.,May 1949)

An *analog*, even when classed as a computor, is basically a model, and is nothing new. The techniques and apparatus involved are familiar to engineers, who have for many years found analogy to be a powerful developmental tool. Recent work has streamlined this tool, employing new techniques to make it flexible and convenient, accurate and fast.

Consider a typical industrial system: a processing unit under controlled operation. To the control designer or instrument engineer, such a system is chiefly a group of physical variables affecting each other in a definite pattern. Whether this unit already exists, or is planned on paper, its dynamic nature may be completely specified in a block diagram showing all influences among its variables. The arrangement and the individual character of the directional casual paths in such a diagram (whereby influences are transferred and propagated) determine the behavior of the system.

The "blocks" of the engineer's diagram correspond to components of the unit being studied: the correspondence being dynamic rather than simply in terms of apparatus. By proper choice, only several different types of block need be used. Thus three fundamental types suffice to represent any system which is *linear*: that is, any system in which effect is proportional to cause. These essential linear operations, denoted by appropriate blocks, are *addition*, *integration*, and *simple proportion*. For nonlinear systems other types of operations and blocks are naturally required, and these include the phenomena of *limiting*, *hysteresis* or backlash, *dead-zone*, etc.

In this, as in other businesses, there are many effective shortcuts, but there is also a sure and straightforward road from the block diagram to a working analog or model. The engineer needs only to assemble a set of standard computor components in strict accordance with his block diagram. The result is a model of the physical system under consideration, but in a far more convenient form.

The responses to chosen initial conditions applied to the model may readily be observed on the oscilloscope screen (as in a radar or televised presentation)*, since a time scale may be chosen such that these conditions may be periodically reapplied more quickly than the persistence of vision. Any parameter of the system may be set in on the calibrated controls of the components, and experimentally altered over a wide range. In short, the analog computor thus assembled presents quantitative dynamic results as fast as engineering data can be supplied to it.

If our engineer is also a mathematician, he may think in terms of differential equations instead of block diagrams, but fundamentally he is only using another symbolism, and is no better off without solutions. If he were also an all-powerful and superhuman calculator, he could solve these equations instantly in his head. This the analog, with Nature's cooperation, does for him.

Perhaps the analog technique is closer to synthesis than to analysis, but it is well adapted both to study and to design. An adequately rapid and flexible tool of synthesis can make unnecessary many purely analytic procedures. It should be added that this technique does not conflict with those of harmonic analysis, but may advantageously be used in conjunction with such methods. In general, however, the analog is not limited to linear systems, as are the methods of frequency analysis. Furthermore, working in the real-time domain, and representing physical variables directly in measurable form, the analog has the closeness-to-nature which is the true character and strength of a model.

A fruitful class of industrial applications is in the technology of what are called *regulatory systems*, where control apparatus is designed and applied to processing equipment and plants. Included are the temperature, pressure, flow, density, and humidity control problems of the process industries, as well as those of speed controls (governors), positional controls (servomechanisms), directional controls (autopilots), stabilizers (shock absorbers and vibration isolators), and hundreds of chemical and electrical control structures.

When several control systems are interconnected, and must cooperate without untoward interaction, the problems of design and application are fantastically increased. In each such case, unstable or sluggish performance implies overall operation which is far from optimum.

By the assembly of analogs or models of such industrial systems, from standard computor components, and by applying the analog as an engineering tool to attack their underlying problems, the way to high stability and high performance may readily be made clear. Proposed alterations in the system under study are represented by rearrangement of the computing assemblage. The addition and interconnection of further analog components extends the model to treat more complex structures as desired. When disturbances are applied at chosen points, the quality of the recovery is immediately made evident to the experimenter, who then determines the optimum parameters for measurement and control by what we may call "lightning empiricism".

The informed control engineer does not evaluate the success of an installation merely by its performance when all adjustments are precisely at optimum, once these settings have been discovered. He also considers its degree of tolerance to those adjustments and to the other dynamic characteristics of the system as a whole. The permanency required of a critical parameter is a source of expense and worry, and if overlooked may lead to a condition which is seriously unsafe. With the analog representation, the *critically* of any characteristic, say a control parameter, may be evaluated easily and immediately by a simple operation. Manual variation of the corresponding adjustment, to fractionally higher and lower values, determines its critical tolerance in terms of the resulting deterioration in transient performance of the important variables under observation. Thus temperamental designs may be rejected in advance.

As to the future, predictions are naturally risky in a field where development has been so accelerated. The current growth of the high-speed analog method, in technique and in application, is gratifying to those of us who have had faith in it. It seems certain that in much less than another decade its potentialities as a tool for development and study will be very generally *known to engineers* and research men. Being fundamental, this tool may be expected to have many new and unpredictable applications, limited only by the imagination and enterprise of those who are and will be engaged in developmental projects.

DESIGN of stable turboprop control systems is greatly speeded up by the WAC electronic analog. This computing device simulates the physical relationship between the five prime variables involved: speed, torque, temperature, fuel flow, and propeller blade angle.

Although as many as 25 design characteristics may be involved in a control system, the analog can determine the optimum values for them in the course of one day.

The analog computing technique might be equally valuable in a number of other uses.

ELECTRONIC ANALOG STUDIES

THE WAC electronic analog is a calculating device which is used to investigate the design of turboprop control systems. When set from calculated or empirical data, the analog presents an oscilloscopic time plot showing the behavior of the important variables during and after applied transient disturbances. To appreciate the usefulness of analog studies some understanding of the turboprop control problem is necessary.

General Problem

In a turboprop engine (Fig. 1) both speed of rotation and torque developed must be controlled since the combination of these variables determines power and efficiency. It is also necessary to restrict the maximum speed, torque, and temperature; for, if any one of these limits is exceeded, damage to the engine will result. Furthermore, in general, maximum power and efficiency are obtained at maximum turbine speed and temperature. One problem, therefore, is to operate in this restricted region without exceeding safe design limits. These three variables – speed, torque, and temperature – can be controlled only by manipulation of fuel flow and propeller blade angle. Thus, the five prime variables involved in the performance of turboprop control systems are speed, torque, temperature, fuel flow, and propeller blade angle. These variables are so interrelated during control that each affects the others; and in addition the problem is further complicated by the inertia and torque characteristics of the turboprop engine.

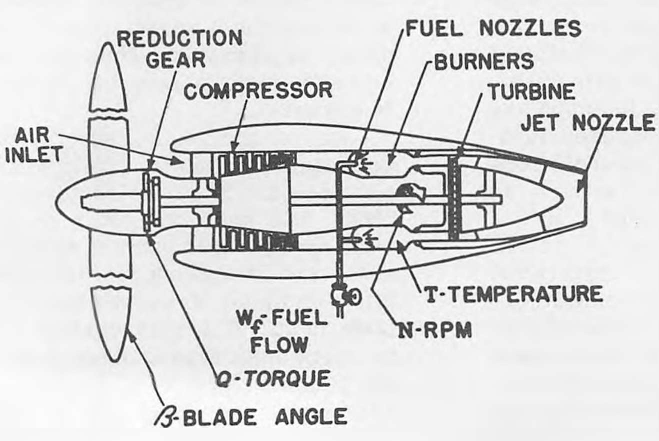


Fig. 1 – Schematic diagram of typical turboprop engine showing component arrangement and principal variables involved for control

Fig. 2 graphically illustrates related influences of the five variables and the possible control loops. The illustration has been purposely arranged to indicate that a change in fuel flow will produce a change in speed, torque, and temperature. Similarly, a change in propeller blade angle also causes a change in speed, torque, and temperature. The complexity of these relationships may be better understood by considering the problem of sighting a gun. In the latter case the inputs corresponding to fuel flow and propeller blade angle are the horizontal and vertical traversing mechanisms. The gun can be positioned accurately by independent adjustment of each input. However, the turboprop control problem is vastly more complicated. In the gun analogy, the horizontal traversing mechanism has no effect on vertical position, but extreme difficulty might be encountered in positioning the gun if the horizontal traverse affected both the vertical and horizontal position. In the turboprop, fuel flow affects three variables that are also affected simultaneously by propeller blade angle. Typical relationships among these variables are illustrated by Fig. 3. These relationships change with external

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FOR TURBOPROP CONTROL SYSTEMS

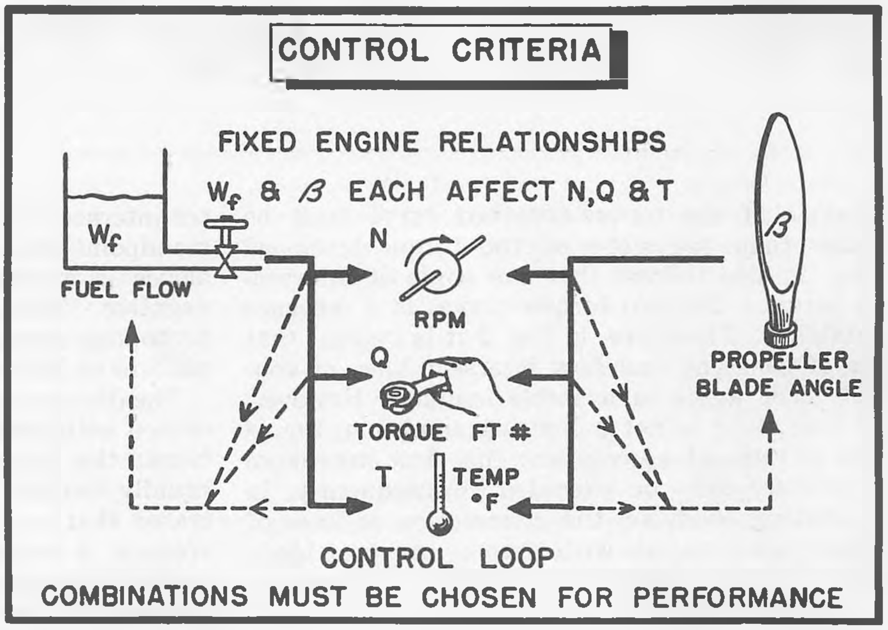


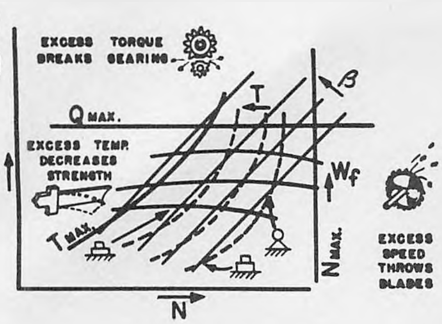
Fig. 2 – Representation of related influences of the five variables involved in control of turboprop engines

effects such as forward speed, entrance air temperature, and ambient pressure.

Fig. 3 also defines the maximum permissible value of the three limiting variables – torque, speed, and temperature – with a pictorial representation of the reason for limitation. Excess torque can damage the reduction gearing, since under certain conditions the engine is capable of developing torques greater than the design value conforming to the necessary weight limitations. Excess speed can damage rotating parts because of the high centrifugal forces. Excess temperature can damage the engine because of the resultant decreased strength of materials.

The stability symbols on Fig. 3 are explained by the simplified charts of Fig. 4. Any change in speed is accomplished by the difference in the torque which the propeller absorbs at a given blade angle and the torque developed by the engine. This results from the relationship that accelerating torque is equal to the inertia times the angular acceleration. Thus, in the left-hand illustration, an engine develops constant torque for any speed; and power is absorbed in a manner such that the torque is proportional to speed. Under these conditions the engine will accelerate to the speed at which the two torques are equal. This is stable equilibrium; any departures from this speed, caused by transient disturbances, are corrected since the torque difference changes speed toward equilibrium. However, if conditions *are reversed* as in the right-hand illustration, instability results since the torque difference would then act to change speed away from equilibrium. Thus, for stability

3



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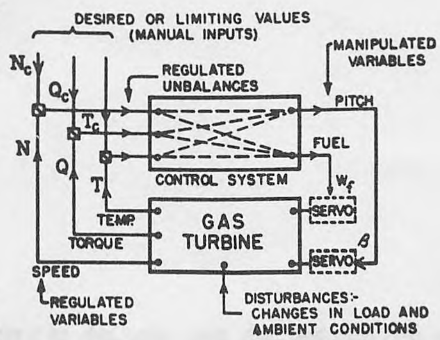

Fig. 3 – Typical relationships of the five variables involved in control of turboprop engines showing limiting conditions

Fig. 4 – Stability criteria. For stability, slope of torque-absorbed curve must be greater than slope of torque-developed curve

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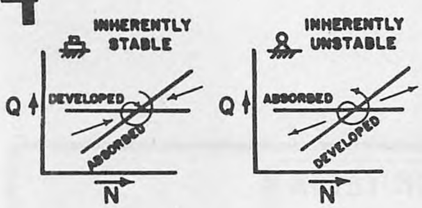

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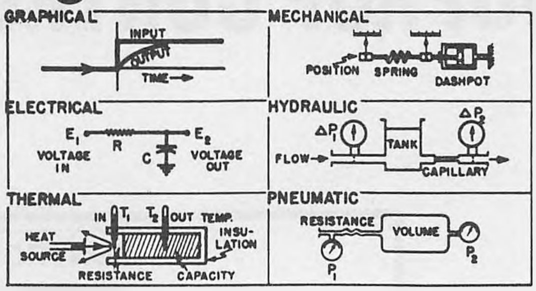

Fig. 5 – General block diagram of turboprop control system

Fig. 6 – Representation of a lagging response in various media

the slope of the torque-absorbed curve must be greater than the slope of the torque-developed curve. It also follows that the angle of intersection between the two torque curves is a measure of stability. Therefore, in Fig. 3 it is evident that lines of constant fuel flow intersect lines of constant blade angle in a stable manner. However, fuel flow alone is not a limiting condition, for as speed is reduced at constant fuel flow maximum temperature may be exceeded. Consequently, in the limiting condition the intersection of lines of constant temperature with lines of constant blade angle determine the inherent stability. These latter intersections are quite poor from the stability standpoint since in most cases the difference in slopes is extremely small and in some cases is negative. Thus, for satisfactory performance, turboprop controls must be superior to those applied to an inherently stable system.

The discussion thus far has been principally concerned with what may be termed static considerations; the dynamics are equally important and equally complex. The previous curves have illustrated that as blade angle is increased, speed decreases. A complicating factor is that the reaction angle is changed at a fixed rate, the resultant speed change lags by a definite time interval known as the characteristic time. The other relationships are similarly complicated by response delays. An important aspect of the problem is to adapt controls to this system so as to give prompt and stable performance over the whole range of operation, and to do so without violating the conditions for safety and efficiency.

Methods of Attack

Solutions for the design of turboprop control systems can be determined by analytical methods, empirical tests, and analogy. The superiority of the latter method will become apparent after consideration of the general block diagram for turboprop controls, Fig. 5. The relationships between control components and engine are complicated by the various responses and modes of control used in each phase. Consideration must be given to individual proportional, integral, and derivative gains and timings. An analytical approach leads to a set

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of seventh-order differential equations such as these:

$$C_7 \frac{d^7\Delta N}{dt^7} + C_6 \frac{d^6\Delta N}{dt^6} + \ldots + C_1 \frac{d\Delta N}{dt} + \Delta N = 0$$

$$C_7 \frac{d^7\Delta T}{dt^7} + C_6 \frac{d^6\Delta T}{dt^6} + \ldots + C_1 \frac{d\Delta T}{dt} + \Delta T = 0$$

$$C_7 \frac{d^7\Delta Q}{dt^7} + C_6 \frac{d^6\Delta Q}{dt^6} + \ldots + C_1 \frac{d\Delta Q}{dt} + \Delta Q = 0$$

Where:

Δ = Difference between instantaneous and equilibrium values
N = Speed
T = Temperature
Q = Torque
t = Time
$C_1, C_2, \ldots C_7$ are evaluated from constants of physical system

Although solutions of these equations are possible, the work is too tedious and time consuming for practical application in investigating the necessarily numerous operating conditions. Consequently, analytical methods are relegated to the important functions of checking other methods of solution and of imparting a better understanding of the control problems encountered. Empirical methods are decidedly necessary and establish final approval of a given design. However, the limitations of test equipment, the cost of operating experimental units, and the time required to investigate the effects of interaction between control components, all prevent the full exploration of the variables which is necessary to establish optimum design. In the electronic analog such solutions are obtained almost instantaneously. Days of calculation and days of test are reduced to minutes.

Analogy is one of the oldest and most powerful problem-solving agencies known to man. The process of reasoning by analogy provides concrete basis and guidance for design; and the model or analog which is employed need not involve the

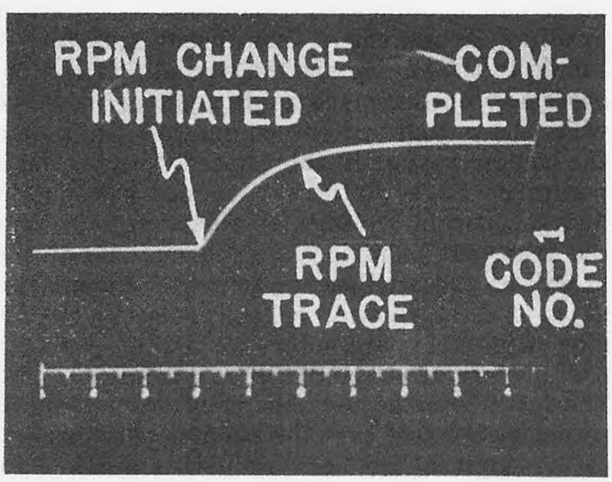


Fig. 7 – Typical oscilloscopic trace of a variable against synchronous time base

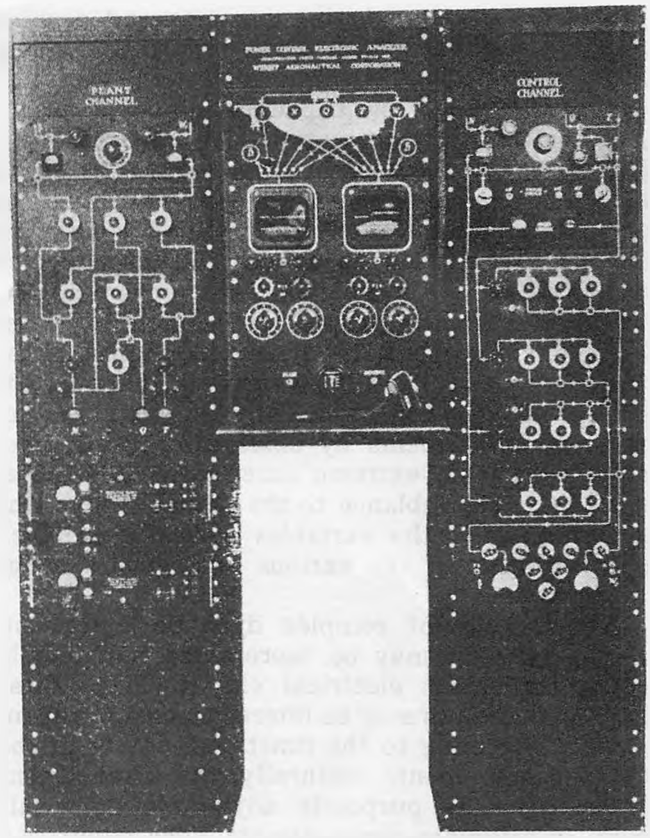


Fig. 8 – WAC electronic analog for investigating design of turboprop control systems

same physical media as the primary system under consideration. Just as the parables of old taught ways of life in terms of everyday occurrences, so can involved physical concepts be simulated by models in simpler media. Thus, by attacking the unknown in terms of the known, qualitative and quantitative solutions are facilitated. In particular, the electronic analog for turboprop control systems makes available several important advances in precision of results, conservation of time and effort, convenience, and flexibility.

The interrelationships of many physical concepts are comparatively well known. For instance, electrical principles are often taught by hydraulic analogy in which voltage is represented by pressure. In much the same manner, useful comparisons are made between inertia and inductance; friction and resistance; and so on. Fig. 6 is presented to illustrate further the facility with which a physical concept can be simulated in various media. The chosen concept is that of a first-order lagging response of a variable responding to a step-function input. Each of the six representations – graphical, electrical, thermal, mechanical, hydraulic, and pneumatic – have the same dynamic response. Any one of these systems may be employed as a true model for any other. Thus, for the convenience of mechanical engineers in understanding the lagging response of the thermocouple employed in a particular control system, the mechanical analog employing a spring and dashpot might be useful. Whereas the electrical engineer might better appreciate the electrical analog consisting of resistor and capacitor.

If the concept that quantitative functional performance can be duplicated in various media is accepted, then the means of accomplishing the simulation will not affect the result. The designer need only concern himself with the output which results from given inputs. Thus, reversing the previous example, if the mechanical engineer were convinced that the dynamic response of the thermocouple faithfully simulates that of the spring system, then he might even simplify complex spring problems by observation of thermocouples. In this extreme case, where the model bears little resemblance to the actual system, the performance of the variables could be as much alike as those of the various analogous systems of Fig. 6.

In the study of complex dynamic equipment, each component may be represented individually by an equivalent electrical circuit. Then, these individual circuits may be interconnected in a manner corresponding to the functional nature of the designed equipment. Naturally, the arrangement should be made purposely adjustable to permit study of alternate arrangements. The result is a physical embodiment of the relevant dynamics, by means of which the whole range of stability and performance possibilities may be explored at will with facility and promptness unobtainable by other means.

Principle of Analog Computer

The advantages of electronic models operating at relatively high speed are readily apparent, provided the transient solutions are furnished in a usable form. In this particular analog, the problem of display is solved by repeating the imposed conditions cyclically 240 times per sec and presenting the resulting variations as an oscilloscopic plot against a synchronous time base as shown in Fig. 7. Thus, the solution is seen as a stationary plot of the transient variation and may be photographed. This technique may be compared to the use of indicator diagrams as an aid in engine study.

The analog computer under discussion is composed entirely of electronic components which are widely adjustable to simulate any particular design of turboprop engine and control system. When applied to automatic control problems, this analog acquires several distinct features. Basically, the computing portions of the instrument are arranged in two channels as shown in Fig. 9, one corresponding to the turboprop engine and the other to the control system. The two channels are reciprocally interconnected to affect one another in exact simulation of the intereffects of control and engine. Each electronic component is provided with calibrated adjustments corresponding to the individual physical constants of the system under study. Means are also incorporated for checking the dynamic relationships which are established by these adjustments. Flexibility is provided not only as to individual *regulating components* but with regard to the arrangement of these components into all conceivable control systems. Since steady-state conditions are only part of the story, means are provided for periodic disturbances to either inputs or outputs of the system. Then, as previously explained, the transient behavior of any variable under control is displayed oscillographically. Thus the influence of a change in any physical constant affecting the system can be instantaneously observed by variation of the adjustment representing that physical constant. In this manner, optimum values are quickly determined. The importance of such evaluation by direct manipulation and observation is self-evident.

There are two prerequisites which aid greatly in any development by analogy. In spite of all its accomplishments, the analog cannot think. Therefore, the first consideration that must be supplied to the analog is information regarding the static and dynamic properties of the turboprop engine. The accuracy of the results is entirely dependent upon the accuracy of the data furnished. Secondly, a definite control plan must be established. The analog will determine values for control design, but adequate data on the available physical components must be furnished. In investigating optimum design, various control plans should be investigated. Therefore, in the Wright Aeronautical analog provisions are made for simulating any feasible control plan by simple switching operations.

In the turboprop control problem there are three regulated variables and two manipulated variables as previously explained. A symbolic representation of the basic parts of the analog is given in Fig. 9. In spite of the multiple nature of the regulated and manipulated variables, an orderly and exacting arrangement has been achieved. The inputs to the engine are the outputs of the control system, propeller blade angle, and fuel flow. Likewise, the outputs of the engine are the inputs of the control system, turbine speed, torque, and temperature. Each of these variables is represented by a measurable voltage, to a fixed scale ratio, and the transient behavior of each may be displayed at will on the oscilloscope screens. The characteristics of the turboprop engine are set by means of nine adjustable parameters consisting of six static partial derivatives and three dynamic properties. The latitude of adjustment thus provided is illustrated by the fact that the performance of reciprocating engines has been simulated by the analog channel intended for turboprop engines. Naturally, such latitude allows many valuable studies of the effect of physical variations on performance under control.

In the control channel, 16 adjustable parameters determine the dynamic character of the possible individual regulatory components. Means are also provided for study of various propeller and fuel control mechanisms. The contribution of any individual component to the overall performance may be determined immediately by manual switching operations.

The repetitive disturbance, necessary to initiate recovery transients, is a square wave or step function which may be imposed at will at any of five separate positions in the system. The disturbances correspond, for example, to sudden shifts by an operator in the control settings of speed, temperature, and torque. In addition, the disturbance may be imposed as change in fuel flow or propeller pitch over and above the manipulation from the control system. Thus the performance and stability of a given system may be evaluated readily for all types of disturbances. Two oscilloscopes are provided so that the transient recovery of any pair of the system's variables may be observed simultaneously. A unique feature employed is that any variable may be plotted against any other; thus eliminating the dimension of time except as a parameter along the trace. The resulting figures, similar to the so-called Lissajou diagrams, are of interest in providing a dramatic visualization of the phase relationships existing throughout the system during transients.

From the foregoing discussion, it is apparent that stability and performance of proposed control systems can be investigated in detail. Not so apparent, however, is the facility for determining, by a few twists of a dial, how critical any parameter of the entire system may be. Thus, tolerances in the dynamic constants of the regulatory devices may be assigned with considerable assurance. This is important in the engineering of controls for maximum reliability and minimum size.

Physical Form of Analog

Fig. 8 shows the computer as installed at Wright Aeronautical Corp. It is entirely enclosed in a steel housing comprising an assembly of standard rack-type cabinets. The overall dimensions are 82 by 20 by 66 in., exclusive of the operating desk in front, and the weight is approximately 1000 lb. The left-hand cabinet contains the plant computing channel as well as the electronic disturbance generator, the ac voltage regulator, and the regulated dc power supplies. On the right is the control computing channel. The central cabinet houses the display channel, with the oscilloscopes, pushbutton selection switches, and their attendant circuits. All electronic components are available from the rear through latched doors, whereas every operating adjustment is brought out for convenience through the front panels.

The electronic construction of the computing channels is contained in a set of 10 semidetachable numbered chassis-panel components, with common power connections. All lengthy signal circuits are completed through coaxial cable. Exclusive of the display channel, 95 vacuum tubes are employed, each being interchangeable with another of its type substantially without effect owing to the thorough employment of relayed feedback techniques in all computing operations.

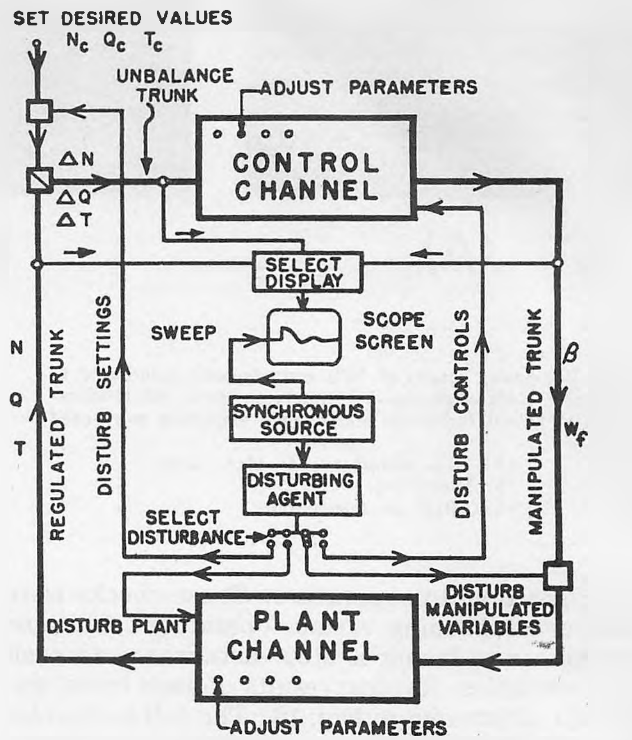


Fig. 9 – Simplified block diagram of WAC electronic analog

An automatic camera is installed between the two oscilloscope bezels on a mount which permits photography of either screen without disassembly, and a registering system is incorporated for convenience in recording.

Numerical calibration of the recorded transients is made possible by the fixed scale factors which are assigned between the real physical variables and their electronic counterparts. In terms of voltage, the relevant range of each of the significant variables is made to correspond directly to 100 v. For the time scale, the 0.004-sec fundamental period in the analog represents in this case 40 sec of real time. Thus in the model, time is accelerated by the factor 10^4 as compared to the physical prototype.

Summary of Results to Date

Correlation between analog results and full-scale engine tests has been established: Control systems predicated on analog information have demonstrated anticipated performance on turboprop engines. In the initial stages of development, analog investigations were made of the independent con-

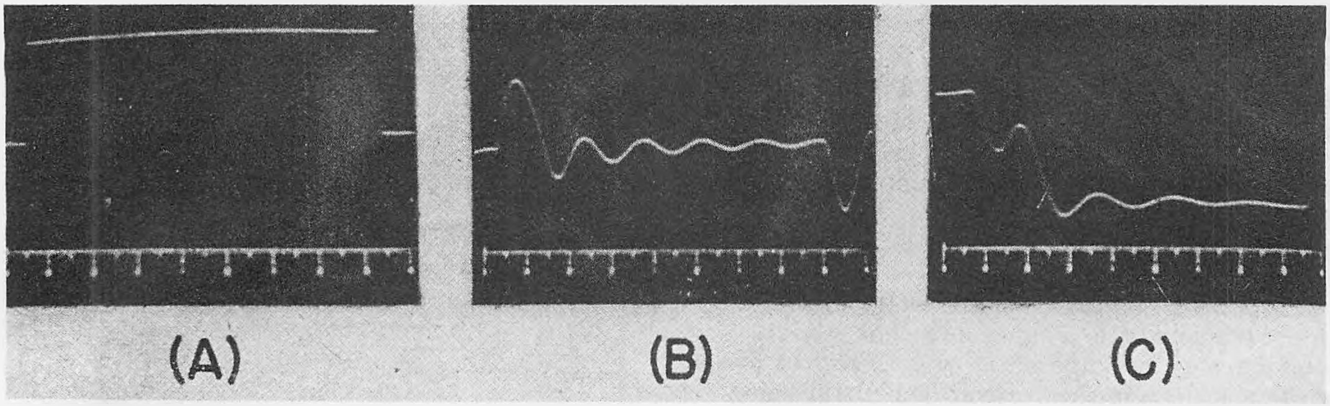


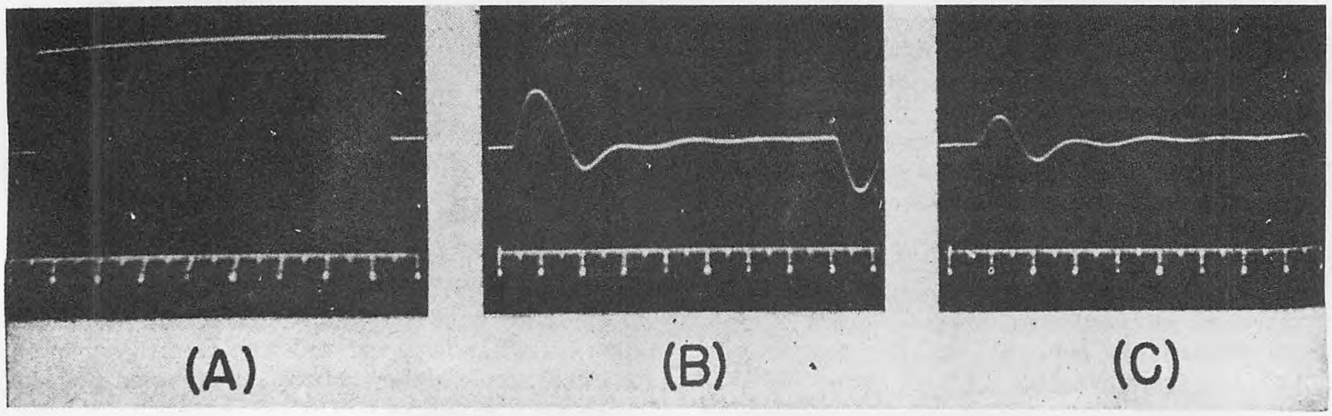
Fig. 10 – Analog results of 10% instantaneous disturbance to propeller blade angle showing effects on speed and temperature of typical turboprop engine when employing only speed control
(A) 10% disturbance to blade angle
(B) Speed error
(C) Effect on temperature

trol of speed and temperature. These checks were made by establishing various operating conditions and then introducing a step disturbance to each of the variables. Typical results of such investigations are illustrated in Fig. 10. The full horizontal time scale for each of these oscilloscopic traces is approximately 40 sec, and the maximum departure of the variable is approximately 10%. Thus Transient (A) shows the applied 10% disturbance to propeller blade angle existing for the full 40 sec, while Transient (B) shows the resultant effect on speed error. Under the action of this particular control, the speed departure is minimized and stably reduced to zero in a period of 5 sec. In a similar manner, Transient (C) shows the accompanying effect on temperature. It is obvious that control of speed has been obtained only at the sacrifice of temperature control.

After investigating fully the various types of independent control, the performance of the most promising combination was checked. The analog results of one combination are presented in Fig. 11. For ease of comparison the scales and arrangement are the same as Fig. 10. The performance displayed in these results indicates that by properly timing and coordinating the action of the various control components, stability and rapid response can be achieved. Selection of the control system for this particular illustration necessitates evaluating 25 design characteristics under the influence of four different types of disturbances and at least eight operating conditions. A complete determination of the optimum values is possible by means of the analog in the course of a day. In contrast, analytical or empirical methods might require months of constant effort. To illustrate the exhaustive analysis required, there are over one million possible combinations requiring investigation. However, since the analog provides a continuous simulation, the evaluation of each characteristic can be immediately scanned and records taken of only the critical values. Thus the entire solution can be obtained in a comparatively short time.

The analog is useful in evaluating the effect on performance of production tolerances and service

Fig. 11 – Analog results of 10% instantaneous disturbance to propeller blade angle showing effects on speed and temperature of typical turboprop engine when employing speed and temperature control
(A) 10% disturbance to blade angle
(B) Speed error
(C) Effect on temperature



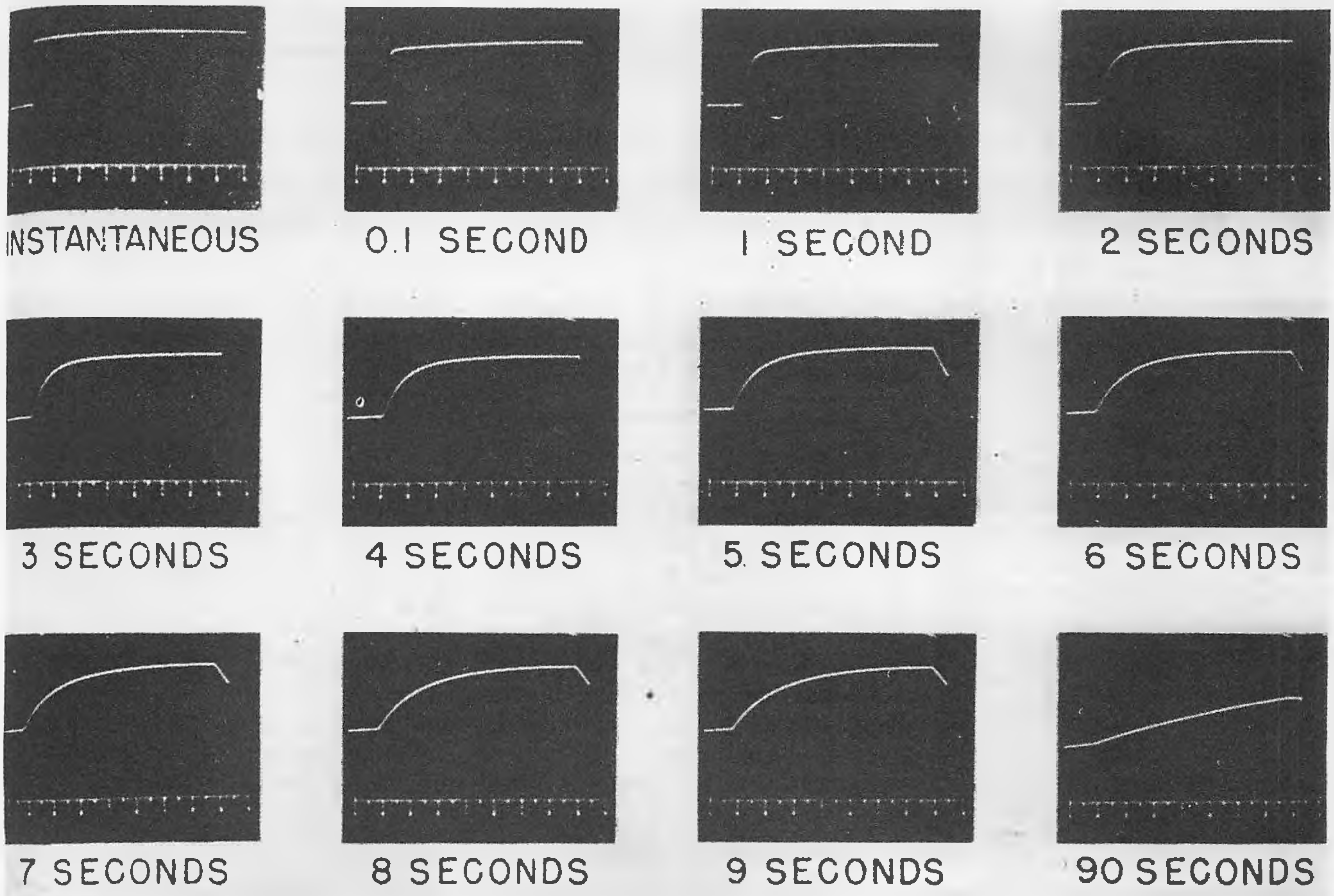


Fig. 12 – Analog demonstration of effect on response of variations in characteristic time for a first-order lag

deterioration of both engine and control. In other words, determinations are made regarding the suitability of the entire system for practical application. The complete engine control system should not be critical to any anticipated variation in characteristics. Too often this important factor cannot be checked in experimental designs, with the result that production difficulties are encountered. The analog provides the optimum means for checking any change in characteristics.

Additional Applications

There are a number of uses to which the analog computing technique may be put beside the control application for which the unit was originally intended. One example is in the technical treatment of dynamic lags, such phenomena being of recognized importance to the stability of regulation. The term "lag" is very broad, and may cover dynamic phenomena of great complexity, as in the higher-order delayed responses, or again may refer to direct time delays. In each case, the flexibility of the analog is a convenience in study. Here we shall show only the response involved in a first-order lag which may approximate that of a relatively fast measuring system. The impressed change is assumed to occur instantaneously, and the resulting transient is an exponential rise to the steady-state equilibrium value. Fig. 12 shows a series of such responses for a number of different characteristic times.

An effective additional facility of the automatic analog equipment is its use as an aid in the explanation of control phenomena. Owing to its embodiment of an analogous control system in high-speed and continuous operation, a familiarity may be imparted which is hard to attain with "paper" exposition, and which is costly and time consuming with full-scale equipment.

Both the engineer and the operating technician can benefit from personal contact with the analog computer, since manipulation of the adjustments shows immediately what effect the corresponding alteration in the real equipment would have on the transient response. Thus the known advantages of "learning by experience" can be afforded on short notice. For the main purpose of the analog computer, however, such pedagogical applications are secondary. The principal application remains, as was intended, that of a research tool in the development of high-performance controls for turboprop engines where neither the methods of analysis as such nor those of cut-and-try are appropriate.

Designing INDUSTRIAL

Electronic computer is adjusted to simulate an industrial operation and its control. Engineer then manipulates system to determine optimum design. To simplify computer construction and increase speed very fast time scales are used in computing circuits

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USE OF ANALOGS makes it possible to experiment readily with devices or phenomena under changes of scale or after transformation of their variables. All models, whether they are the small-scale replicas used by civil engineers, model airplanes in the wind tunnels of aerodynamic engineers, miniature boat hulls in the towing tanks of naval architects, or the equivalent circuits used by acoustical engineers to study microphones, are analogs. Dynamic analogs can be highly complex assemblies such as differential analyzers, abstractions such as mathematics itself, or direct simulations of the process.

The great advantage of analogs as devices for solving engineering problems is that they are simple. Electrical analogs of mechanical, thermal, or other systems can be assembled and adjusted quickly and easily. For example, in designing a pneumatic control, the analogous electrical network of resistors and capacitors of Fig. 1A was built. As a suitable design evolved from experiment a more formal network was constructed. Finally, after experience in the laboratory under many control circumstances, the actual pneumatic control of Fig. 1B was built. Much time and costly machining were saved using the easily modified electrical analogy.

To facilitate making electrical analogies and to perform the broader functions of analog computers in problems dealing with automatic controllers, the Analaut has been developed. It is a flexible electronic instrument for study and demonstration of regulatory systems such as industrial process controls, servomechanisms or position followers, navigational controls, and stabilizers for power plants.

Designing Controllers by Analogs

As long as a process remains in the steady state its analysis is relatively simple. About two decades ago engineers in the process industries, particulary those concerned with instrumentation, became concerned with the dynamic nature of their processes and equipment, especially under automatic operation. Owing to the complexity of such problems, early studies were empirical. Mathematical analyses and syntheses of idealized systems were made. Hydraulic analogs of thermal systems were built from which transient behavior could be studied readily by direct measurement.

Beginning in 1936 the writer developed a complete computational Automatic Control Analyzer based on interconnected high-speed models of both process equipment and its associated controller, which took the form shown in Fig. 2A. Different masks depicting the processes and controls being studied were superimposed on the panel to facilitate visualizing the system; the in-

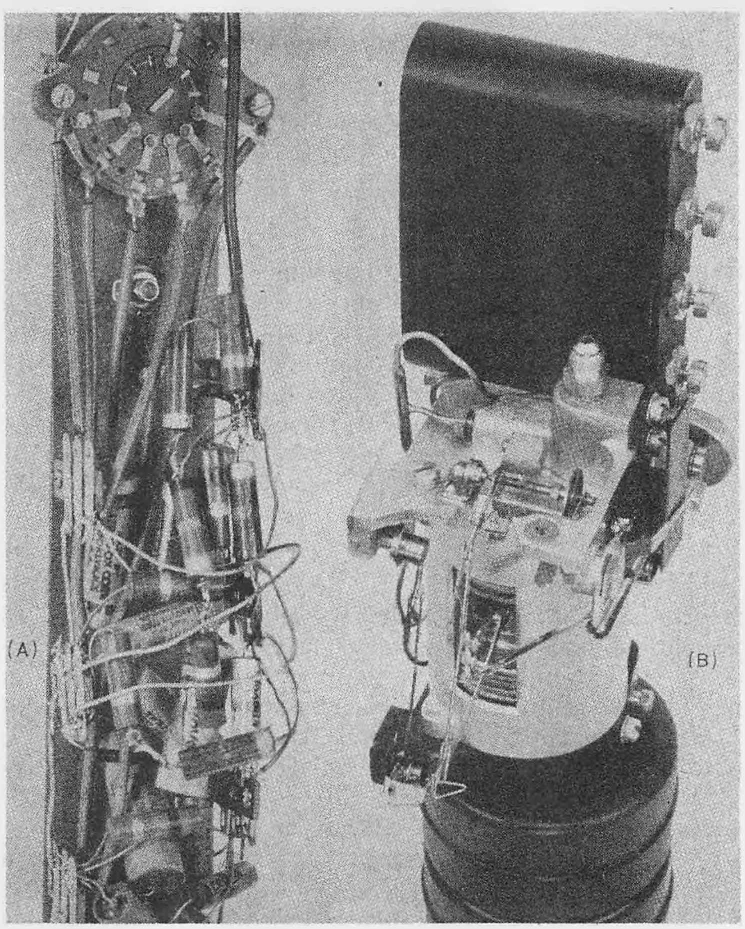


FIG. 1—Electrical analog (A) simplifies design of pneumatic controller (B)

CONTROLLERS By Analog

strument is still in use. The same basic technique, developed to a higher degree, is employed in the modern instrument shown in Fig. 2B. It is used for designing controls and also for predicting the necessary type of control for a proposed installation and the adjustment for optimum performance of complex systems.

Whereas controllers can be designed by mathematical analysis provided the system is not prohibitively complex or by testing in the completed plant if adjustments to the system can be made safely and economically, it is simpler to represent the closed control-process loop by an analog. The heavy lines of Fig. 3A show the loop whose properties are to be studied; the rest of the diagram shows the elements of the analog analyzer. The control manipulates the plant input m in recognition of the unbalance u so as to cause the regulated variable v to follow its desired value v^*, thus reducing the absolute value of the unbalance u to a minimum near zero. All the variable and parameters in the analog are the counterparts of those in the actual plant.

In the analog computing system, the controller and the plant are represented by electronic model assemblages, a basic circuit of which is shown in Fig. 3B. The essential loop variables are transformed into measurable voltages, each of which can be related to the corresponding plant variable by an appropriate scale factor such as pounds per square inch per volt (to convert to pressure in a pneumatic control).

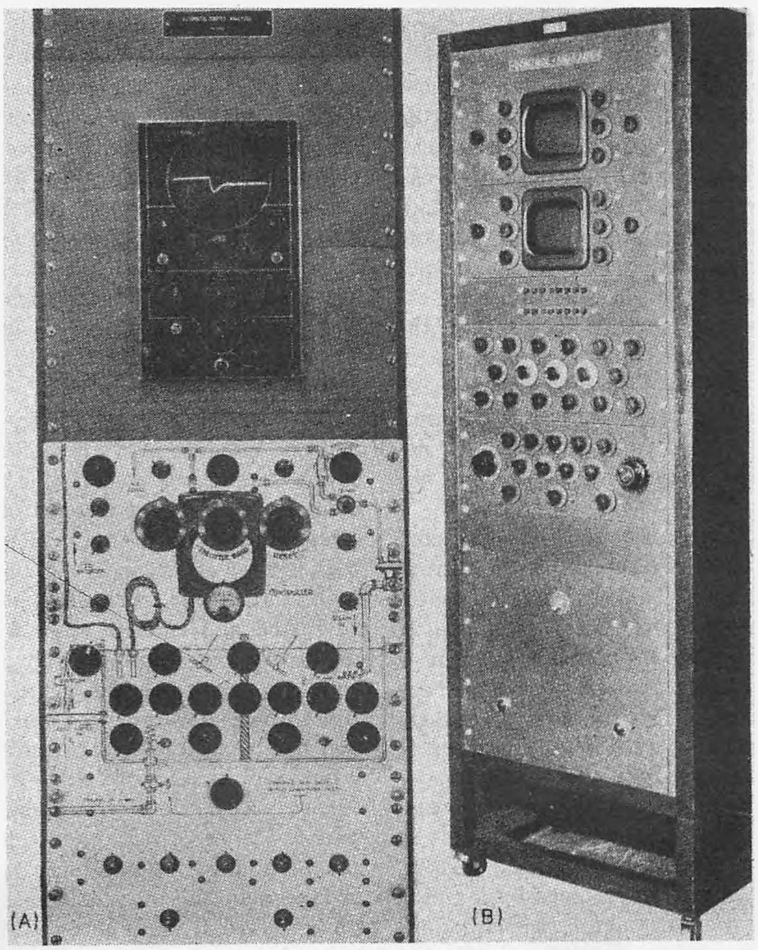


FIG. 2—Circuits of analog computer simulate plant and controller

For repesenting the desired value there is a manually adjustable steady component and an optionally inserted variable component for disturbing the system. The flexibility of the instrument permits comparing controlled and uncontrolled responses of the simulated system, studying hysteresis and excursion limit effects, inserting conventional regulating functions with proportional, derivative, integral, and second integral effects, and inserting special features from external circuits. Response of the analog is determined by disturbing it with a recurrent pulse and observing the transient on an oscilloscope. The time scale of the analog is made short so that the loop will have returned to equilibrium before the next pulse and so that the computing elements, especially the capacitors, can be conveniently small. The disturbance can be inserted at any desirable point in the loop.

Usually the variations around the simulated loop are displayed as functions of time on the oscilloscope, with suitable timing markers if necessary. However, by plotting one variable against another parametric plots of great interest can be obtained. Figure 4 shows curves plotted against time, and a parametric curve (for a more complex system) by way of comparing the two types of displays. The parametric method shows the stability and phase relations among significant loop variables.

With such an analog of the process an analog of the appropriate controller can be developed and its suitability observed from the transient response obtained. By manipulating plant or control parameters that are likely to vary during operation, critical conditions can be found and evaluated. With this information the control is practically designed. *The fast operating* time of the analog permits observing the complete transient response as an adjustment is made, so that a complete study of a system can be completed quickly.

Basic Circuit

For special purposes the analog might be arranged differently than the one described here, but the same basic circuit can be used. Most of the complete analog system is based on conventional electronic techniques and so need not be reviewed. However, it should be pointed out that, of the possible mediums for building analogs, the convenience and flexibility of electronic circuits makes them excellent for experimental purposes. If one stays well above the noise and drift thresholds, there is no practical limit to the precision that can be obtained if the needs justify the effort. At the opposite exreme, tube noise can be employed for random excitations where statistical evaluations are to be made.

Figure 3B shows a useful general-purpose circuit for use in electronic analogs. Considered as an amplifier, the circuit is directly coupled for handling direct current but can operate to frequencies that are high compared to the fundamental frequency employed in the disturbance. The input impedance as seen from e_1 is very high. The internal impedance of the circuit is also relatively high so that for reliable results substantially no current can be drawn from the output by the load. Thus, because no current can be drawn at the output e_2, the circuit is usually followed by another of the same kind.

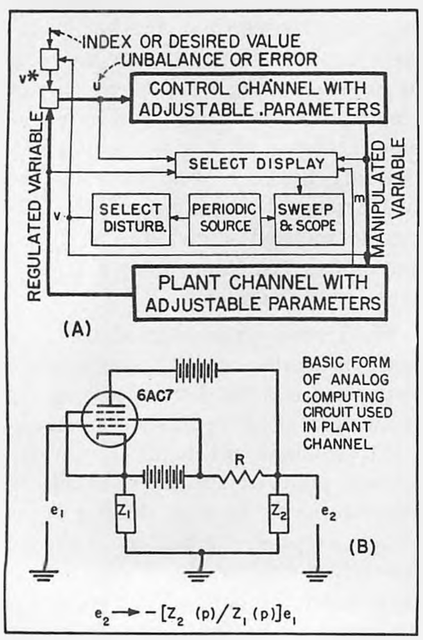


FIG. 3—(A) Block diagram of automatic control computer, and (B) basic circuit of used in the analog computer elements

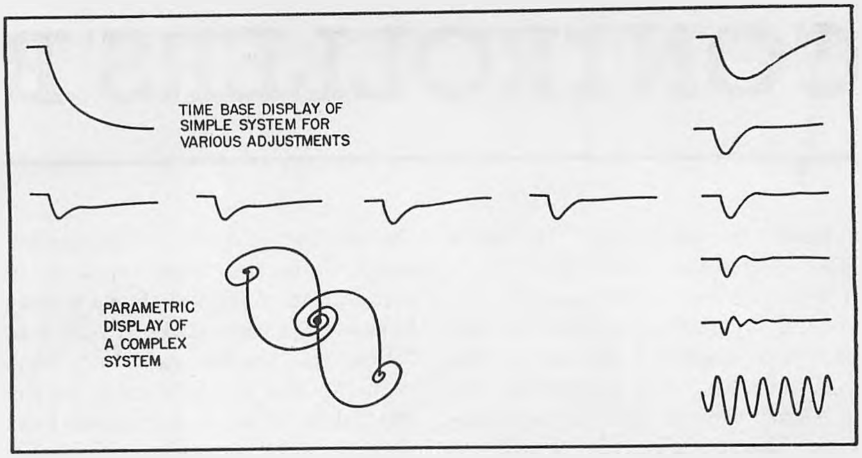


FIG. 4—Reproduction from oscilloscope tracings show how optimum response of plant can be determined by systematic adjustment of various controller adjustments

A fixed source of screen excitation is provided, giving constant gain to zero frequency. The same voltage source provides a reverse current mode of operation in the computing portion of the circuit. Dropping-resistance R is chosen near the average effective d-c plate resistance of the tube. A peculiarity of the circuit is that there are no paths from the tube electrodes to ground other than those through the elements Z_1 and Z_2, thus the currents through these elements are equal and opposite. As the grid voltage approaches cutoff, current circulates through Z_1 and Z_2 in that order, making the output e_2 positive. At the opposite extreme, the current circulates in the reverse direction making e_2 negative. Because the voltage across Z_1 follows e_1, the output e_2 is dynamically related to e_1 in a manner dependent almost entirely on the values of Z_1 and Z_2.

If Z_1 is purely resistive, the current in Z_2 corresponds to the input voltage e_1. This property is useful in various ways; for example, Z_2 can be the input terminals of a four-terminal filter, in which case the current into the filter is directly manipulable with no expenditure of input energy.

If Z_2 is also purely resistive and equal to Z_1, reversal of sign or "minus one" operation results. With Z_1 and Z_2 replaced by a single linear potentiometer, a distortionless inverting amplifier having a useful adjustment is obtained. With the tap in the center, the gain or transfer function is nearly unity. Deflection of the tap in one direction gives a transfer or gain of G and an equal deflection in the other direction gives a gain of $1/G$.

With Z_1 still purely resistive, if Z_2 is purely capacitive, the circuit is a reasonably good integrator with a time constant R_1C_2. In the control analog computer for which this circuit was developed, the computing interval is typically four milliseconds, so that the time constant of the integrator can be made long compared to the computing time using components of reasonable size. If the elements are reversed the circuit is a differentiator. In fact there are numerous dynamic characteristics that can be obtained using different combinations of impedances for Z_1 and Z_2. The nominal equation for the circuit is given in Fig. 3B.

In operating the circuit, care must be taken to prevent saturation of the tube or components. For example, a typical fast integrator will integrate to a limit in a millisecond with one volt remaining on the input. However, such a device can be tested and calibrated by applying a square wave of about five volts amplitude to the input, with an additive adjustable d-c bias. The bias can be set to bring the effective input level to zero and will keep the output within the limits of saturation. Under these conditions a sharp and straight sawtooth will be produced in the output by a sharp square wave at the input; the amplitude of the output will be dependent

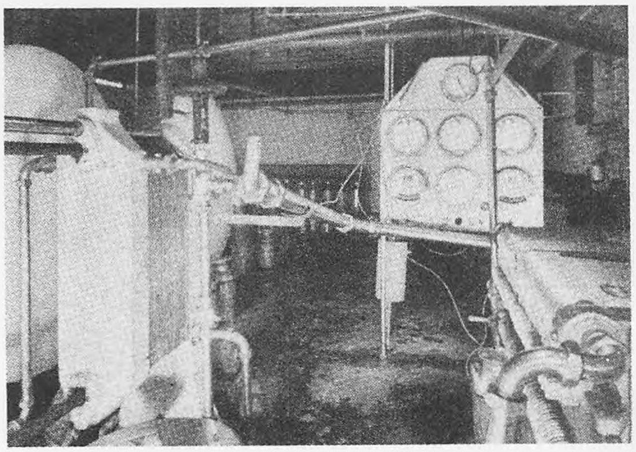


FIG. 5—Control for heat exchanger in this pasteurizing plant was designed by means of electrical analogs

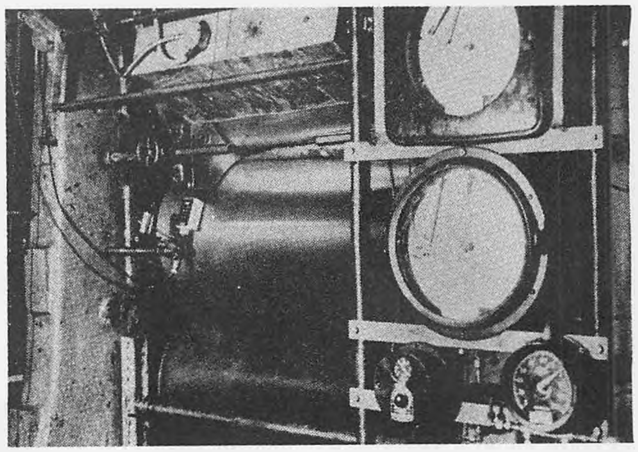


FIG. 6—Floating thermometer head on plastic calendering roller actuates automatic process controller

on the amplitude of the input, its period, and the time constant of the integrator. Other types of computing networks require other techniques for calibration and adjustment, but this example illustrates the simplicity of the methods.

The combinations possible with this basic circuit provide a powerful general technique for constructing computers and control analogs. Most dynamic conditions can be reproduced with this circuit and combinations of passive networks. For a small project, or for initial experimentation, the basic circuit using batteries is especially appropriate because well-regulated power supplies are unnecessary. As used in the control analog computer, common power supplies and auxiliary switching and calibration circuits are necessarily added to the basic circuit.

Industrial Applications

The first step in using the analog computer for designing an automatic control for an industrial plant or process is to reduce the actual system to its elecrical model. In many processes it is possible to recognize the electrical analogs from the equipment and to compute parameters from known data or by simple tests. Distributed parameters can usually be represented to useful accuracies with a few lumped sections.

As mentioned above, if a direct approach is not feasible the dynamic response of the plant can be determined by introducing a known disturbance at the input or manipulated variable and observing the disturbance produced at the output or regulated variable. The plant must remain in a sufficiently undisturbed condition, aside from the intentional disturbance, or the measurement must be repeated often enough to eliminate random effects. Where the response depends on the condition of the load or there are other nonlinearities, a series of tests may be necessary. The record of plant response is then duplicated to a much faster time scale on the control computer, with especial attention to duplicating delay and the initial portions of the response. Once the plant response has been provided in the analyzer, the appropriate control can be quickly determined.

Two typical problems illustrate more specifically how the analog method of designing controllers is carried out in practice. Figure 5 shows a portion of a high temperature pasteurizer; the main heat exchanger is at the right and the instrument panel in the near background. Several interlocking controls are included in the plant to assure holding every drop of milk at a maximum temperature for a minimum interval, avoiding overheating. The crucial regulation problem is to control the hot water temperature in the final milk heater stage at a point chosen for its significant relation to the milk temperature by manipulating a steam valve elsewhere in the system. Under manual operation with water replacing the milk to avoid accidents, a record was made of the temperature variations resulting from a sudden known change of the steam valve. From this information the settings for a proportional derivative-integral control were determined on the analog computer. High performance was obtained from the predicted settings and further adjustments were unnecessary.

In another type of problem the crucial regulated variable was the surface temperature of the central roll of a plastic calender. The temperature was measured electronically by the floating head shown in Fig. 6 and recorded on a self-balancing capacitor bridge instrument. The manipulated variable was steam pressure under control of an auxiliary or cascaded regulator. By making a manual change in the steam pressure, the plant response was obtained on the temperature recorder. The analog of the plant was then set to duplicate this response and several control methods studied. The best type control mechanism thus determined was installed and set to the predicted dynamic adjustments, giving satisfactory control immediately.

Besides providing a design and operating tool in the field of automatic control, this type of analog has also proved useful in instructing plant personnel and as a college lecture room demonstrator and laboratory test set. Acknowledgement is made to the engineers of The Foxboro Company for whom the early developments of these techniques were made, and to Prof. J. A. Hrones of MIT for encouragement in their application to the pedagogy of automatic controls.

ELECTRONIC ANALOG METHODS

The reader is now in the position to glimpse into the intimacies of problem set-up and scaling on electronic analogs.

We start with an exposition of the methodology behind analog problem solving (Sheingold) followed by some useful techniques of scale-factoring (Reswick).

This is topped off by the "whipped cream" of an application expertly detailed (Foster of the RAE).

In this sequence of three articles all the essential steps are detailed. The beginning reader is then in the position to read all the remaining documents with greater perspective and understanding.

(It can also be recommended that some of the earlier material be re-read for fuller comprehension.)

THEORY AND APPLICATION OF HIGH-SPEED ELECTRONIC MODELS

D. H. Sheingold

Presented at New York
January 19, 1954.

I. INTRODUCTION - The Need

A vast field for the application of analogs is now being revealed in the solution of chemical dynamics problems, in studies of biology, anatomy, and physiology, in economics and sociology, in addition to the increasingly widespread uses of models in engineering and physics.

Many excellent descriptive papers[1-10] and at least one book[11] have been written about Electronic Analog Computation, relating to applications in specific problems or to construction of computing equipment. These efforts have largely employed language familiar to electrical engineers. Because there are fruitful possibilities for the use of analog methods in so many diverse fields, there has been and remains the need for a generalized electronic model approach which is completely divorced from the minutiae of conventional electronic techniques. Among the terminology necessary to be discarded are such details as the connection of electrical elements to operational amplifiers, the loading and unloading of potentiometers, and the internal wiring of operational amplifiers.

There has been developed a program for research and design by analog methods at repetitive speeds without resort to overtly electronic techniques.[12] The advantage of this type of approach is that it virtually eliminates the need for specialized electrical knowledge in the computation laboratory. The scientist or engineer, whatever his background, is at liberty to apply his undivided attention to the problems requiring solution, without needing to detour into the particularities of electronics.

The Realization

The fundamental construction plan for any electronic analog or indirect model[13] is the block diagram. It is the "vigorous equation" in which physically realizable functional operations on a set of variables are represented by unidirectional blocks. Such operations may be linear or nonlinear, and need not necessarily be expressible in mathematical terms. By connecting these operational blocks in a causal configuration which satisfies the known or postulated relationships relating to a physical or mathematical system, a "block diagram" model of that system is obtained. A block diagram for a very simple physical system, together with the equation which it also represents, may be seen in Fig. 1. Any system which constitutes a faithful physical realization of the functional operations exemplified in the blocks is a model or analog of the prototype system, and can be electrical, mechanical, hydraulic, pneumatic, thermal, chemical or any combination.

An electronic model may comprise a universe of operational blocks in which voltages with respect to a common reference represent all time-dependent variables, and time represents a continuous independent variable. Integration, at present, may be performed readily with respect to time only, although this is not a fundamental limitation. Solutions occur as voltage histories measured (1) on voltmeters, (2) plotted on chart paper, or (3) displayed on the screen of a cathode-ray oscillograph.

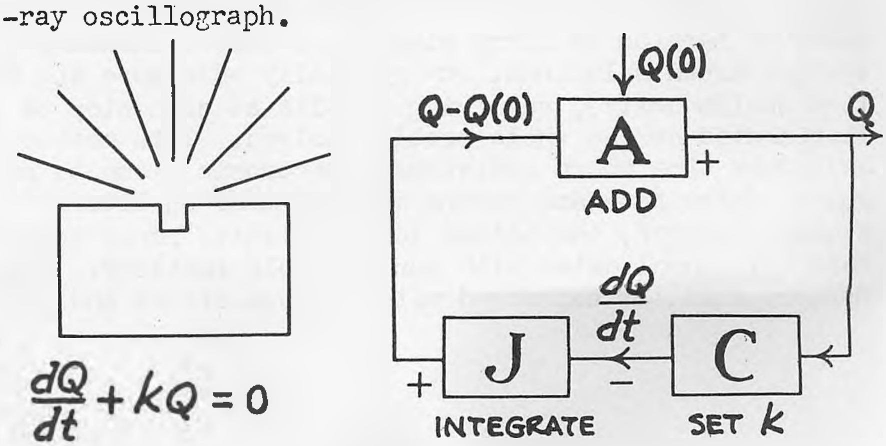


Figure 1. Simple physical example. The equation describes radioactive decay. Q is a measure of the intensity of radiation.

The method of computation at "high speed" uses solution times that are sufficiently short to allow solutions to be presented repetitively on the screen of a cathode-ray oscillograph, so that the effects of continuous adjustment of system parameters may be seen immediately. New techniques using an automatically calibrated d-c oscilloscopic display permit accuracies of representation and measurement, for voltages and time, of better than ½% for a large number of simultaneous traces.

The same operational equipment may be used alternatively for fast-repetitive and for low-speed or "real time" computation, if the physical operational blocks are capable of performing at speeds represented by frequencies from zero (d-c) to substantially more than 10 kilocycles.

II. HOW PROBLEMS ARE SOLVED[14,15,16]

The process involved in the solution of a design or synthesis problem by high-speed analog methods is embodied in the following sequence of operations:

a. Presentation of the problem as an expression of the relationships (known or suspected) existing at each point in a prototype system by writing the local equations, or by expressing local relationships in function-plot form. (5-40% of total time for the study)

b. Expression of the relationships as a causal block diagram, utilizing operational blocks. (5-15%)

c. Reduction of the numerical quantities in the problem to a form suitable for computation, i.e. scaling or normalizing. (10-25%)

d. Connection of the physical embodiment of the operational blocks (the computing components) by means of cables, applying all stimuli and initial conditions, setting all parameters, and bringing to the display system all the variables one desires to observe. (5-15%)

e. Analog solution (15-75%)

- i. Verification of analog behavior: check against known or expected performance, in special cases.
- ii. Exploration: adjustment of parameters to determine immediately those areas in which settings are of interest. Closer exploration in these regions.
- iii. Criticality studies; determination of effects of changing the configuration.
- iv. Conclusions: optimum values where significant.
- v. Check of conclusions against other work, if available; computed verification of specific cases by numerical or graphical methods.

f. Application of results to analysis or design

With experience systems may be modelled directly, with a minimum of equations or block diagrams. Many problems of a quite high degree of complexity may be dealt with in very thorough fashion by first studying a more elementary system. Then, constantly checking against known solutions, one gradually adds more and more degrees and orders of complexity (and nonlinearity), verifying results at each step of the way, until the entire system is represented or the whole problem solved. This method might be prohibitive in terms of available time where individual electronic elements must be placed and displaced, or at low speed, where runs and reruns must be made on literally miles of chart paper. At repetitive speeds, however, the method leads to quite rapid solutions, since trial settings may be made and consolidated with considerable facility. Physical operational blocks may be added, removed, or exchanged with minimum effort and negligible electrical or electronic

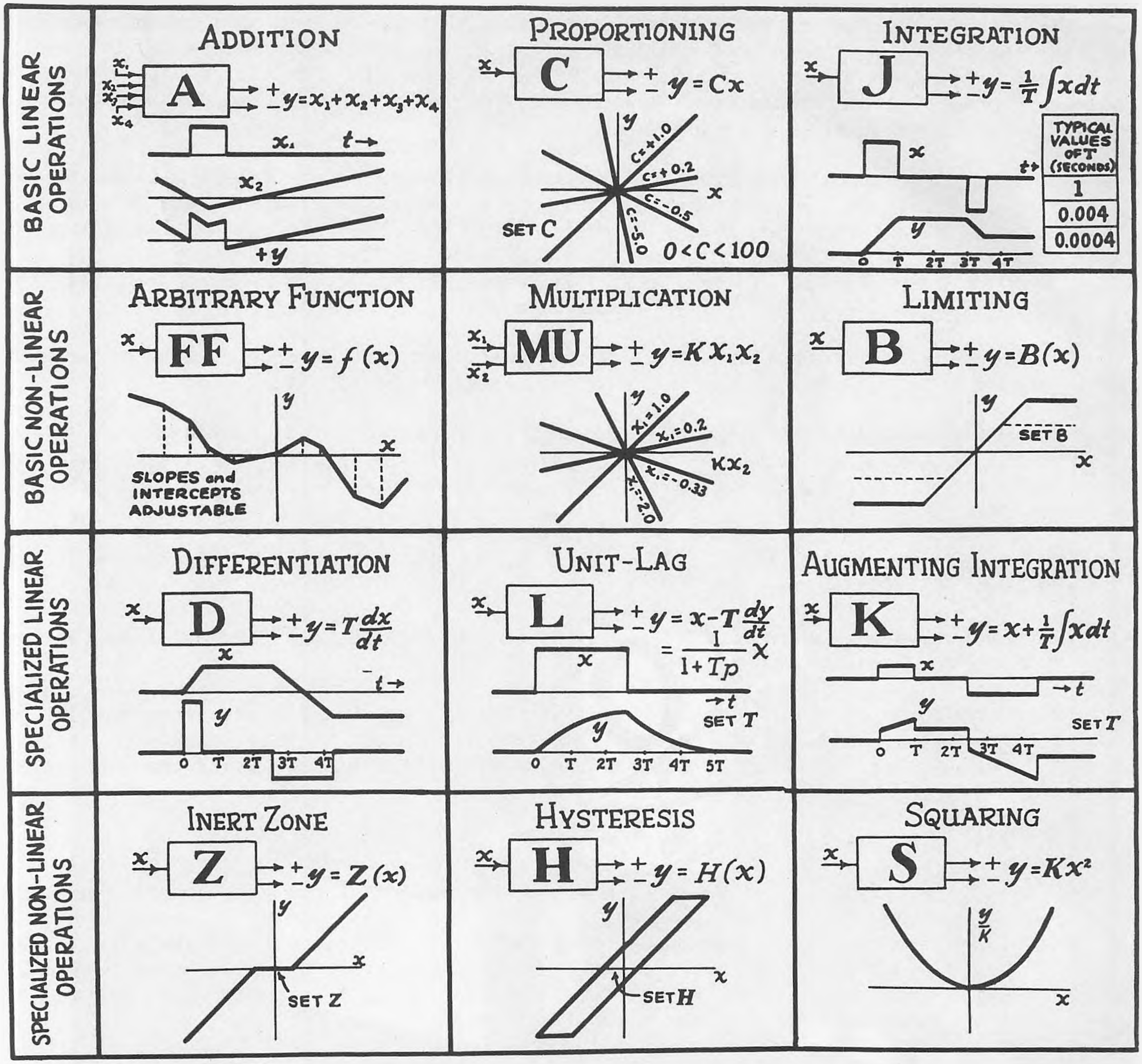


Figure 2. Table of operational block equations and responses.

(a.)

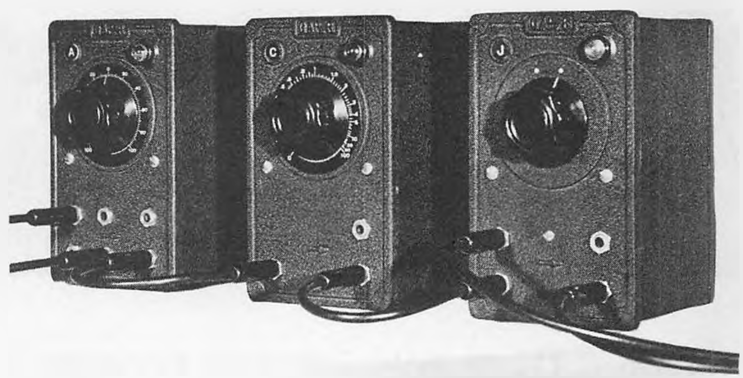


Figure 3. Typical computing components. (a) Adding component, Coefficient component, and Integrating component.

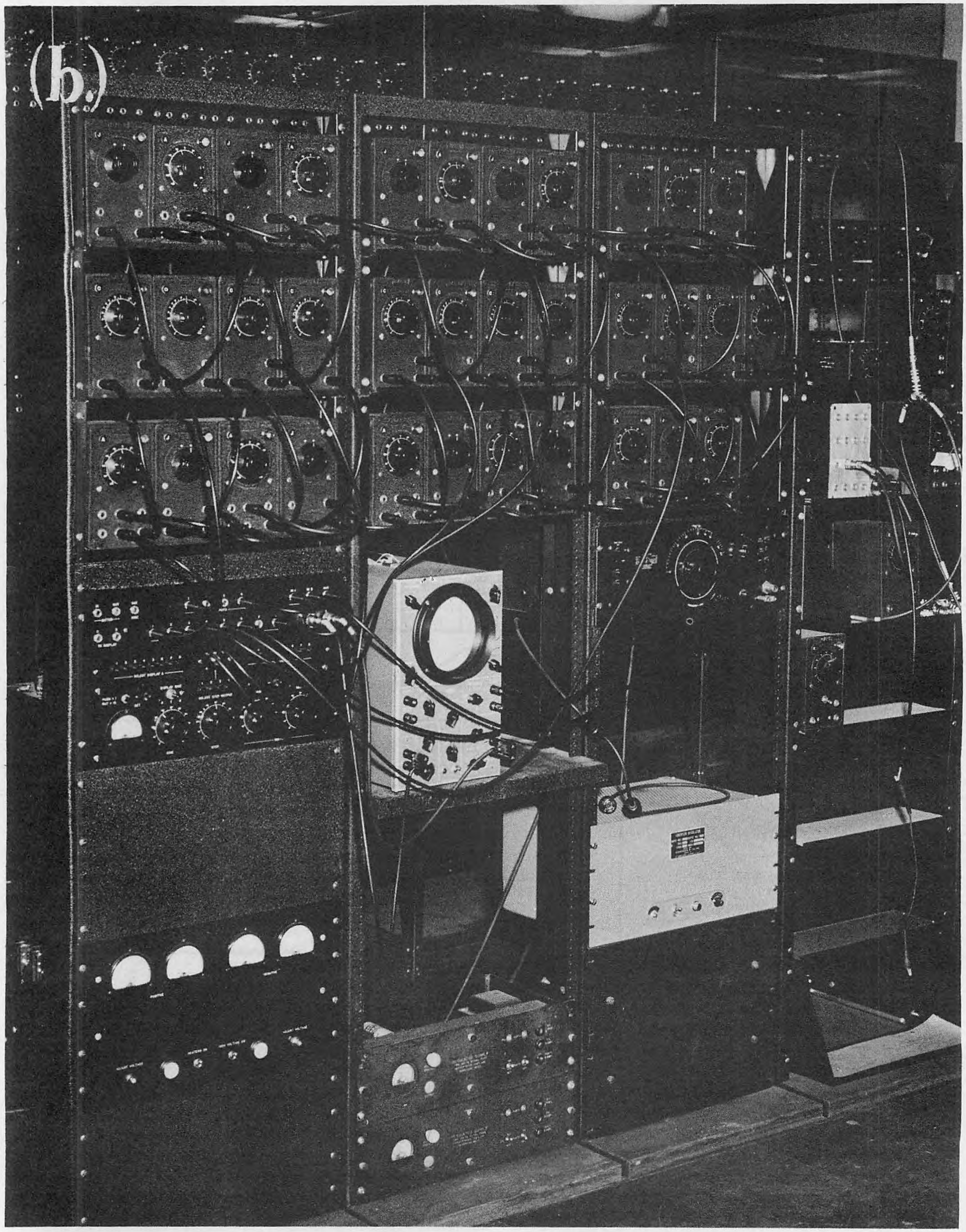


Figure 3.(b) Computing equipment in a typical rack assembly.

knowledge. Such a technique, moreover, is quite enlightening in demonstrating to the researcher the behavior of his system in all its ramifications, and leads to complete confidence in the model.

III. THE TOOLS[17]

The most important requirements for a set of operational blocks are, as might be expected, faithfulness of representation at all significant speeds and freedom from parasitic variation over the period of computation. Experience has shown that both criteria may be satisfactorily fulfilled in repetitive operation.

In addition, it is desirable to make positive and negative outputs both available from each block, since inversion is a trivial but necessary operation. It is also important that the blocks be causally unidirectional with "high-energy" outputs and "low-energy" inputs. Each block may be physically portable and capable of being supplied from a common source of external power. An indication of limiting at the outputs should also be included to insure that the blocks are not unintentionally driven into regions in which they cease to embody the true operations.[18]

The most common functional operations, their equations and responses, and their block diagram symbols are shown in Fig. 2, together with numerical characteristics for the commercially available embodiment of these blocks. A photograph of typical physical blocks is shown in Fig. 3.

There are a number of other operations which would extend the range of application of high-speed model techniques, but they are not currently in commercial production. Among these are the generalized nonlinear function of two or more variables, the integrator with respect to an arbitrary variable, the generalized linear response function, and its special case: the finite time delay.

For introducing stimuli, a waveform generator may be used. Solutions are observed on the screen of a d-c cathode-ray oscillograph. For added convenience, an automatically calibrated d-c electronic display system provides high accuracy in reading results. Excellent permanent records may be obtained by photographic techniques. A typical Land camera record of responses of a system to a stimulus is shown in Fig. 4. In addition to the highly quantitative automatically calibrated oscilloscope display which plots its own coordinates, a technique widely used, especially in the study of nonlinear systems, is the function cross-plot. By this technique, the phase relationship between two variables of interest may be observed. Typical displays of this sort are shown in Fig. 5.

IV. THE BLUEPRINT

A block diagram is a pictorial equation which specifies the indirect model of a system. The general technique of forming a block diagram is to:

a. Establish the highest derivative in each local differential equation as a function, to be subsequently obtained, of the lower-order derivatives and of known quantities.

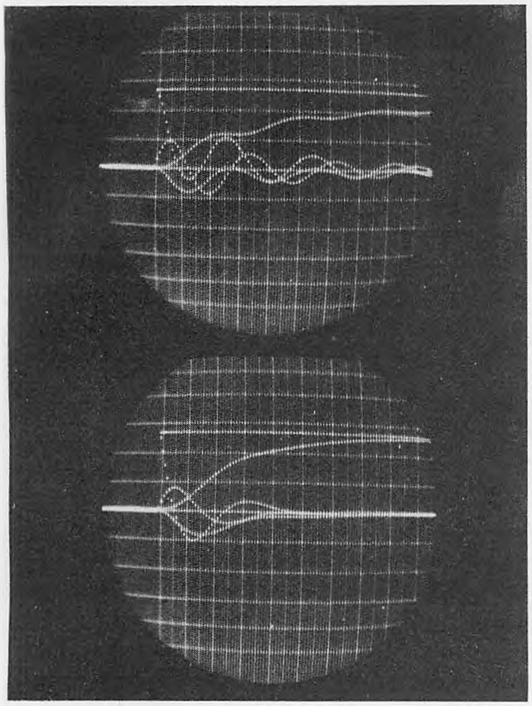
b. Integrate successively (with blocks) until the term of lowest required order is attained in each equation.

c. Perform whatever functional operations are necessary to obtain the functions needed for the highest-order derivatives.

d. Append initial conditions where they occur in the physical system.

Two examples which illustrate the technique are shown in Figs. 6 and 7. The solu-

Figure 4. High-speed solutions.



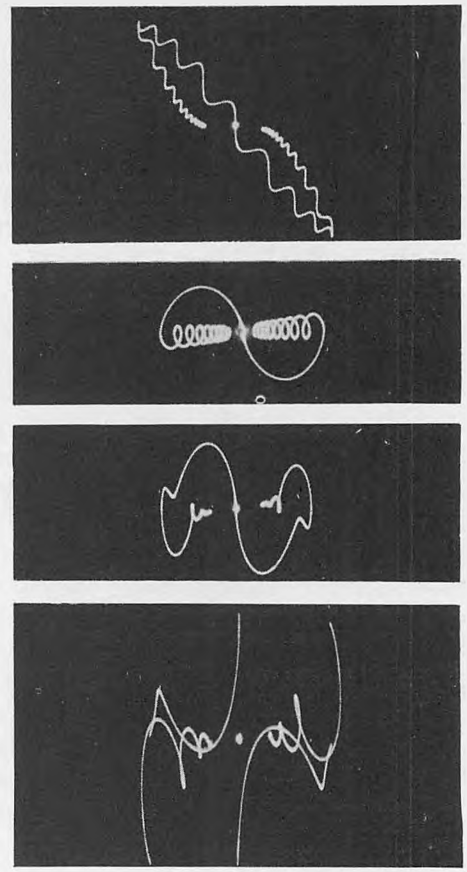
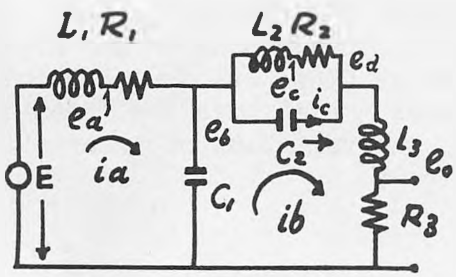


Figure 5. Phase-space plots on an oscilloscope.



$$e_a = i_a R_1 + e_b$$
$$i_a = \frac{1}{L_1}\int (E - e_a)\,dt$$
$$e_b = \frac{1}{C_1}\int (i_a - i_b)\,dt$$
$$i_b = \frac{1}{L_3}\int (e_d - e_o)\,dt$$
$$e_o = i_b R_3$$
$$e_d = e_b - \frac{1}{C_2}\int i_c\,dt$$
$$i_c = i_b - \frac{1}{L_2}\int (e_b - e_c)\,dt$$
$$e_c = (i_b - i_c) R_2 + e_d$$

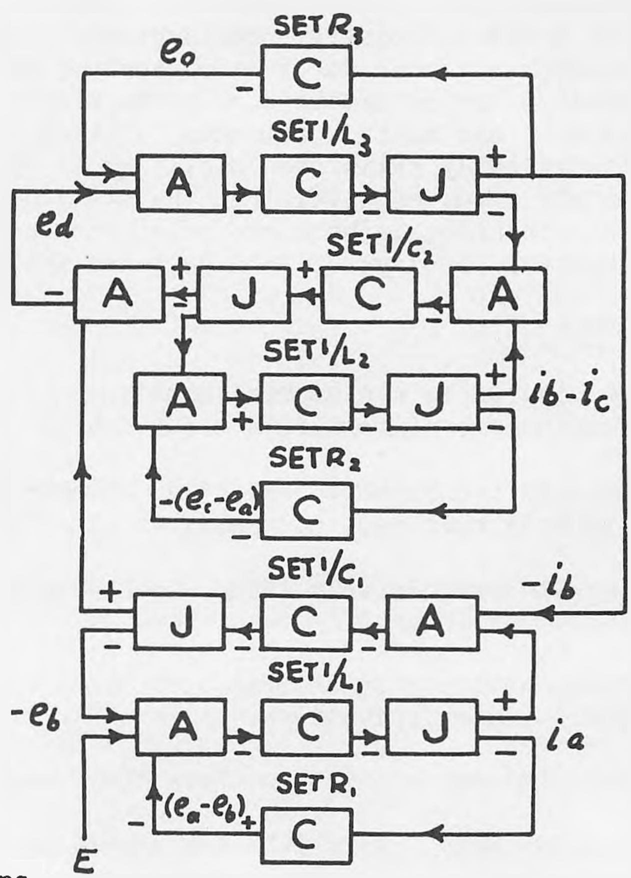


Figure 6. Electric filter, its equations, and its block diagram.

tion of a sample problem with many important applications in diverse fields is described in the Appendix.

In forming block diagrams, the most successful procedures utilize the equations as they are obtained directly from each part of the system without mathematical combination. If overall performance data rather than the equations of portions of a system are available, block diagram techniques, because of their informality and generality, may nevertheless be applicable. The controlled plant of Fig. 8 is a case in point. Whatever linear or nonlinear blocks may be found necessary are utilized to approximate the plant response. The characteristics of a practicable controller which will regulate the plant successfully may then be found by analog study. High-speed computation, in which the effects of changes in parameters may be observed continuously, is of considerable advantage in this case, because the nature of the plant and the controller must both be determined largely by empirical methods.[19]

Partial differential equations and purely algebraic systems are susceptible to operational block diagram methods, but at present a more generally successful and less costly approach is through the use of analogs of more direct nature. These include network analyzers, passive circuits, and certain operational amplifier assemblies; they are not covered here.

It is evident that block diagrams may be inverted, employing differentiation instead of integration as the primary dynamic operation, but usually a more accurate physical embodiment is attainable with integration. This concept becomes clearer when it is observed that while the theoretical step response of an integrator involves a pole at infinite time, the differentiator has a pole at the instant of application.

V. THE SCALE MODEL

High speed electronic analog techniques utilize voltage excursions of $\pm$ 50 or more volts and maximum time durations of 2-4 milliseconds or greater. The physical systems which are represented, though, might have scales for variables in such diverse dimensions as grams, minutes, inches, dollars, centuries, neutrons per microsecond, mols of CO_2, or births per year. To be quantitatively useful, this information must be translated into the dimensions of the electronic model, or scaled. There are many methods of establishing the numerical analogy, all of which are successful to some degree, and all of which lead to similar coefficient settings, if the following basic principles are observed.

It is convenient to choose scales which give coefficient settings near unity for the principal parameters of a system, for a number of reasons. For example, an extreme setting for a "main line" parameter inevitably means that, while a voltage in one portion of a model may tend toward limits, some voltage elsewhere in the system will try the lower limits of resolution of the physical blocks. Also, transformations to numbers near unity are usual in studies by graphical or numerical methods. Finally, the unit setting of a physical coefficient block is at the center of a quasi-logarithmic scale extending from zero to extremely large ratios, and the greatest accuracy of setting and reading is at the central portion of the range.

A mechanized method of choosing coefficients, which embodies the best features of the most effective techniques now in use, is the method of normalizing[20], which utilizes dimensionless ratios for all variables:

a. Obtain typical "unit" values for all variables as closely as may be known or estimated.

  i. For dependent variables, their maximum estimated values.
  ii. For time, the largest significant time constant of the system or 1/10 the maximum "interesting" time, or the maximum value of a *variable divided by* its maximum rate of change.

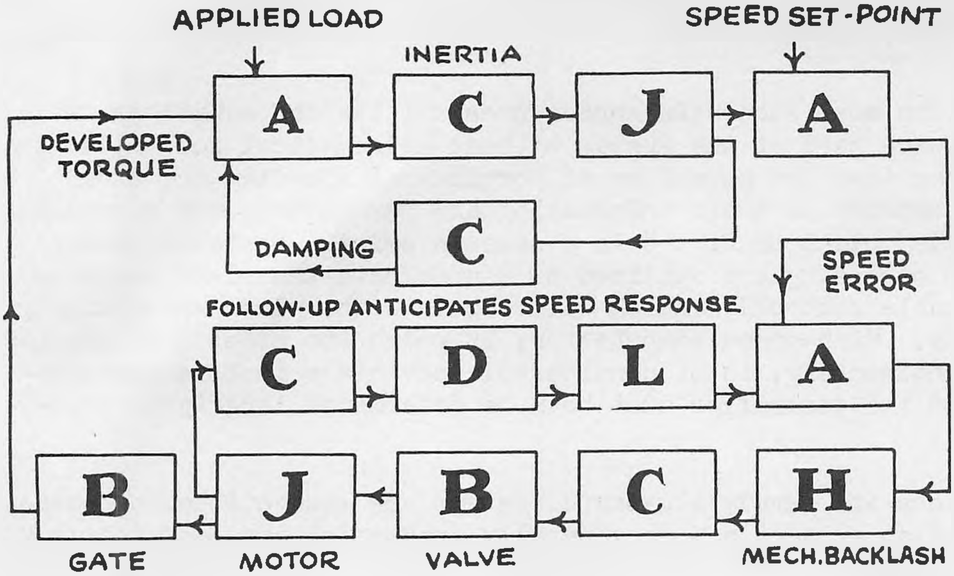


Figure 7.Hydro governor block diagram.

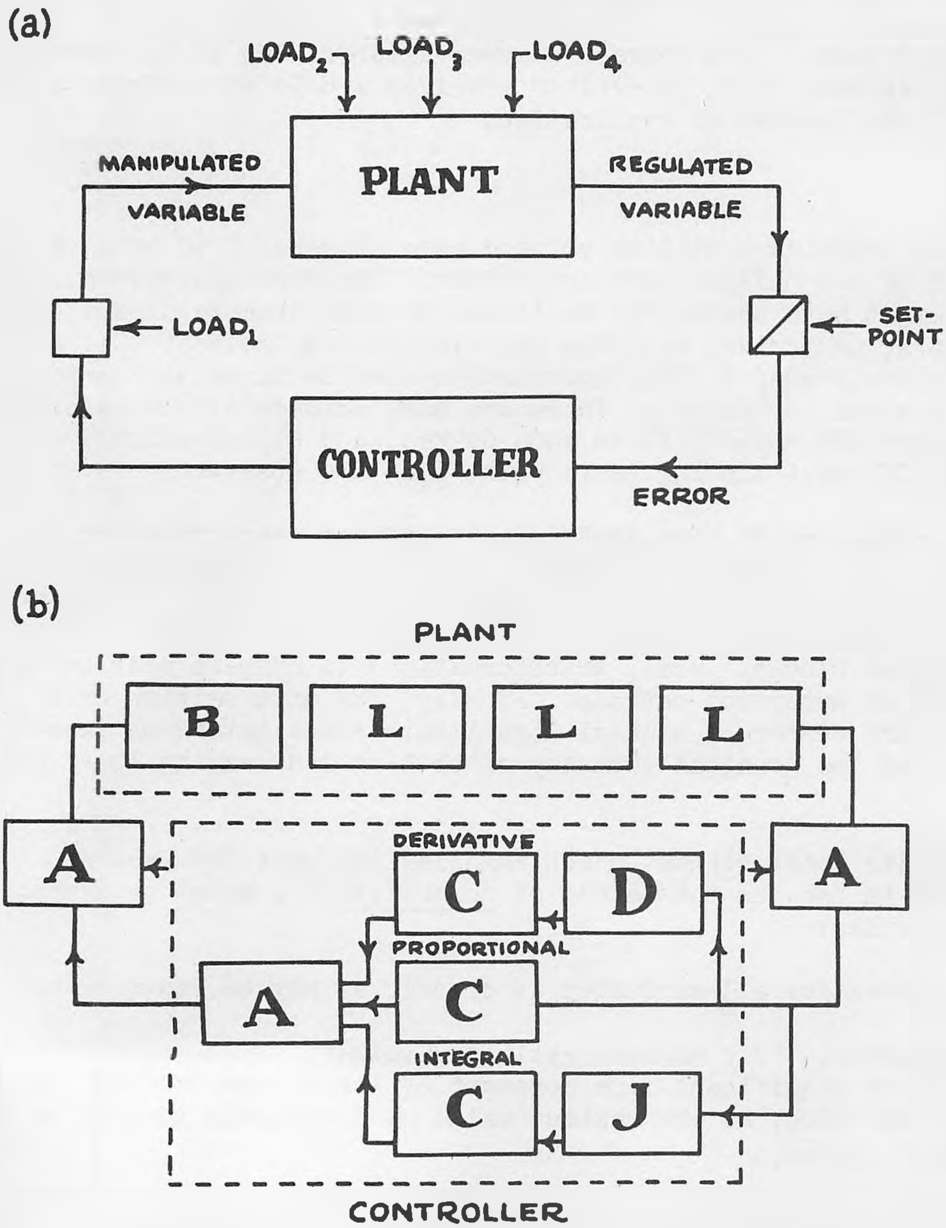


Figure 8.Representation of controlled plant.(a) General plant block diagram. (b) Operational equivalent of typical plant and controller

b. Rewrite the system equations, using the ratio of each variable to its unit as a new normalized variable, and absorbing the multiplied unit value in the coefficient of the term.

In the physical analog of the system, if voltages are read in terms of the voltage unit, and time is measured in terms of the model's time unit, the equations of (b) describe directly both the system under study and the model. The analog's coefficient settings will be the coefficients of the normalized system. The normalized equations of the system described in the Appendix are shown with the block diagram in Fig. 10.

APPENDIX

Application of the Method

The methods described in the preceding text are best understood in terms of their application to an actual problem. The problem chosen is one in which two inertial driving sources, independently loaded, are coupled together. It is desired to study the manner in which load is shared in the transient state, and to determine (a) the most rapid method of returning the system to equilibrium, (b) how best to render the systems independent in the steady state, and (c) how to minimize stress on the connecting member.

Here the mechanical system of Fig. 9 is studied. By a simple transformation, a study could be made of many analogous paralleled systems, such as the parallel connection of two power systems[21], or the connection of a common member between two water supplies. By an extension of the techniques used here, systems with more than two sources could be studied.

The derivation of a block diagram for the system is as follows: (see Fig. 10)

a. The sum of the torques at each inertial member is obtained in an adding block, and a coefficient block is used to divide by the inertia, thus obtaining acceleration.

b. One integration of each acceleration gives the velocity of each end of the coupling member. For linear damping torque, proportional to speed, a coefficient block is inserted to multiply by the damping factor, and the resulting torque is brought to the appropriate torque adder.

c. Because the shaft is spinning at some nominal velocity, the actual position of each end is of less interest than the difference in position, since the stress on the shaft is proportional to the differential displacement. To this end, the net velocity is obtained in an adder, one of the velocities being applied negatively. Then the differential velocity is integrated to obtain relative angular position.

d. The differential position, multiplied by the stiffness of the shaft, gives the restoring torque at each end, as caused by the difference in position.

e. The differential position is measured and applied to a controller which affects motor torques at the sources, thus relieving the shaft of stress in the steady state. A simple linear controller utilizing proportional, integral, and derivative control is shown.

Appropriate disturbances might include sudden application of either or both loads, or a change in the driving torque of either source. These are applied as shown in the block diagram. Responses of any variable may be observed at the output of the appropriate block.

The effects of nonlinearities may be studied by inserting a nonlinear operational block in place of, or in series with, the appropriate coefficient block. Typical nonlinearities include limited torque in the driving sources, torsion of the shaft beyond the elastic limit, friction in the bearings, and so on.

Typical responses of the linear system are shown in Fig. 11.

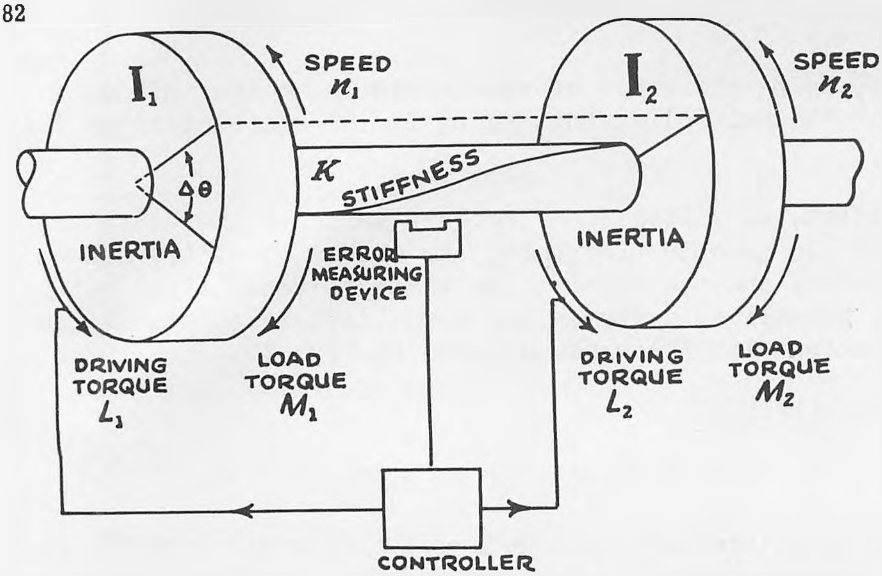
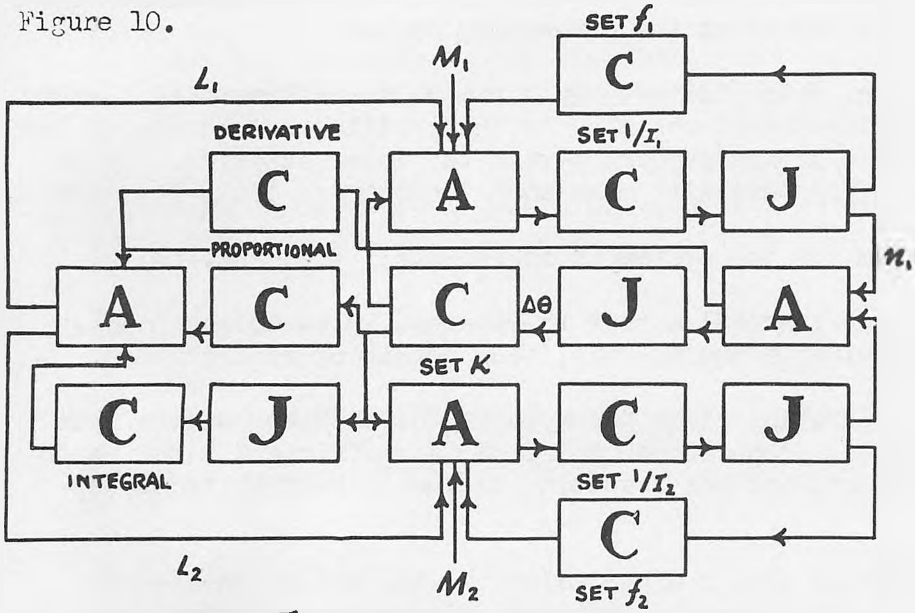


Figure 9. Coupled mechanical system.

Figure 10. Block diagram and, normalized equations for the system of Fig. 9, showing proportional, derivative, and integral control.

Figure 11. Responses for the system of Figs. 9 and 10 to step change in M_1. Shown are differential velocity and differential position.
(a) Uncontrolled response, equal inertias.
(b) Response under control, equal inertias.
(c) Uncontrolled response, $I_2 = 5\ I_1$
(d) Controlled response, $I_2 = 5\ I_1$

Figure 10.



NORMALIZED EQUATIONS:

$$I_1' \frac{d\bar{n}_1}{d\bar{t}} = \bar{L}_1 + \bar{M}_1 - f_1'\bar{n}_1 + K'\overline{\Delta\theta}$$

$$I_2' \frac{d\bar{n}_2}{dt} = \bar{L}_2 + \bar{M}_2 - f_2'\bar{n}_2 - K'\overline{\Delta\theta}$$

$$\overline{\Delta\theta} = \int(\bar{n}_2 - \bar{n}_1)\,d\bar{t}$$

$$-\bar{L}_1 = \bar{L}_2 = T_D'(\bar{n}_2 - \bar{n}_1) + C'\overline{\Delta\theta} + \frac{1}{T_I'}\int\overline{\Delta\theta}\,d\bar{t}$$

VARIABLES:

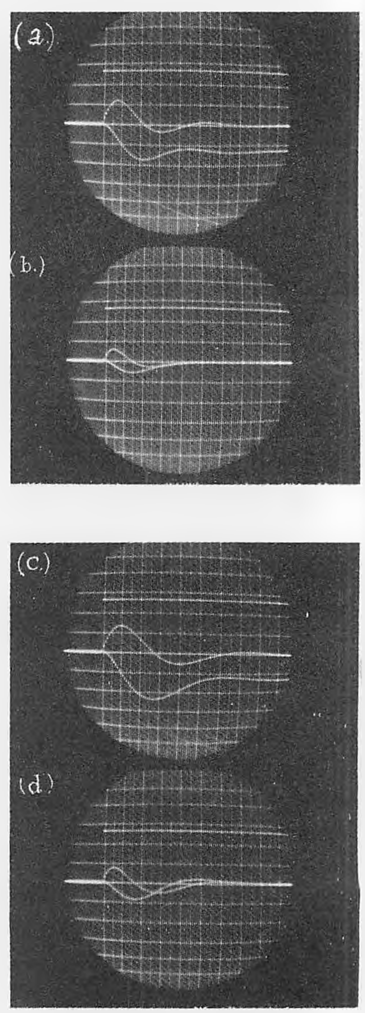
$\bar{n}_1 = n_1/N_0$
$\bar{n}_2 = n_2/N_0$
$\bar{L}_1 = L_1/L_0$
$\bar{L}_2 = L_2/L_0$
$\bar{M}_1 = M_1/L_0$
$\bar{M}_2 = M_2/L_0$
$\overline{\Delta\theta} = \Delta\theta/\theta_0$
$\bar{t} = \frac{t}{T_0}$

CONSTANTS:

$I_1' = I_1\ N_0/L_0T_0 = 1$
$I_2' = I_2\ N_0/L_0T_0 = \frac{I_2}{I_1}$
$f_1' = f_1\ \frac{N_0}{L_0} = \frac{f_1}{\sqrt{I_1K}}$
$f_2' = f_2\ \frac{N_d}{L_0} = \frac{f_2}{\sqrt{I_1K}}$
$K' = K\theta_0/L_0 = 1$

SIMPLIFYING RELATIONSHIPS:

$T_0 = \sqrt{\frac{I_1}{K}}$
$L_0 = K\theta_0$
$\theta_0 = N_0T_0$



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Product Engineering—March, 1954

Scale Factors for ANALOG COMPUTERS

A technique is developed which permits a simple visualization of the scale factor problem in analog computers. This method allows the operation of time scaling to be considered apart from scale factor determinations. Both operations become routine, simple, and fast. A typical problem is shown to illustrate the technique.

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Assistant Professor
Massachusetts Institute of Technology

THE ELECTRONIC ANALOG COMPUTER is now a widely accepted tool for research and development. Many papers and articles have described the manifold types in commercial production as well as those which have been built by research groups. Suffice it to say here that in an analog computer, measured quantities, usually voltages, are caused to vary in the same way as do variables in a physical system. Thus the analog "computes" the solution to a problem by faithfully reacting exactly as would the original counterpart. Results produced by an analog may be applied to an actual problem with an accuracy that depends on the precision of computation and the degree to which the analog models the counterpart. It is this second factor which is to be considered here.

To model a system, an analog must have the same functional form (form of differential equation) and its parameters must be numerically determined. To interpret results, factors relating analog variables to system variables must be established. Most references on this subject concern themselves with techniques for providing functional similarity but sometimes leave the problem of actually determining the numbers to the ingenuity of the user.

One approach to the problem is to make all system parameters and variables dimensionless. This technique is not difficult to apply and often reveals many basic factors of importance in a study. At the same time, however, the effect of a particular parameter may sometimes be obscured. Design problems require the ultimate return to dimensional values. In particular, the application of non-dimensional techniques when certain nonlinearities are present is often difficult. For these reasons it appears that a procedure based on direct equivalences between analog voltages and system variables has practical value.

The Dual Analog technique is such a procedure. The electronic analog representing a physical system may be made to operate either at the same speed as its physical counterpart or it may operate either faster or slower. An analog operating at physical system speed is called a true-time analog. The name fast (or slow)-time analog is applied to the analog which does not run at the same speed as the original system.

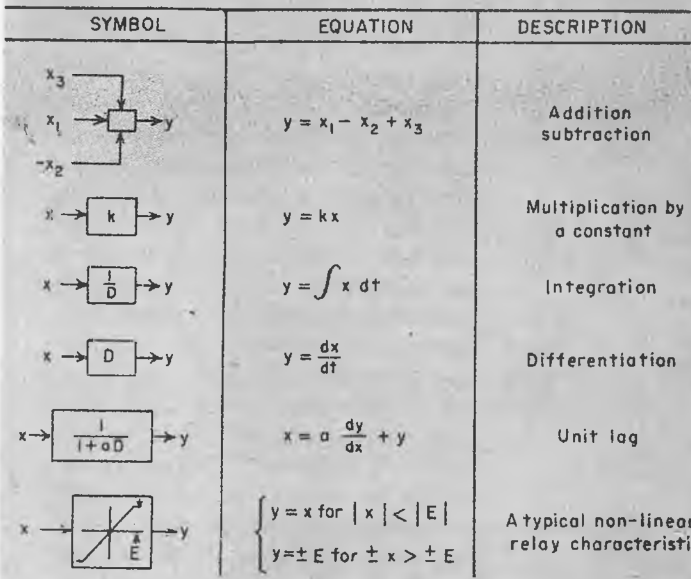
The importance of this time classification is derived from the fact that scaling of time does not occur between the true-time analog and its counterpart. Thus one need only concern himself with the determination of dimensional scale factors, determination which becomes a completely algebraic process.

It will be shown that once the true-time analog has been considered and scale factors determined, it is comparatively easy to account for slow or fast time operation.

The steps of the technique are then:

(1) Formulate a hypothetical true-time analog and determine scale factors and parameter settings.

(2) Transform the true-time set-up into an appropriate speed of operation.



Symbol	Equation	Description
	$y = x_1 - x_2 + x_3$	Addition subtraction
	$y = kx$	Multiplication by a constant
	$y = \int x\, dt$	Integration
	$y = \frac{dx}{dt}$	Differentiation
	$x = a\frac{dy}{dx} + y$	Unit lag
	$y = x$ for $\lvert x \rvert < \lvert E \rvert$; $y = \pm E$ for $\pm x > \pm E$	A typical non-linear relay characteristic

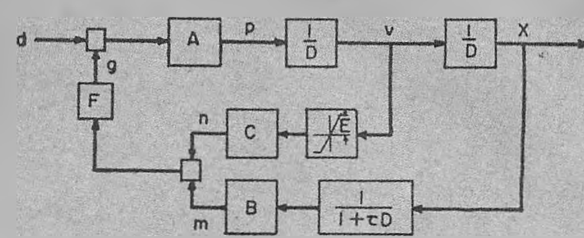
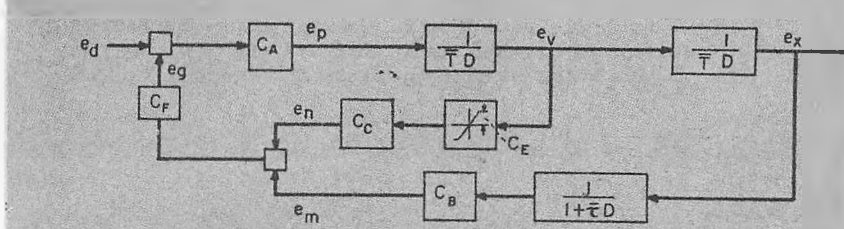


Fig. 1—Functional block diagram of a typical computer system. This represents a typical problem.

Fig. 2—An electronic analog of the functional system shown above, represented as a block diagram.



This procedure will be outlined in a somewhat general example and then illustrated with a numerical example.

Consider a typical problem involving addition and subtraction, multiplication by a constant, integration, simple or unit lag, and certain nonlinearities. Fig. 1 shows a functional block diagram of a hypothetical system. The various operational symbols are explained in the insert and some variables are indicated at appropriate points on the diagram.

In the physical counterpart, variables must be measured in a consistent set of engineering units. The system parameters will then be defined in terms of the units of the variables which they relate. Thus, for example, since

$$n = Cv$$

where $[n] = [\text{lb}]$, $[v] = [\text{ft/sec}]$

then: $$[C] = \left[\frac{\text{lb}}{\text{ft/sec}}\right]$$

where [] = units of the parameter enclosed by the brackets.

An electronic analog of the system of Fig. 1 can be represented in a similar block diagram. However, it will have two important differences:

(1) Voltage magnitudes are used to represent all system variables and;

(2) Constants having the units [time^{-1}] appear before each operation of integration. (See Table I.)

The analog block diagram might then appear as Fig. 2, where

C_A, C_B, C_C, C_F are analog coefficient settings,

$e_d, e_p, e_v, e_x, e_m, e_n, e_g$ are voltages representing variables,

$\bar{T}$ is the time constant of voltage integration ($\bar{T} = T$ on True-time analog

$\bar{\tau}$ is the time constant for electronic unit lag, ($\bar{\tau} = \tau$) on true-time analog

C_E is a limit in volts, and

$k_d, k_p, k_v, k_x, k_m, k_n, k_g$ are factors relating system variables to analog voltages.

Consider now the true-time analog of the hypothetical system. The diagram of Fig. 2 becomes such an analog when $\bar{T} = T$ sec and $\bar{\tau} = \tau$ sec assuming the physical system variations are measured in seconds.

Visualize the analog block diagram operating in a plane which is parallel to the system diagram. If each block diagram, system and analog, is disturbed at the same time, each voltage in the analog will be in phase with its analogous variable in the physical system. In fact, if each voltage were fed into a "transducer" having an appropriate constant and unit transformation, the outputs of these "transducers" would be at every instant equal to the variables in the physical system. These "transducers" may be represented by operational blocks and may be assigned values which are identically the scale factors to be determined.

If these scale factor blocks are added to the parallel plane diagram of Fig. 2, a three dimensional block diagram results which contains all relations involving both scale factors and parameter settings. It is very significant that such a diagram will be dimensionally homogenous throughout its entirety. Such a diagram is shown in Fig. 3.

The relations which involve the analog parameter settings (C_n, etc), the scale factors (k_n, etc) the physical system parameters (n, etc), and the analog integrator time constants (T) may be formulated by equating the "gain" obtained by moving between equal signs via the physical system, to the "gain" obtained by moving between the same equal signs via the analog. For *example in Fig. 3:*

Between two and three

$$A = 1/k_d \cdot C_A \cdot k_p$$

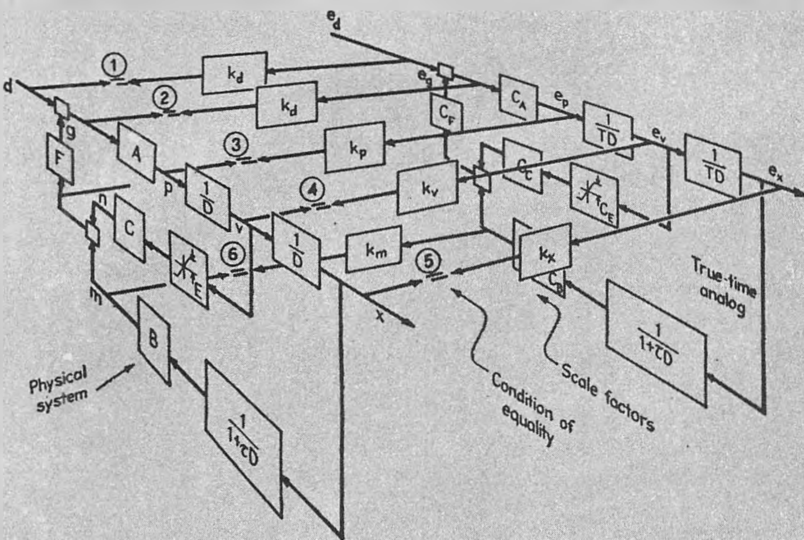


Fig. 3—Three-dimensional block diagram illustrates relationship between physical system and its true-time analog. It is not usually necessary to construct a diagram of this complex type to utilize the method involved.

similarly between two and four

$$A \cdot 1/D = 1/k_d \cdot C_A \cdot 1/TD \cdot k_v$$

and between four and five

$$1/D = 1/k_v \cdot 1/TD \cdot k_x$$

In most problems, the system parameters will be known numerically in terms of consistent units, however, the scale factors and analog parameter settings are to be determined. Since there will remain many more unknown than equations of the above type, it is necessary to designate certain unknowns and use a trial and error procedure to arrive at a final set of values. This operation is better illustrated in the numerical example which follows.

Some important conditions apply to the analog parameter settings which considerably restricts their range. These may be formulated as two requirements to be met in each setting:

(1) That it is a reasonable parameter setting, i.e., lies in the linear, accurate range of the coefficient unit, and,

(2) That it does not produce voltage variations which are too large to be handled by the computer or too small to be accurately measured.

Fortunately, both of these requirements are met by the single consideration that all coefficient settings be as close to unity as possible. Although this may seem a rather arbitrary requirement, when the action of most real systems in nature is formulated mathematically with an accuracy and significance consistent with the accuracy of measurement and determination of parameters, the so-called loop gains in a system will have similar orders of magnitude. The gain can usually be adjusted to a value near unity when time units are defined appropriately. In other words, a unit lag having a time constant 1/100th of another in a system will have little effect on the system's dynamic performance, or alternatively, a part of a system having a natural frequency 100 times that of another part will have small effect on studies made near the lower frequency. If, in an actual case, it is impossible to obtain coefficient settings near unity, (between say, 0.1 and 10) there is a strong possibility that some part of the problem is being formulated inconsistently.

The first step of the technique is now complete and there remains the problem of applying the results obtained for the hypothetical true-time computer to a fast or slow-time analog.

The key to this transformation lies in the fact that the analog can be considered as operating in any time units that may be desired. Thus it is con-

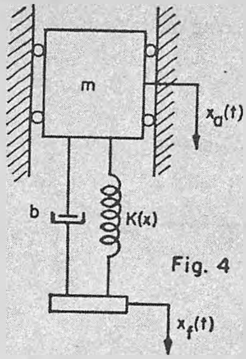
Table I — System Integration vs Analog Integration

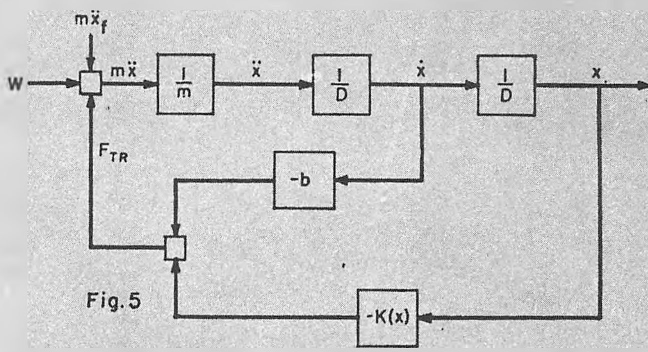
	System Integration of Velocity *	Analog Integration of Velocity **
Equation	$x = \int v\,dt$	$e_x = \frac{1}{\bar{T}} \int e_v dt$
Operation	$x = v \frac{1}{D}$	$e_x = e_v \frac{1}{\bar{T}D}$

* Units are so defined that constant of integration is not required.
** *Constant* of integration required for dimensional consistency.

Units

$x = [\text{in.}]$
$v = [\text{in./sec}]$
$e = [\text{volts}]$
$\bar{T} = [\text{sec}]$
$D = [\text{sec}^{-1}]$
$D = d/dt$
$\frac{1}{D} = \int dt\ [\text{sec}]$





Parameters and Inputs

$\frac{1}{m} = 386$

$b = 2.04$

$K(x) = 16 \rightarrow 30$

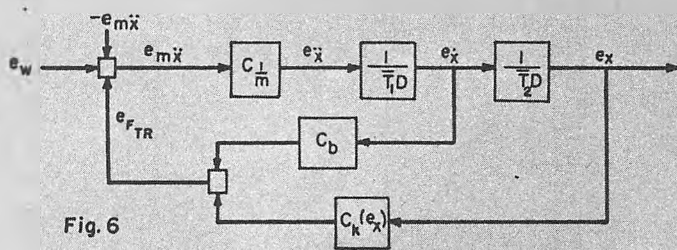
$m\ddot{x}_f =$ input

$W = 1.0$

Fig. 4—Schematic of the shock mounted damped structure used in text example.

Fig. 5—Functional block diagram for the system shown schematically in Fig. 4.

Fig. 6—Analog block diagram corresponding to the functional diagram of Fig. 5, above.



venient to define a new unit of time for the analog—

the *asec* (for analog-second)

let 1 sec $= k_t \cdot 1$ asec

thus $k_t = \frac{1}{1} \frac{\text{sec}}{asec}$ and is the ratio of true-time to analog time.

Consider an analog operating (integrating, differentiating, etc.) in *asec* units. Where the differential operator D had units of [sec^{-1}] in the physical system and true-time analog, it will now have units of [$asec^{-1}$].

As explained previously, Table I, it is necessary for the time constants associated with electronic integration to have the same units as those ascribed to the operator, (D). Thus, when all time constants in the fast or slow-time analog are given the same numerical values as those in the true-time analog, but in terms of asec unit, the resulting block diagram will be identical to that of the true-time analog. Accordingly, all scale factors and parameter settings previously arrived at, may be applied directly. One asec then represents one second.

The choice for k_t (time scale factor) usually depends on a knowledge of the true length of time over which events are to be studied as compared with the length of time events are to be studied on the analog. Often this is the ratio of the periodic disturbance duration in the analog to the length of time for a system to reach equilibrium.

Suppose, for example, that a 40 millisecond square wave disturbance is to represent 10 seconds of real time. Then,

$$1[asec] = 4 \text{ [milliseconds]}$$

and,

$$k_t = \frac{1[\text{sec}]}{1[asec]} = \frac{1[\text{sec}]}{.04\ [\text{sec}]} = 25$$

If each time constant of integration in the true-time analog *is* T [sec] then each similar time constant in the fast time analog should be $\overline{T}$ [*asec*] or $T/25$ [sec]. Other time constants as

Table II — Calculations for True-Time Analog

Path	Relationship	Chosen Values	Determined Values
(3) → (1)	$\frac{1}{k_t} \cdot C_b \cdot K_f = 2.04$	$k_f = 1 \left[\frac{\text{lb}}{\text{volt}}\right]$ $k_{\dot{x}} = 1 \left[\frac{\text{in/sec}}{\text{volt}}\right]$	$C_b = 2.04$
(1) → (2)	$\frac{1}{k_f} \cdot \frac{C_1}{m} \cdot k_{\ddot{x}} = 386$	$k_{\ddot{x}} = 100 \left[\frac{\text{in/sec}^2}{\text{volt}}\right]$	$\frac{C_1}{m} = 3.86$
(2) → (3)	$\frac{1}{k_{\ddot{x}}} \cdot \frac{1}{T_1 D} \cdot k_{\dot{x}} = \frac{1}{D}$		$T_1 = \frac{1}{100}$ [sec]
(4) → (1)	$\frac{1}{k_x} \cdot C_k(e_x) \cdot k_f = (16 \rightarrow 30)$	$k_x = \frac{1}{10} \left[\frac{\text{lb/in.}}{\text{volt}}\right]$	$C_k(e_x) = 1.6 \rightarrow 3.0$
(3) → (4)	$\frac{1}{k_{\dot{x}}} \cdot \frac{1}{T_2 D} \cdot k_x = \frac{1}{D}$		$T_2 = \frac{1}{10}$ [sec]

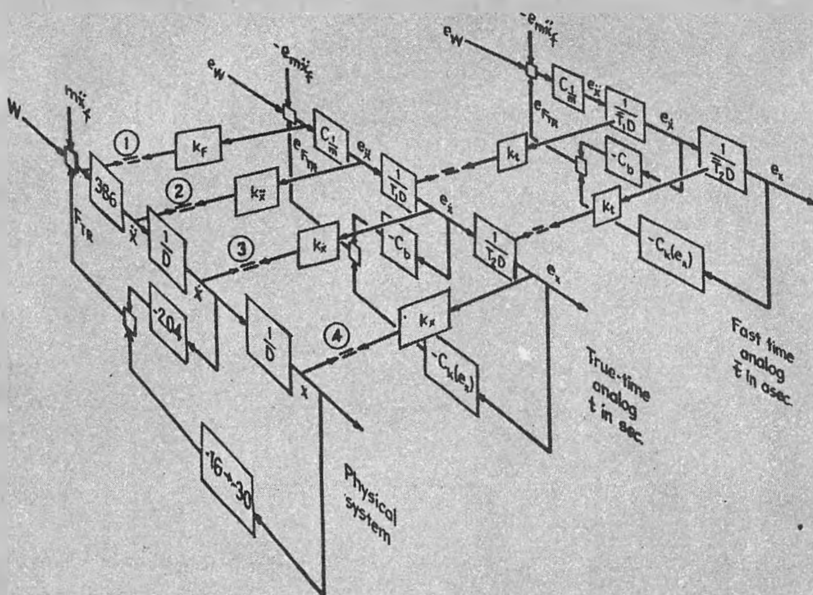


Fig. 7—Three-dimensional block diagram for illustrative example represented by Fig. 4. Physical, true-time and fast-time systems are shown.

in a unit lag should have the same numerical value in asec units and therefore be reduced accordingly.

In the cases where a desired time constant may not be set on an integrator, a convenient value may be set and thought of as the product of the desired time constant and a numerical coefficient. The introduction of this coefficient will merely change the actual setting of some series coefficient and possibly a local scale factor. Conversely, time constants integration and consequently time scale factors may be changed by introducing a numerical gain before each fixed time constant integrator. Of course, time lag time constants must be actually changed.

The time scale transformation which is concerned only with the time constants may also be indicated by operational blocks connecting the true-time block diagram in one plane with the fast or slow-time block diagram in a third parallel plane. This possibility has been indicated in the diagram for the numerical example which follows:

EXAMPLE:

A mass m is supported on a shock-mount structure having essentially constant damping but non-linear spring characteristics, Fig. 4. The frame is to be subjected to a time function, x_f (t), as recorded at an actual location. It is desired to perform an analog study which will yield solutions for the total force (F_{TR}) transmitted across the shock-mount, (not including steady weight) as a function of time.

The data given below are the parameters to the example shown in Fig. 4. All the necessary information to set-up the problem on the computer is included.

Given data:

a) $m = \frac{1}{386} \frac{\text{lb sec}}{\text{in.}}$; $(W = 1 \text{ lb})$

b) ζ (Damping Ratio) $= 0.2$;

$$\left(\zeta = \frac{b}{2\sqrt{Km}}\right)$$

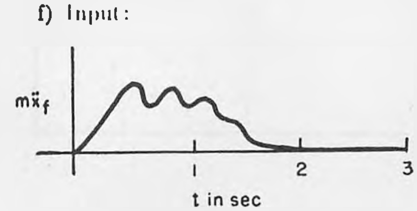
c) x_0 (static deflection) $= \frac{1}{16}$ in.

d) $b = 2.04 \left[\frac{\text{lb sec}}{\text{in.}}\right]$;

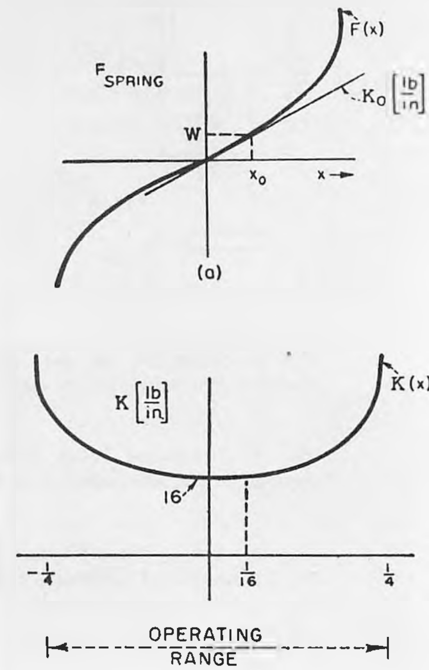
(Based on $K = 16 \left[\frac{\text{lb}}{\text{in.}}\right]$ at x_0)

e) $K = 16$ to $30 \left[\frac{\text{lb}}{\text{in.}}\right]$ (for the operating range)

f) Input:



g) Spring characteristics:



Letting $x_a = x + x_f$, then $m\ddot{x}$ (acceleration) $+ c\dot{x}$ (velocity) $+ K(x) + K(x_0) = m\ddot{x}_f$ (acceleration) $+ W$, the functional block diagram for the system can now be drawn (Fig. 5).

The analogue block diagram for either "true-time" or "*asec*-time" is given in Fig. 6. Fig. 7 shows the three-dimensional block diagram for this system. Calculations for the true-time data are given in Table II.

There are, of course, many solutions depending on chosen values. Other information such as a predetermined scale factor (k_f) for an input function generator might lead to a different set of results.

Time Scaling For Fast-Time Analog

If the integrator being used may be set at 0.0004 and 0.004 [sec], it is convenient to let

$\bar{T}_1 = 0.0004$ sec and $\bar{T}_2 = 0.004$ sec.

Then, $k_t = \frac{0.0100}{0.0004} = 25$ and

1 [*asec*] = 0.04 [sec] = 40 [millisec]

Thus, a square wave disturbance of 40 millisec duration corresponds to 1 sec in real time. In this example, the natural period of vibration is about 1/20 sec. Thus about 20 cycles could appear on the analog display. This should be satisfactory for the given damping ratio.

Editor's note:

The following remarkable document has had only limited circulation in the United States. It arrived unheralded at GAP/R headquarters and constitutes in our minds an outstanding example of proficient application.

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Technical Note No. M.S.12

July, 1953

ROYAL AIRCRAFT ESTABLISHMENT, FARNBOROUGH

The Philbrick Electronic Analog Computor, and its use for the Solution of the Dynamical Equations of an Aircraft-Plus-Autopilot

by

D. J. Foster, B.Sc.(Eng.), Grad.R.Ae.S.

SUMMARY

This note describes the Philbrick Electronic Analog Computor, and explains how problems may be scaled and expressed as block diagrams ready to be set up using computor components. A list of the components in the possession of Maths. Services Dept., R.A.E. is included, and a specimen problem is worked through to indicate the capabilities of the calculator. The Philbrick Analog Computor is very suitable for problems involving a set of differential equations with time as the independent variable, where solutions for a large number of different parameters are required and where an accuracy of 5% is sufficient.

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1 General

The Philbrick Analog Computor* is a flexible calculating machine, specially suitable for the solution of sets of ordinary differential equations (linear and non linear) with time as the independent variable. The components work entirely electronically, having no moving parts such as motor drives, servo gears or mechanical recorders. This enables a high speed of operation to be employed, so that a problem set upon the computor is solved repetitively 50 times a second. The user can explore the solutions of the equations set upon the computor as fast as he can adjust parameters on the instrument scales.

The equations are built up using a combination of individual units, which each performs one particular function by operating upon voltages representing the dependent variables. For example:-

The Coefficient Unit performs the operation

$$\pm y = C.x. \text{ (C variable from 0 to 100)}$$

The Adding Unit performs the operation

$$\pm y = x_1 + x_2 + x_3 + x_4 + C \text{ (C variable from -10 to +10 volts)}$$

The Integrating Unit performs the operation

$$\pm y = \frac{1}{T_o} \int x \, . \, dt$$

where $T_o = 0.0004$ or 0.004 secs.

Most linear differential equations may be built up using a combination of these three units alone, but the other units are supplied for non linear work, such as multipliers, function generators, hysteresis components and inert zone components.

In its physical form, each component is unidirectional; information flows only from input to output. The output signal capacity of each component is such that the load imposed by the input of another component is negligible. Any component may therefore instruct any number of others without correction. Initial conditions are established by an external voltage wave - usually a step - applied as a stimulus, and the appropriate variable voltages are displayed on an oscilloscope screen to give the solutions.

2 Quantitative data and accuracy

The nominal voltage range for all variables is 100 volts, minus 50 to plus 50, while the normal computing epoch is 5 milliseconds. Problems have to be scaled to allow them to conform to these computor requirements (see Scale Factors, para.4).

The following figures relating to the accuracy of the machine are taken from the makers handbook[1]. No direct check has been made on them, but some remarks on the overall accuracy of a particular problem are included in conclusions, para.11.

* Manufactured by Geo. A. Philbrick Researches, Inc., 230 Congress Street, Boston 10, Mass., and known by the makers as the "GAP/R".

Errors due to electronic variations,	1%
Tolerance on circuit elements,	1%
Fidelity to waveform,	0.1%
Resolution,	0.1%
Reproducibility of parameters between successive solutions	0.01%
Thermal drifts within computing epoch	Negligible

3 The computor components

3(a) The K.3 series of components

Each K.3 component is a self-contained operational unit, housed in a cast aluminium case. At the back, 5 pin input and output connections supply power, and permit cable connections in cascade from one unit to the next. On the front, one to four input jacks and two output jacks provide for computing signal connections via standard cables. The output jacks afford direct and inverted signals, and are usable simultaneously. An indicating dial on each component serves for setting characteristics, and a lamp lights when the output signal reaches its limiting value of plus or minus 50 volts. A list of K3 components owned by Maths Services department R.A.E. is given in Table I, together with their functions.

3(b) The K.4 series of components

These components perform operations which would normally require 4 or more interconnected K.3 components, or else perform a mathematical operation which necessitates more electronic components than will fit into a K.3 size box. A list of K.4 components owned by Maths. Services Dept., R.A.E. is given at the end of Table I, together with their functions.

3(c) The central component

This device supplies a calibrated and adjustable initiating signal (or stimulus) of special type, called the Delta wave by the makers, but best described as a double step wave (see Fig.1). It is fed via push buttons to any one of four outputs. The step wave is made up of four sections each of 5 milliseconds duration; a neutral at zero and positive and negative departures from it. This signal is used to impose initial conditions, and will serve to determine the magnitudes of computor solutions by comparison on a cathode ray oscilloscope.

The Central component has a d.c. metering system of dual sensitivity, which may be used for d.c. voltage measurements or for the provision of steady voltages for test or computation. It also has a 25 c.p.s. electronic switch, which preserves d.c. and provides for the synchronized display of any two variables.

There is a variable amplitude 50 c.p.s. sinusoidal signal which appears to have its main application with the K.4 Multiplying and Function Fitting components. A double push-button switching system is provided so that any of 8 selected signals may be carried to either of 2 oscilloscope inputs, recovered to zero d.c. average for stability of viewing. Lever switches permit substitution of either the step or sine wave in place of either signal being displayed. Standard input and output jacks are provided for all signals to and from this unit.

3(d) Regulated power supply

This unit supplies power to operate the computor, but only the d.c. output is regulated. It was designed to run off the American supply system of 115v. 60 c.p.s., but the unit at the R.A.E. has been run quite successfully on 115v. 50 c.p.s., a 2-1 step-down transformer being interposed between the main supply and the power pack. The output from the unit is 115v. 50 c.p.s. (directly coupled to mains input), ± 300v. 400 mA_{max} d.c. regulated. A time delay switch is built into the unit, so that after switching on the 115v. a.c. (which feeds all valve heaters throughout the computor), two minutes elapse before the 300v. d.c. can be switched on.

With 1 power-pack, 1 central component and 25 k-3 components in circuit, the power consumed was 8.2 amps at 115 volts ≏ 1 K.W. As this is approximately half the total equipment, for all the equipment to be in use at once, two 1 K.W. transformers of 2-1 step down ratio will be required.

4 Scale factors and initial conditions

4(a) Time scale

In general, the problem time interval with which one is concerned will be greater than the time taken for the computor to solve the problem set up on it. This necessitates the choice of a time scale factor to govern the transition from problem time to computor time. The most usual case will be the one in which the integrating units are being fed with a 100 c.p.s. square wave clamping signal, while the 50 c.p.s. step function is being used to apply successive positive, zero, negative and zero stimuli to some component. Thus the time for which the computor gives a solution is $\frac{1}{2} \times \frac{1}{100} = \frac{1}{4} \times \frac{1}{50} = 0.005$ secs, the integrator outputs being clamped at zero while the clamping wave is positive and the input stimulus is zero (see Fig.1).

Let t denote the computor time variable

" τ denote the problem time variable

" T_o denote the integrating unit time constant ($T_o = 0.004$ or 0.0004 secs)

" τ_o secs problem time be equivalent to T_o secs computing time.

Then t secs computing time are equivalent to $t \cdot \frac{\tau_o}{T_o}$ problem secs, which is denoted by τ .

The integrating units perform the operation

$$\pm E_y = \frac{1}{T_o} \cdot \int E_x \, dt$$

where E_y and E_x are voltages representing the variables y and x .

Therefore
$$\pm y = \frac{1}{T_o} \int x \cdot \frac{T_o}{\tau_o} \, d\left(\frac{t\tau_o}{T_o}\right)$$

$$\pm y = \frac{1}{\tau_o} \int x \cdot d\tau$$

which is the integration it is desired to perform, divided by the factor τ_o.

The maximum value of t is 0.005 secs, while the minimum value of T_o is 0.0004 secs. Suppose it is required to observe the solution of a problem for 20 minutes. Then

$$20 \times 60 = \frac{0.005\ \tau_o}{0.0004}$$

$$\text{or } \tau_o = 96 \text{ secs.}$$

Hence all integration signs in the problem equations must have a coefficient of $\frac{1}{96}$ applied to them.

4(b) Electrical variables scale

Case (i) For a linear problem in which only one of the initial conditions is non-zero, it is possible to avoid calculating the scales of the voltages representing the variables in the problem. The stimulus to the problem is increased in magnitude until one of the units is on the point of overloading, and then the magnitudes of the solutions can be expressed as a fraction of the magnitude of the stimulus, by comparison through the electronic switch to an oscilloscope. It is advisable, after finding the maximum initial conditions, to check the signal at all outputs with an oscilloscope to see that no important variable is represented by a very small voltage. If this does occur, it may be possible to improve matters by a simple rearrangement of order of the components, e.g. interchanging a coefficient unit and an integrator.

Case (ii) For a linear problem in which there are two or more non-zero initial conditions, the magnitudes of these must be related according to the scale of the associated variables in the problem itself. Comparison may still be made directly between the magnitude of the solution and the magnitude of the initial condition.

Case (iii) For a non-linear problem, it is necessary to assign extreme values to the variables of the problem. Generally each such variable will correspond to a voltage in the computor, and the extremes will be represented by values of voltages not exceeding, and preferably close to, plus and minus 50 volts. Scale factors for variables are thus already determined as - say - 0.01 radian/volt. If the initial value of a certain angle was required to be 0.1 radian, then a step function of magnitude 10 volts would have to be added onto the computor signal representing the angle.

5 Block diagrams

In order to facilitate the arrangement of computor components to simulate the problem represented by the given set of equations, block diagrams are drawn. Each block of a diagram represents one elementary mathematical operation, such as addition, multiplication or integration. Since computor components are available which also perform one elementary mathematical operation, the transition from block diagram to computor set-up is direct.

In setting up block diagrams, one may almost always proceed thus:- Assume the highest derivative (order n) of a function is available as a signal, and integrate it n times, by using n integrating blocks. This gives all the lower derivatives including the zeroeth (i.e. the function itself). With these signals in combination, one supplies the assumed nth derivative as expressed by the differential equation solved explicitly for that quantity. With simultaneous differential equations, cross connection between equations will of course be required.

Example 1 It was required to find a circuit which would shift the phase of a quantity without altering its magnitude. It was known that by placing a capacitor, C, and a variable resistor, R, in series, the potential at their junction may be changed from 180° out of phase to in phase with the voltage by varying the resistor from zero to infinity. By considering this circuit, the equation relating the input x and the output y is found to be:-

$$\frac{1}{RC}\int (x - y)\, dt = (y + x)$$

For an input of form $x = a \sin \omega t$, with $|x| = |y|$,

the phase difference is $2 \operatorname{Tan}^{-1} (\omega RC)$

The block diagram for this equation is given in Fig.2 of the appendix.

Example 2 To solve

$$\ddot{x} + P\dot{y} + Qy = 0 \;\ldots\ldots\; \text{(i)} \qquad \text{given that } \dot{x} = \dot{y} = y = 0), \; x = X) \text{ at } t = 0$$
$$\ddot{y} + R\dot{x} + Sx = 0 \;\ldots\ldots\; \text{(ii)}$$

From (i) $- P\dot{y} - Qy = \ddot{x} \;\ldots\ldots\; \text{(iii)}$

From (ii) $- R\dot{x} - Sx = \ddot{y} \;\ldots\ldots\; \text{(iv)}$

Equations (iii) to (iv) are shown as block diagrams in Fig.3 of the appendix.

Since the integrators produce an output which starts from zero at t = o, whereas it is given that $x = X$ at this instant, the output of the $\dot{x}$ integrator must be regarded as $(x - X)$, and the constant term X must be added in separately. Actually, the external source shown, and the coefficient box are not strictly necessary, as the adder boxes contain an adjustable source of constant voltage.

6 Repetitive solutions and clamping

If a problem is set up using computor components, and fed by the double step wave, then every step of the wave will supply a stimulus to the problem, causing a corresponding disturbance in the solution. If at the end of one computing epoch, the solution displayed has not reached a zero value, then the succeeding solution will be in error, as it will not have started from zero. If the solution is divergent, then the initial stage of the divergence, which is of interest, is completely lost after the first solution. To overcome this difficulty, the output of each integrating unit may be clamped at zero for the duration of alternate computing epochs. This is performed by feeding the upper input of each integrator with a 100 c.p.s. square wave, which alternates between plus and minus 50 volts. During the positive part of each cycle, the integrator output is clamped at zero, while during the negative part of each cycle free integration is allowed. The 100 c.p.s. square wave is generated from the 50 c.p.s. double step wave in such a manner that the square wave is positive during the zero intervals of the step wave. A special component is supplied by the Philbrick Company for this purpose, but by an oversight was not ordered. A unit has been made in the R.A.E. to fulfil this function.

An illustration of the relative phases of stimulus, clamping signal and a typical solution are given in Fig.1.

7 Introduction

I.A.P. Department, R.A.E., wished to investigate the response of an aircraft and the behaviour of a gyroscope used in an automatic pilot, either when there was a sudden demand for an attitude change through the autopilot, or when the aircraft flew into a horizontal gust. For each case, there were two constants relating to the strength of gyro monitoring signals. Seven different values were allotted to each of these constants and it was desired to observe the effects of all possible combinations. The total number of cases to be investigated can be seen to be of the order $2 \times 3 \times 7 \times 7 = 294$. To have solved the dynamical equations by classical methods would have taken about a year, allowing one day for each case. The use of a fast machine to solve the equations was essential, and enabled solutions to be obtained in just over two weeks.

8 The dynamical equations

The equations of motion in the plane of symmetry are:-

$$\dot{U} = x_u . U + x_w . W - K . \theta$$

$$\dot{W} = z_u . U + z_w . W + q + z_\eta . \eta$$

$$\dot{q} = - x . U - \chi . \dot{W} - \omega . W - \delta . \eta - \nu . q$$

$$\dot{\theta} = q$$

$$\dot{\varepsilon} = - m . \varepsilon - \left(\frac{m - a}{K}\right) \dot{U}$$

$$\eta = G(\theta - \theta_D - \varepsilon)$$

Where the variables are:-

$$U = \frac{\text{Incremental forward velocity}}{\text{Initial forward velocity}}$$

$$W = \frac{\text{Vertical velocity}}{\text{Initial forward velocity}}$$

θ = Pitch angle

q = Angular velocity (radians/airsec)

η = Elevator movement

ε = Gyro error

θ_D = Demanded pitch change.

The 'dot' notation implies differentiation with respect to time in airsecs, where in this problem 1 airsec = 3.69 secs.

The constants are (a) aircraft

$x_u = -0.022$	$z_u = -0.48$	$x = 0$	$\delta = 104$
$x_w = -0.020$	$z_w = -2.9$	$\chi = 1.75$	$\nu = 2.72$
$K = 0.15$	$z.\eta = -0.2$	$\omega = 57$	

(b) autopilot

G = 1 or 2, m = 0.0154, 0.0308, 0.0615, 0.0923, 0.123, 0.1845, 0.369

a = 0, 0.25m, 0.5m, 0.75m, m, 1.25m, 1.50m.

It was desired to observe the long period oscillation of the aircraft, which meant that a solution would be required for about 15 minutes problem time.

Now 15 minutes = 900 secs = 244 airsecs.

Therefore
$$244 = \frac{0.005\ \tau_o}{0.0004}$$

$$\text{or } \tau_o = 19.52.$$

For convenience in rescaling the problem, τ_o was taken as 20. Then since T_o secs computing time is equivalent to τ_o secs problem time, and the computing interval is 0.005 secs while T_o = 0.0004 secs, then the computing interval is equivalent to $20 \times \frac{0.005}{0.0004}$ airsecs = 250 airsecs.

As explained in paragraph 4(a), all derivatives with respect to time must be divided by τ_o, so that $\dot{U}$ becomes $0.05\dot{U}$ etc. Taking the case when G = 1, the rescaled equations become:-

$$0.05\dot{U} = -0.022U - 0.02W - 0.15\theta \quad \text{(i)}$$

$$0.05\dot{W} = -0.48U - 2.9W + q - 0.2\eta \quad \text{(ii)}$$

$$0.05(\dot{q} + 1.75\dot{W}) = -57W - 104\eta - 2.72q \quad \text{(iii)}$$

$$0.05\dot{\theta} = q \quad \text{(iv)}$$

$$0.5\left\{\dot{\varepsilon} + \frac{m - a}{0.15} . \dot{U}\right\} = - m . \varepsilon \quad \text{(v)}$$

$$\eta = (\theta - \theta_D - \varepsilon) \quad \text{(vi)}$$

Now, the output obtained by integrating equation (ii) is 0.05W, while a term of 57W is required for substituting in equation (iii). This necessitates a gain of $\frac{57}{0.05}$ = 1140; a similar gain is required to obtain η from q via θ. As a computing component will overload when its output voltage reaches 50 volts, then a random voltage of only $\frac{50}{1140} \simeq 0.05$ volt in the output giving 0.05W is sufficient to overload the coefficient components forming 57W. As all computing components contain d.c. amplifiers, which are liable to drift, a random voltage of 0.05 volt is quite likely to be found. The above equations were set up on the computor, and several units became overloaded even without any input stimulus being applied.

There were two possible solutions to this difficulty:-

(1) To increase the computing time interval, thereby reducing the scale factor τ_o,

(2) To approximate the equations, which are known to contain long and short period oscillations, to eliminate the short period terms.

By using the electronic switch, a 25 c.p.s. square wave can be obtained, which, if used for clamping the integrators, would increase the computing interval 4 times. In a less extreme case this might be a solution, but in this case would still necessitate a gain of 250, which is too high. (Trouble is likely to be experienced if gains of over 100 are required.)

The second approach must therefore be adopted. The approximate dynamical equations are:-

$$\dot{U} = x_u \cdot U + x_w \cdot W - K \cdot \theta$$

$$\dot{W} = z_u \cdot U + z_w \cdot W + q$$

$$W = -\frac{\delta}{\omega} \cdot \eta$$

$$\eta = G(\theta - \theta_D - \varepsilon)$$

$$\dot{\varepsilon} = -m \cdot \varepsilon - \left(\frac{m - a}{K}\right) \dot{U}$$

$$\dot{\theta} = q$$

It can be seen that the large coefficients δ and ω have become amalgamated into a single coefficient of their quotient. Eliminating η and q from the equations, putting in numerical values for the coefficients, and rescaling, we have:-

$$\dot{U} = -0.44U - 0.4W - 3\theta$$

$$0.1(\dot{W} - \dot{\theta}) = -0.96U - 5.8W$$

$$W = -1.82(\theta - \theta_D - \varepsilon)$$

$$0.1\left\{\dot{\varepsilon} + \frac{(m - a)}{0.15}\dot{U}\right\} = -2m \cdot \varepsilon .$$

The block diagram for these equations is given in Fig.4, while a typical set of solutions is shown in Fig.5.

9 Display and recording of solutions

The solutions of the equations were displayed on a blue screen cathode ray oscilloscope, and photographed on 35 mm film, using an F.73 camera. The fastest exposure permissible was found to be $^1/50$ sec at f3.5 on Ilford H.P.3 film developed in Kodak D.76 developer, with the camera 6" from the screen. The film was subtitled at intervals along its length, indicating the change in parameters. The subtitles were written on paper, held in contact with the oscilloscope screen and illuminated by an 0.80 watt bulb placed just behind the camera, when it was found that the same exposure could be given.

10 Further developments

The frequency response of the aircraft to sinusoidal gusts will be determined by removing the clamping signal from all integrating components, and feeding the computor from an oscillator.

11 Conclusions

The Philbrick Analog Computor is readily adapted to the solution of many problems which may be reduced to sets of ordinary differential equations. It is very easily set up, and an inexperienced person will become accustomed to operating it in a few days.

The calculator may not be able to solve problems requiring a long time scale and having simultaneous equations involving large gains between equations. That is, equations representing a mixture of very fast and very slow modes of motion; however these equations are usually separated into fast and slow parts.

The number of computing elements in the possession of Maths. Services Department, R.A.E. will be found sufficient for the solution of quite complex problems, but it seems that there are insufficient Coefficient, Integrating and Adding Components for large problems. If large sets of linear equations are to be solved, it is suggested that a further 2 Integrating Components, 2 Adding Components and 5 Coefficient Components be purchased. For non-linear problems, integrators are likely to be the most urgently needed units.

From a digital check on some results obtained from the computor, the accuracy appears to be within 5% with the set up used for the autopilot problem. Probably more important is how accurately the solutions may be determined after being photographed from a C.R.O. screen, but for most purposes the accuracy appears to be ample.

For problems involving a complex set of differential equations, (linear or non-linear) with time as the independent variable, which require solutions for a large number of different parameters, where an accuracy of 5% is sufficient, the Philbrick Analog Computor is ideally suitable.

TABLE I

List of Philbrick Computing Components in Possession of Maths. Services Department, R.A.E.

Name of Unit	Designation	Number owned by R.A.E.	Remarks or Operational Equation
Regulated Power Supply	RS	2	Input 115V. 50 c.p.s. Output 115V. 50 c.p.s. " 300V. d.c. regulated.
Central Component	CC	2	Generates 50 c.p.s. Delta wave, 50 c.p.s. Sine Wave, Incorporates 25 c.p.s. electronic switch, and provides for display of two outputs selected from up to eight inputs.
Adding Component	K3-A	8	$\pm y = x_1 + x_2 + x_3 + x_4 \mp X$ where X = d.c. voltage up to 10 volts.
Coefficient Component	K3-C	15	$\pm y = C \cdot x$ C variable from 0 to 100, with centre dial posn. of 1.0.
Integrating Component	K3-J	6	$\pm y = \frac{1}{T_o} \int x \, dt$ $T_o = \infty$, 0, 0.0004, 0.004 or 1.0 secs.
Augmenting Integrator	K3-K	2	$\pm y = x + \frac{1}{T_o} \int x \, dt$ T_o variable from 0 to 0.0004 secs or $T_o = \infty$.
Differentiating Component	K3-D	1	$\pm y = T_o \frac{dx}{dt}$ T_o = 0.0004 secs. Also approximate derivative setting available.
Unit-lag Component	K3-L	2	$y + T_o \frac{dy}{dt} = \pm x$ T_o variable from 0 to 0.0004 sec.
Bounding Component	K3-B	1	$\pm 2y = \lvert x + B \rvert - \lvert x - B \rvert$ B is adjustable from 0 (no output) to 50V. (unbounded output).

TABLE I (CONTD)

Name of Unit	Designation	Number owned by R.A.E.	Remarks or Operational Equation
Backlash Component	K3-H	1	This unit transmits a signal acted upon by a travelling inert zone. Output follows input, after a sufficient change, but remains behind a prescribable amount. Upon reversal of the input, the output is stationary until the input has proceeded by the prescribed amount in the reverse direction. The degree of backlash is continuously adjustable on a 0-100 linear dial from zero to a maximum of 10% of the full excursion.
Component Inert-zone	K3-Z	1	$\pm 2y = 2x + \left\lvert x - \frac{D}{2} \right\rvert - \left\lvert x + \frac{D}{2} \right\rvert$ This unit suppresses a central band of variation of the input signal, outside of which surplus variations are transmitted at unit sensitivity. The zone or band is adjustable by means of a 0-100 linear dial from zero to 10% of the full signal range.
Clamping Unit (The Philbrick Co. supply an Absolute Value Component which performs this function)	- K3-V	1	Converts a step wave to a 100 c.p.s. square wave of amplitude $\pm$ 50V. which, when fed into the clamp (upper) input of the integrating unit, maintains the output at zero for the +ve part of the cycle and permits free integration over the -ve part of the cycle. Essential for use with problems having divergent solutions.
Connecting jack-box	K3-4	4	Has four rows of plug sockets, each row consisting of four sockets with common earths and common plug connectors.
Dynamic Component	K4-DY	1	$A_2 T_o^2 \frac{d^2y}{dt^2} + A_1 T_o \frac{dy}{dt} + A_o y = x(t)$ where T_o is 0.0004 secs and A_o, A_1, A_2 are directly calibrated on 0-100 Coefficient type dials. The input is x, and outputs of $\pm$ y, $\pm \dot{y}$ and $+\ddot{y}$ are obtainable. It is difficult to use this unit in conjunction with others because there is no provision for clamping the output, but some problems reduce directly to the above form.

TABLE I (CONTD)

Name of Unit	Designation	Number owned by R.A.E.	Remarks or Operational Equation
Multiplying Component	K4-MU	2	This component comprises a pair of independent multiplying units $\pm \frac{y}{10} = \frac{x_1}{10} \cdot \frac{x_2}{10}$; $\pm \frac{\nu}{10} = \frac{U_1}{10} \cdot \frac{U_2}{10}$ With the multiplier one may vary the coefficient of a variable in an equation in dependence on another variable, and non-linear arrangements may thus be assembled. With 2 units working together, the product of 5 quantities may be obtained, or digital powers up to the 5th order computed.
Functional Component	K4-FF	2	This 'Function Fitter' is designed to cause the output voltage to follow an assignable function of the input voltage. It utilizes an approximation based on ten connected line segments, each of which is adjustable both as to slope and length. The equation is $y = f(x)$ where $f(o) = 0$. The makers claim many applications for this component in the synthesis of non-linear systems. Whenever the corners are admissable it is possible to incorporate functions like sine, arc tangent, $^2/_3$ power etc., but perhaps more significant is the ability to fit empirical functions which originate from experimental data.

FIG. 1.

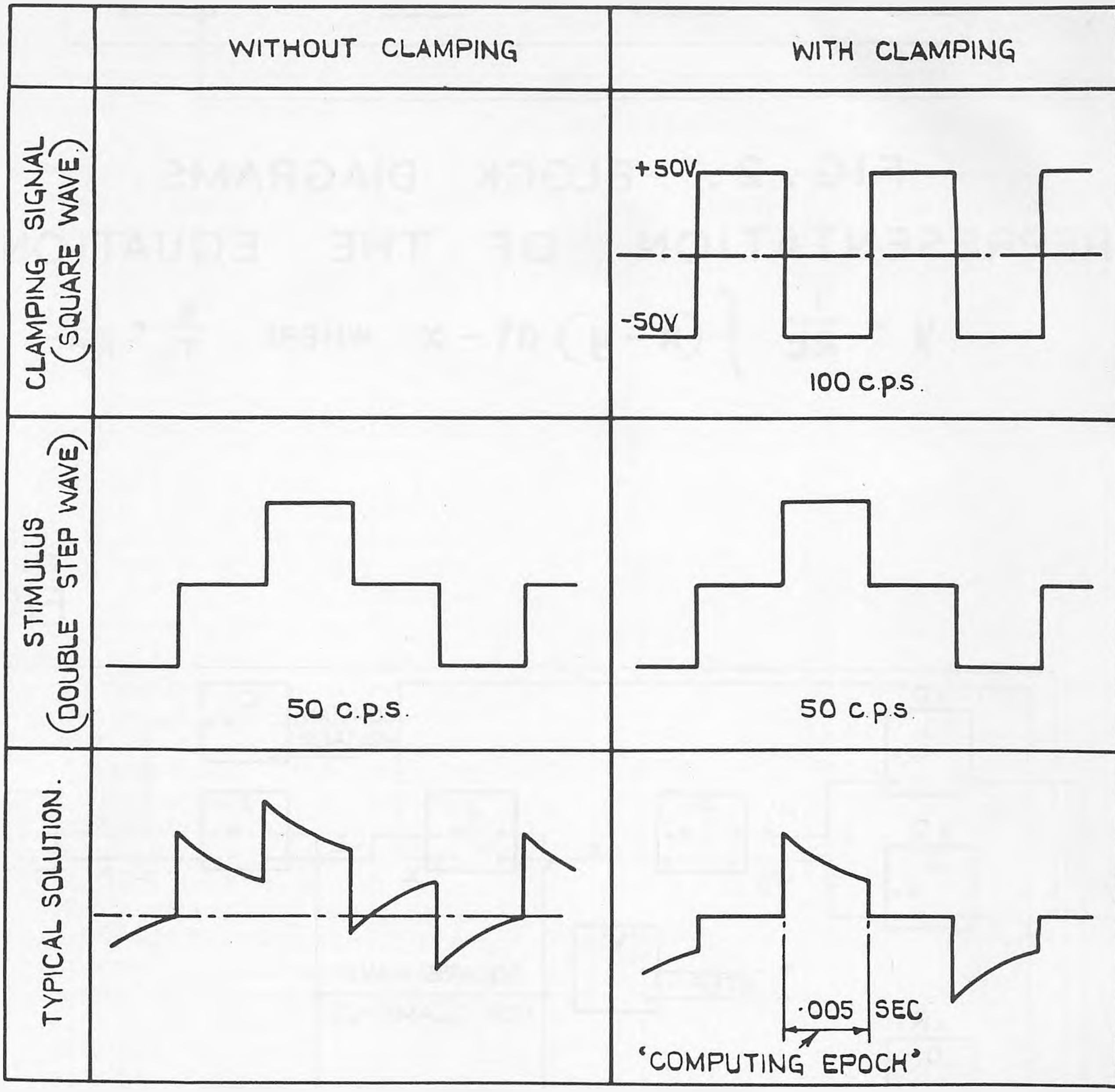
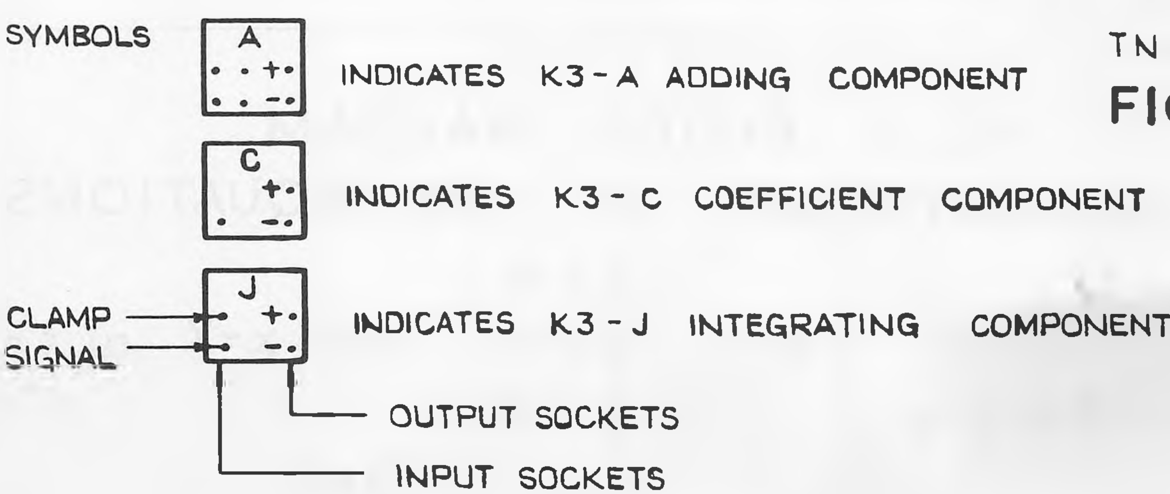


FIG. 1. RELATIVE PHASES OF CLAMPING SIGNAL, STIMULUS AND SOLUTION.

TN MS. 12.

FIG. 2.



ETC.

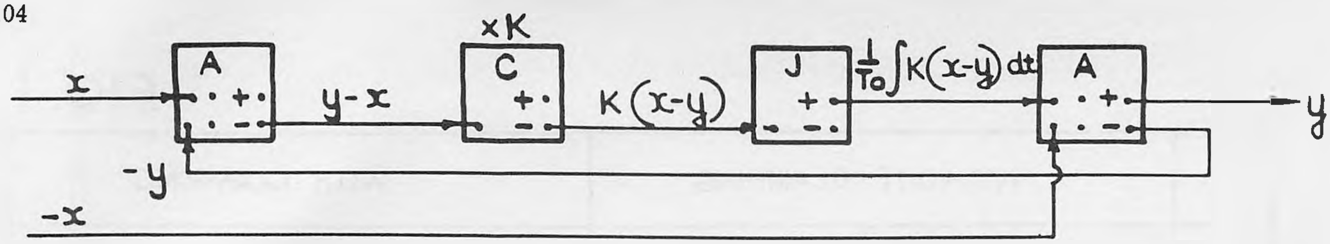


FIG. 2. BLOCK DIAGRAMS.
REPRESENTATION OF THE EQUATION

$$y = \frac{1}{Rc}\int (x-y)\,dt - x \quad \text{WHERE} \quad \frac{K}{T_o} = \frac{1}{Rc}$$

T.N MS 12.
FIG. 3.

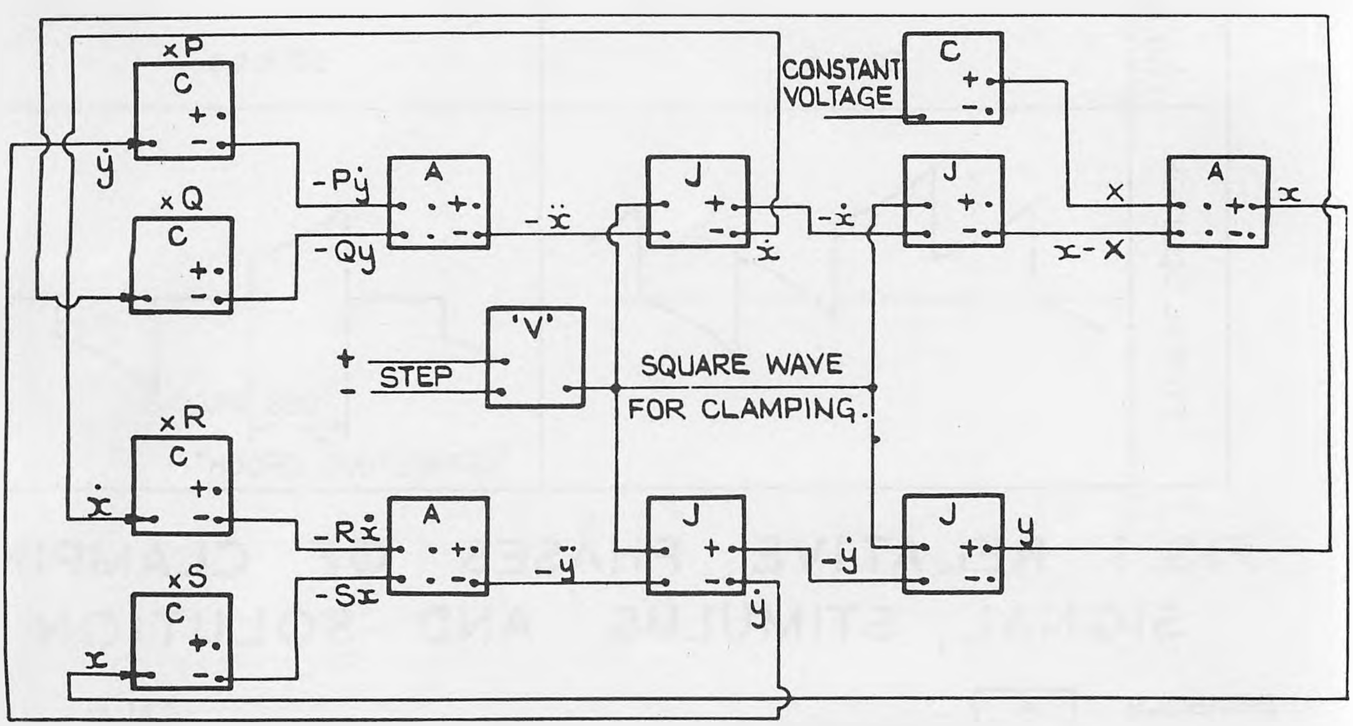


FIG. 3. BLOCK DIAGRAM.
REPRESENTATION OF THE EQUATIONS

$$\left.\begin{array}{l} -P\dot{y} - Qy = \ddot{x} \\ -R\dot{x} - Sx = \ddot{y} \end{array}\right\} \text{WITH} \quad \begin{array}{l} \dot{x} = 0 \\ \dot{y} = 0 \\ y = 0 \end{array} \quad \text{AND} \quad x = X \ \text{at}\ t = 0$$

$T_o = 1$

FIG. 4.

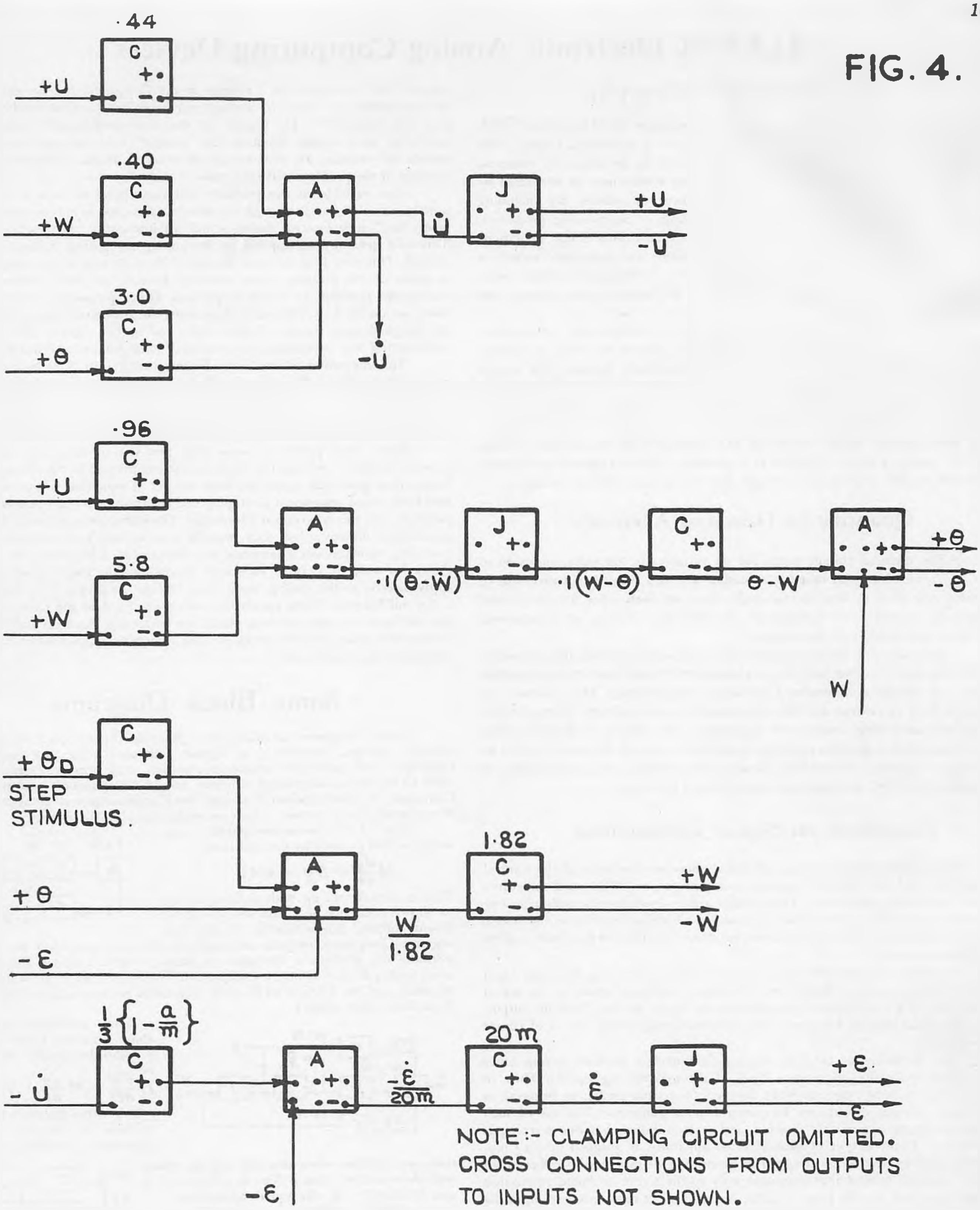


FIG. 4. BLOCK DIAGRAMS FOR AIRCRAFT -PLUS- AUTOPILOT DYNAMICAL EQUATIONS

$$\dot{u} = -.44u - .4w - 3\theta$$

$$.1(\dot{w} - \dot{\theta}) = -.96u - 5.8w$$

$$w = -1.82\,(\theta - \theta_D - \varepsilon)$$

$$\frac{\dot{\varepsilon}}{20m} = -\frac{1}{3}\left\{1 - \frac{a}{m}\right\}\dot{u} - \varepsilon$$

GAP/R Electronic Analog Computing Devices

The Component Philosophy

Patient study and trial has yielded the optimum set of operations which combine to make up the desired range of computing structures. Central rôles are given to operations of *addition, multiplication* by an adjustable constant, and *integration* with respect to time. Any linear system may be embodied by the connection of Components with these three operations. By including simple nonlinear elements, such as have limiting or suppressing properties, many important nonlinear systems are added. Certain other linear operations have been built into Component form for economy and simplicity, such as a differentiator and a first-order lag. Again, for combinations which recur, larger Components have been developed, as for second-order systems and functions of one variable.

In its physical form each Component is unidirectional: information flows only from input to output. This does not prevent the study of bidirectional actions, since the two paths may be individually formed. The output signal capacity of each Component makes negligible the load imposed by the input of another, so that it may "instruct" any number of others without correction. Electronically, all computing signals are instantaneous DC voltages with zero neutrals. Initial conditions are established by an external voltage wave — usually a step — applied as a stimulus, and the responding solutions displayed on the oscilloscope through the appropriate variable voltages.

Quantitative Data and Accuracy

The nominal voltage range for all variables is 100 volts, minus 50 to plus 50. A useful decimal unit for excursion is thus 5 volts. It is convenient to consider unit time as 400 microseconds, since ten such units fill the normal computing interval of 4 milliseconds, and the time settings of Components then have unit maxima or fixed-values.

Feedback is so heavily applied that electronic variations affect sensitivities by less than 1%. Precision circuit elements of highest quality, and specified within that tolerance, determine Component characteristics. Thus normal calibrations, fixed or central, are also maintained to such accuracy. Thermal drifts within the computing interval are negligible. The *fidelity* to dynamic form, as in integration, is built in to 0.1%; *resolution* is also of this order, giving an advantage frequently overlooked. Finally the *precision,* or reproducibility of parameters, is 0.01% or better between successive solutions.

Comments on Signal Connections

Plug cables normally carry all signals between the jacks of the Central Component and the various Computor Components, to CRO's, and sometimes to and from other equipment. These cables always lead from an output to one or more input. The pattern of interconnection naturally depends on the system being represented or the problem under solution. A variety of examples is given in subsequent pages.

It should be mentioned again that the removal of a plug from any input automatically grounds or makes zero the input involved, owing to the action of the jack. If a free cable is plugged into an input, leading from no output, then that input may be far from zero and errors may result; this is of course to be avoided.

The flexibility of GAP/R Analog Computation permits, among other things, the application of various types of external driving signals. Thus, in particular, it is quite simple to make frequency-spectrum studies by bringing in oscillating voltages of variable frequency from a generator. The use of such signals as inputs is perfectly feasible, and is encouraged, but one warning is appropriate. If the output of such external apparatus is plugged into a Component, special care may be required in grounding. It may not be sufficient, in certain cases, to ground that apparatus only to the sleeve or shield connection of the input jack, via the plug or cable. A clip-lead, or other connection, to the shell of any plug in a Component output does, however, assure a trouble-free ground.

Choice of Scale Factors

Scale Factors, *in a model* or analog, relate its variables and parameters to those of a primary system under representation. They are the quantitative factors in the *transformations,* stated or implicit, which convert the equations of the primary system into those of the analog assemblage, or *vice versa.* Even when equations are not used, when on the basis of familiarity with the Components and the problem one passes directly from system to analog, it is still necessary to interpret, i.e. to *scale,* the computed solutions. There are many known methods for determining scale factors, and all will work. It is intended here chiefly to give some guiding principles.

In so far as they are known or suspected, extreme values should be assigned for the variables of the problem to be solved. Generally, each such variable will correspond to a voltage in the Computor, and the extremes will be represented by values of voltage not exceeding, and preferably close to, plus and minus 50 V. The ranges will then correspond and be coextensive, or nearly so, zero voltage denoting the "neutral" value of each variable. Scale factors for variables are thus already determined. If reassignment is advisable, solution of the problem will soon make it evident.

One variable in the problem will correspond to time in the analog; usually it is also time, although not always. One chooses conservatively a maximum "real" time interval during which all interesting phenomena will occur. This will generally correspond to the analog computing interval of 4 milliseconds. One may thus calibrate electronic time, as well as the voltage signals in terms of the problem under solution. Further, all time parameters in the analog are readable in direct proportion. Other parameters, as for example those set on K3-C Coefficient Components, are ordinarily interpreted through the assigned scale factors of their input and output signals. Most often, the unity-setting will correspond to a nominal value for such parameters.

The imposition of initial conditions is commonly done by means of the Step or Delta signal. This signal will also be scaled to the system under study. As already stated, it is advisable to employ as large a Delta as possible as long as limits are not exceeded during computation, since this minimizes the CRO gains to be used and maximizes the attainable resolution of all results.

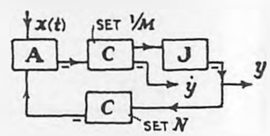
Some users prefer to work with two sets of equations: one for the primary problem, and one for its electronic counterpart in the Computor. Scale factors then guide the transition from one set of equations to the other. It has also been found convenient to employ one equation or set of equations for both purposes, and this has certain advantages. The technique is to make the original equation(s) dimensionless. One method is to express each variable as a ratio involving its maximum excursions: the time included if present. A convenient choice for unity is 1/10 of each such maximum. The modified equations will apply directly to the analog, since "unit voltage" is simply 5 V, and unit time is 0.4 milliseconds. Time parameters which are fixed in the Components, and the maximum settings for those which are adjustable, have the unit value. The Component equations themselves are also somewhat simpler when expressed in this non-dimensional form.

Some Block Diagrams

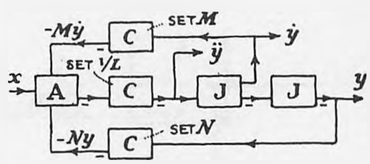
Block Diagrams, so-called, have become the accepted shorthand for dynamic systems, particularly as regards computors. They are equivalent to equations, with additional causal information, and provide a stepping-stone close to an analog computing structure and its Components. In setting up the Computor, if added realism is sought, the Components may be assembled and interconnected in a manner which resembles the block diagram.

One of the commonest physical situations is that covered by the equation:

$$M\frac{dy}{dt} + Ny = x(t)$$



This is directly set up with 4 Components as shown. Outputs of y and its derivative, among others, are available. (Note that negative Component outputs are employed wherever possible.) An equivalent assembly not giving the derivative is simply a K3-L Unit-lag Component in series with a K3-C Coefficient Component. The K3-L is set at M/N in appropriate units, and the K3-C is set at $1/N$. (Hereafter we may refer to a Component by its final initial alone.)

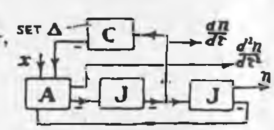


Going one notch beyond the above 1st-order system, we show a 2nd-order system. Its equation is:

$$L\frac{d^2y}{dt^2} + M\frac{dy}{dt} + Ny = x(t)$$

It is often desirable to be able, as here, to set each parameter separately. However, simplifications are possible when liberties may be taken with the variables. Thus if $Ny = \eta$, $t(N/L)^{1/2} = \tau$, and $M/(LN)^{1/2} = \Delta$, the equation becomes:

$$\frac{d^2\eta}{d\tau^2} + \Delta\frac{d\eta}{d\tau} + \eta = x\left(\sqrt{\frac{L}{N}}\,\tau\right)$$



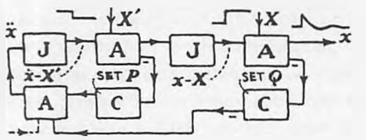
and results in the simpler block diagram shown.

In setting up block diagrams for equations as above, and hence also for Computors, one may almost always proceed as follows: Assume the highest derivative (order n) is available as a signal, and integrate it n times. This gives all the lower derivatives including the zeroeth. With these signals in combination, one supplies the assumed n-th derivative as expressed by the differential equation "solved" explicitly for that quantity. This method generalises satisfactorily for sets of equations. Sometimes other tricks are necessary, but the technique is quite universal. It is standard Differential Analyzer practice, for example.

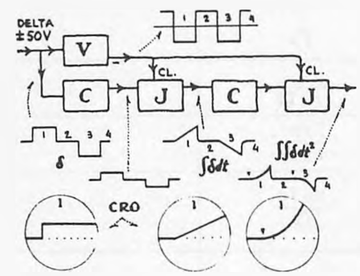
INITIAL CONDITIONS In physical systems there is usually an input variable which, as *stimulus,* determines initial conditions. The cases above are simple examples, with non-homogeneous equations. There was no question of how one embodies the initial values of the dependent variable(s) and the derivatives thereof. Formal mathematical equations are frequently presented however, in homogeneous form, with specific values for all but the highest derivative. For instance consider (the dot-notation for derivatives is used):

$$\ddot{x} + P\dot{x} + Qx = 0\,, \quad \text{and} \quad x(0) = X\,, \quad \dot{x}(0) = X'$$

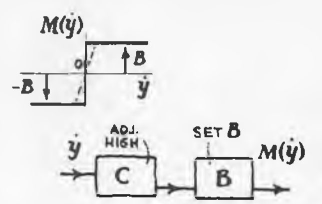
This type of situation may be handled straightforwardly by the addition of step inputs in the loop, as the accompanying diagram shows. With this arrangement, it is still possible to include any "forcing function" to cover the non-homogeneous case by adding it in as usual ahead of the highest derivative. If this input were zero prior to the initial instant (and of course never infinite) it will not influence the initial conditions cited above.



CLAMPING Although it might or might not have been required in the case of the above block diagram, a clamping signal could have been applied to each integrator. Clamping returns operations to an enforced neutral equilibrium during the zero intervals of the Delta Wave. It permits unstable or weakly stable systems to be solved on a cyclic time base. The result is shown here for the case of two J's in chain. (The right-hand switch position may be employed on each J.) In more complex systems the effects of clamping do not differ from those in this simplified case. In some assemblages it is not necessary to clamp all J's, even for unstable solutions, but it is a safe policy.



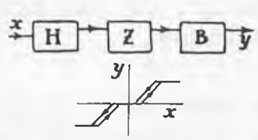
STEP-FUNCTIONS OF DEPENDENT VARIABLES Suppose that the second diagram of this section, with its equation, represented a mass-spring-dashpot structure (see K4-DY), and that not viscous but *dry* friction were present. Then the C used for setting *M* must be replaced by a combination giving the function $M(\dot{y})$, as illustrated, rather than simply $M\dot{y}$. This may be realized by including a B Component after the C, and setting the latter to as high a gain as possible without producing hysteresis, as observed on a CRO cross-plot. The dotted line is shown for lower sensitivity. It is important to note the causal order of the Components. One thus obtains a close approximation to Coulomb friction, so-called, in which only the sign of the velocity is involved. The magnitude of the force is adjusted on the B. It is evident that compound or mixed damping may be represented as well, by employing a parallel and additive path for each type.



ORDER OF K3-J and K3-C COMPONENTS in SERIES The sensitivity or time-factor of the J is normally fixed, so that it is frequently applied in series with a C. Since these operations are linear, it may appear unimportant in what order the two Components are placed. However if the C is to be set to zero at times in order entirely to remove the integrating effect from the system, then it is better for C to follow J than *vice versa*. It is in the nature of the J that a stable zero input may not give an output of the same sort.

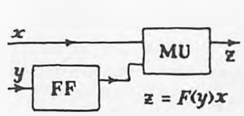
NONLINEAR K3 COMBINATIONS

Many relaying devices, such as *pilot valves* in various control systems, have characteristics similar to that shown in the adjacent plot. This may be represented by a chain comprising H, Z, and B in that order. Query: How does interchange of H and Z in the chain alter the characteristic, if at all?



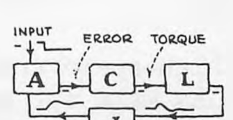
FUNCTIONAL COEFFICIENTS

Constant coefficients in equations are well handled by the C, properly installed and adjusted. When the coefficient of any variable depends on some other variable, itself either dependent or independent, then a Multiplier (MU) may replace the C. The other MU input is supplied with the appropriate function of that other variable.

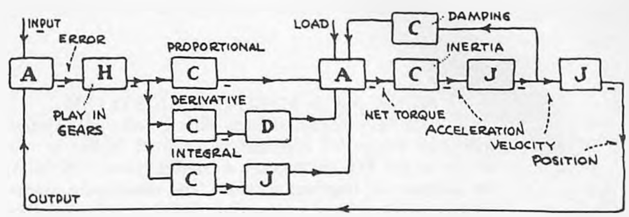


SIMPLEST SERVO LOOP

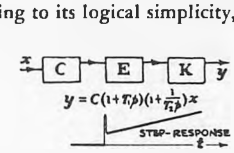
An unrefined but complete servomechanism may be assembled from 4 Components, as shown. Servo experts may then be made in *one lesson.*



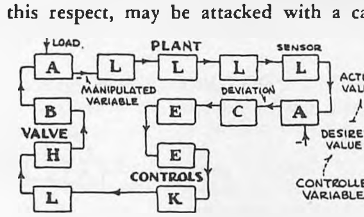
MORE REFINED SERVO Advanced studies are begun by adding more elaborate corrective dynamics, as for example integrals or derivatives: pure or adulterated. Inertia and damping may be separately adjusted, as properties of the displaced member. A load change may be represented. The presence of imperfectly meshing gears in a differential is embodied by insertion of an H. Further possibilities include: lags, bounded torques, Coulomb friction, tachometer-feedbacks, filters, higher derivatives, and multiple servomechanisms.



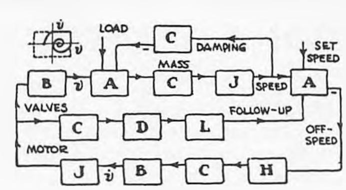
FACTORABLE CONTROL MECHANISM Part of the last diagram shows an additive form of control mechanism. This form is sometimes preferred owing to its logical simplicity, and because each effect is independently reducible to zero. Another form, equally effective in practice and simpler physically, is that obtained with the series K3 arrangement illustrated. With another each of E and K, the complete CO characteristic is obtained. It is not possible here to reduce the proportional effect completely to zero without nullifying the whole control. For most applications, this is unimportant.



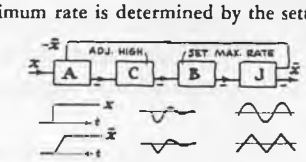
CONTROLLED INDUSTRIAL PLANT Most thermal, chemical, and hydraulic plants resist control in some degree by virtue of lags. Modeling of the dynamics of plants, in this respect, may be attacked with a cascaded series of L's, set either through advance calculation or by experimental determination of equivalent time-constants. Here is an example which should be self-explanatory.



ISOCHRONOUS GOVERNOR A governor is usually a hydraulically-operated automatic speed controller. The assemblage shown will manifest most of the ideosyncrasies of governing mechanisms, theoretical study of which is made obscure by the non-linearities they contain. The analog, on the other hand, gives the answer instantly, by co-operating with Nature. Admittedly, only one form of governor is here assembled.



BOUNDING THE RATE OF ANY SIGNAL By bounding the input to a J in a loop, it is possible to follow an input signal faithfully up to a predetermined speed, beyond which pursuit is continued at the constant maximum rate until capture. This maximum rate is determined by the setting of the B. Adjustment of the C toward higher gain is continued to the point of instability, and then somewhat retracted, for the closest following.



TWO GENERAL FEEDBACK METHODS As in the last example, feedback performs a myriad of tricks. A pair of general techniques are worth outlining. Frequently a desired operator may be put in the form $[1 \pm f(p)]^{-1}$, where $f(p)$ is assembled from Components. The diagram shows how this transformation is obtained, and if necessary multiplied by $f(p)$. Another valuable routine is that of *reciprocation*, employing a loop to "reverse cause and effect". This impossible ideal is approximated as shown, when the C gain is made reasonably large. Sometimes more elaborate dynamics are needed *for* optimum performance and stability.

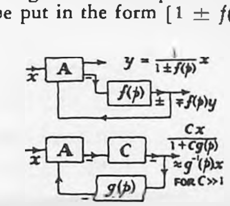
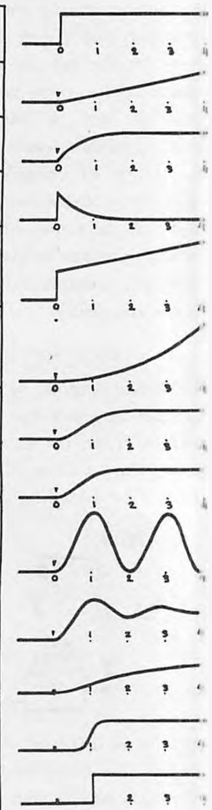
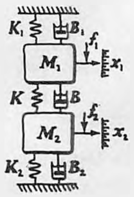


Table of Operators and Responses

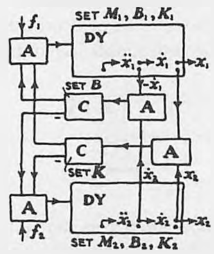
¶All the forms given here are dimensionless. ¶The initiating time-function is assumed to be the unit-step, and the operators should not be confused with Fourier or LaPlace transforms which imply a unit pulse. ¶All time responses are zero for t less than zero. ¶The time derivative and the time integral (the definite integral with zero lower limit) of each response yields a new response, based on multiplication or division of the corresponding operator by $p = d/dt$. ¶To obtain frequency spectra, simply replace p by $i2\pi f$.

$\varphi(p)$	$\psi(t)$
1	1
$\frac{1}{Tp}$	$\frac{t}{T}$
$\frac{1}{1+Tp}$	$1-e^{-t/T}$
$\frac{Tp}{1+Tp}$	$e^{-t/T}$
$\frac{1+Tp}{Tp}$	$1+\frac{t}{T}$
$\frac{1}{(Tp)^2}$	$(\frac{t}{T})^2$
$\frac{1}{(1+Tp)^2}$	$1+(\frac{t}{T}-1)e^{-t/T}$
$\frac{1}{(1+Tp)(1+\alpha Tp)}$ $\alpha \neq 1$	$1-\frac{1}{1-\alpha}\left[e^{-t/(\alpha T)}-\alpha e^{-t/T}\right]$
$\frac{1}{1+(Tp)^2}$	$1-\cos\frac{t}{T}$
$\frac{1}{(1+\beta Tp)^2+(Tp)^2}$ $\lambda^2=1+\beta^2$	$1-\lambda e^{-\frac{\beta t}{\lambda^2 T}}\cos\left[\frac{t}{\lambda^2 T}+\tan^{-1}\beta\right]$
$e^{-\sqrt{Tp}}$	$\int_0^{\sqrt{t/T}/2} e^{-q^2}dq$
$(1+\frac{T}{n}p)^{-n}=L(p)$	$\int_0^{nt/T} r^{n-1}e^{-r}dr$
$e^{-Tp}=\lim_{n\to\infty} L(p)$	$\frac{1}{2}\left[1+\frac{t-T}{\lvert t-T\rvert}\right]$



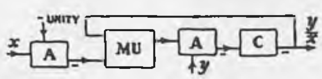


MULTI-MASS MECHANICAL SYSTEM We have shown already how a 2nd-order system with one degree of freedom is attained, either at one stroke in the DY or through a combination of K3's. As the degrees of freedom increase, the computing assemblage grows in direct proportion (unlike analysis). Two DY Components are shown here for a two-mass system, with 6 added K3's for the coupling details. Naturally, the latter types could have been used throughout, if desired. The study of *very* complex systems requires only a good supply of Components (or special equipment), but need not otherwise be feared.



RECIPROCALS These may be generated by the Function Components, or by an MU in a high-gain feedback loop. A more satisfactory method is based on the identity $1/x = (1/x)(1-x)+1$
Only multiplication and addition is needed; a more general case is described below.

RATIOS OR DIVISION Here again the method of applying an MU in a high-gain feedback loop is possible, but may present a stability problem owing to the high and changing gain. As above, a more recondite method is available which involves an implicit operation. In this case we have: $y/x = (y/x)(1-x)+y$



This is the branch of magic wherein the answer is utilized in arriving at the answer. Electrical scale factors are not included in the equations given. If both inputs are at 10 V, the output should duplicate them with the C set at unity, or removed with a reversal of sign.

PROBLEM FORMULATION FOR ANALOG COMPUTORS

The next three papers serve the needs of those interested in general techniques which will aid in the establishment of system performance equations and more particularly, in the formulation of a suitable analog model of a physical process.

The Nolan article is actually more generally applicable than its title would indicate, and its ambitious bibliography should prove particularly helpful.

The two documents following this, are concerned with so-called "direct modelling" of physical processes by passive and active electronic models, with the second primarily concerned in passive modelling (for more of which see the works of G. D. McCann) while the first deals with direct modelling using active operations such as adding, proportioning and integrating.

As one becomes more familiar with high-speed analog machines these direct links to the physical problem make for eminently successful synthesis of new designs by direct experimentation and manipulation of the computor model.

Analog Computers and their Application to Heat Transfer and Fluid Flow — Part 1

(Later Parts and Bibliography will be published in forthcoming issues)

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The purpose of this paper is to survey the field of analog computers with particular reference to their application to heat transfer and fluid flow problems and to present a bibliography.

A computer is an information processing device. It has been defined as a "device which accepts quantitative information, may arrange it and perform mathematical and logical operations on it, and makes available the resulting quantitative information as an output".[15] (The superscript number is the number of the reference listed in the bibliography.) This is a broad definition which includes slide rules, desk calculators, differential analyzers, industrial controllers, telephone exchanges, and large-scale digital computers.

Computers are generally classified as digital, analog, or a combination of both. Digital computers are discrete variable devices which represent numbers by counting discrete objects in space or discrete events in time such as holes in a paper card or tape, the teeth of a gear wheel, or electric pulses in a circuit. Fundamentally, they are machines which perform arithmetical operations on numbers, such as desk calculators or telephone exchanges. Analog computers are continuous variable devices which represent numbers by measuring some physical quantity such as shaft rotation, voltage, resistance, or position of a hand on a scale. Slide rules, differential analyzers, and industrial controllers are analog computers. A computer which employs both digital and analog computing devices would also include converters to change information from one form to another.

History

In order to gain a proper perspective, it would be good to briefly review the history of computers.[15] One of the oldest forms of computers is the abacus of ancient times. The seventeenth century saw the slide rule and an adding machine invented by Pascal to assist his father in checking accounts. During the first half of the nineteenth century the planimeter came into being and Charles Babbage worked out the concepts of digital machines. In 1876 a ball-and-disk integrator was invented by James Thomson and a harmonic analyzer was conceived by his brother Lord Kelvin. In the last half of the nineteenth century, adding machines, comptometers, calculating cash registers, and billing machines were developed. In World War I appeared the application of rudimentary electrical computing techniques to anti-aircraft fire control. In 1925 the first large differential analyzer was made by Vannevar Bush at the Massachusetts Institute of Technology. Various electric bridges, field plots by means of poorly conducting liquids and potential probes, and applications of d-c and a-c network analyzers were made during the 1920's. In the 1930's occurred the development of servo methods, potentiometers, resolvers, and feedback amplifiers which gave impetus to electrical analog computation.

Today there are a number of large scale digital and analog computer installations, a number of simple differential analyzers on the market, many available components which a user can assemble to fit his own needs, and a bewildering variety of small special purpose computers for process control and other computing purposes. Analog and digital techniques are used now to supplement each other; a simple analog computer to understand the problem and a digital computer for detailed investigation of areas of the problem are a powerful combination.

Computers in Design and Research

Computers have been a vital factor in the revolutionary progress in the level of complexity of practical science and engineering during the past century. They make it possible to solve problems faster, to handle problems which formerly required more effort than could economically be expended, and to find solutions to problems which were previously incapable of solution. Today the computer is well established along with the slide rule, balance, and test tube as an essential tool of the industrial researcher.

The technique of simulation is a powerful aid to the system designer. One form of simulation is the study of a system by the cut and try examination of its mathematical representation by means of an analog or digital computer. A modified program of system design would probably include the following steps:[11] (1) description of system inputs, (2) *first order design*, (3) component analysis, (4) experiments for required parameters, (5) systems analysis, (6) system simulation, and (7) modification toward an optimum design by a repetition of steps (2) through (6).

The competitors of simulation are the mathematical analysis and the "build and discard" techniques. When systems are complicated, simulation shows definite advantages over the other two techniques. The building and testing of prototypes is costly and time consuming. The use of the mathematical analysis technique can result in a disproportionate expenditure of engineering time on extensive calculations and in tests to prove the assumptions underlying these calculations. When applied properly, computers can supply adequate solutions to the problems at hand with a saving in time, money, and engineering personnel.

Analog Computers

By applying the principle of analogy, whereby various simple laws of nature and various parameters in different physical systems can be related to each other, a designer can translate a given problem from one physical system in which design computations are difficult and test models expensive, to another physical system in which low-cost models with continuously variable parameters can be quickly produced and tested.[38] A physical system is an assemblage of physical elements which may include mechanisms, electric circuits, chemical processes, heat processes, etc. Within the range of operating conditions in which known laws of design apply and by the proper application of conversion factors, data obtained in the analogous system are applicable in the original system.

A physical system can be represented by a physical analog or by an operational analog.[38] If each element or component in an original system is replaced by its analogous element or component in the model system, and if all interactions between elements are appropriately expressed so that the dynamic performance characteristics of the two systems are similar, then the model system is the physical or direct analog of the original system. The chief advantage of this direct technique is that the engineer does not have to write explicit equations for either system, which would be impossible to write for very complex systems. It is only necessary that the analog be a true and valid model and that the applied forces and boundary conditions be known. Data obtained by running tests on the analog, when directly translated back into terms appropriate to the original system, will establish the performance of the original system.

For example, a delta area in California has the ocean on its west, the Sacramento River entering from the north, the San Joaquin River entering from the south, and the Mokelumne River entering from the east. A pumping plant lifts the water out of the channels at the south end of the delta to supply the San Joaquin Valley. An old network of channels carries the flow through the delta. The problem arose of how to bring the Sacramento River water across the delta to the San Joaquin side while maintaining a pattern of flow in the channels which will hold intrusion of ocean salinity in check and thereby permit the transfer to be made without danger of contamination. An analog computer was set up to study this problem.[144] The analogous hydraulic and electrical relationships were developed and quantity of flow was represented in the analog by electric current, water surface elevations were represented by voltage, inertia by inductance, storage by capacitance, frictional drag by resistance, and time by time. However, the analog runs through five hundred days of actual tide changes in each second of operating time. The analog can reproduce the square-law relation between friction and velocity that is characteristic of fluid flow. Gate keepers are represented by rectifier circuits. Stream flows and currents are represented by controlled d-c currents added to and subtracted from certain points in the analog network. The wave motions associated with tides are represented by a-c voltages of specified magnitude. Net current flows are read on d-c milliammeters and tidal amplitudes and phase differences are read on cathode ray oscilloscopes. The analog has proved to be an effective means for expediting the work of finding the flow distribution patterns in a network of channels. It is particularly effective when tidal effects must be included in addition to gravity flows. The results obtained with the analog have checked well with those obtained by other means.

It is not always necessary, and frequently not desirable, to replace each individual element of an original system with an equivalent in the analog system. Analog models can often be greatly simplified and their usefulness broadened if they reproduce only the functional operations of the original system. This approach is particularly appropriate if the performance of the original system can be formulated mathematically. A model system which can reproduce the mathematical operations implicit in the mathematical formulation of the original system, without regard to the actual nature or elements of that system, is an operational analog. The building blocks of this operational analog may be actual physical elements with near ideal performance or relatively complex devices developed specifically to perform certain mathematical operations. Differential analyzers, mechanical or electronic, are one type of operational analog which have been put to a great multitude and variety of uses.

The dynamic performance of mechanical, hydraulic, thermal, magnetic, and acoustic systems and complex systems containing components in several of these fields can often be reproduced with simple analogous electrical systems. Electrical models are used more often than other type models, such as air-flow or water-flow models, because they can be produced at low cost in minimum time, and, once assembled, they can easily be modified with the many electrical components mass-produced to fill radio, television, industrial control, and armed services requirements. Also, the computational and recording devices now available are designed to accept the output signals of voltage and current produced in electrical models; complex sensing and conversion devices are required if equivalent measurements are to be made in mechanical, thermal, or fluid systems. Many important mathematical manipulations such as algebraic summation, trigonometric resolution, differentiation, integration, and other data-processing steps can be carried out rapidly with electrical analogs of the mathematical operations.[33, 47, 55]

Planimeters, slide rules, automobile speedometers, industrial controllers, linear equation solvers, fire control computers, network analyzers, and differential analyzers are examples of analog computers. They can be applied to a great variety of problems. They can be used to solve abstract mathematical equations. They can be used to find solutions to systems of linear simultaneous equations where the speed of solution is essential. In the form of pneumatic, mechanical, electrical, electromechanical, or hydraulic industrial regulators they solve continuously, day after day, equations expressing the desired behavior of processes or plants under control. In the automatic control field, they serve as low-cost bench-type prototypes which can be tested conveniently in the laboratory under a variety of conditions. They may be used to represent a part of an over-all system (such as an airplane and its aerodynamic controls) while the actual piece of equipment to be tested (such as an autopilot) completes the system to be studied. They can be used to test systems or devices whose equations or transfer functions are known but which have not yet been built. They can be used to determine transfer functions of systems built up from physical analogs. By generating or displaying a large number of solutions for instant examination on cathode ray tubes or other graphic recorders, optimum system parameters can be determined. They can be used to solve a wide variety of problems whose boundary or initial conditions or both are complex, such as occur in the flow of heat in irregularly shaped bodies or the flow of fluids in reservoirs.

The accuracy to which an analog computer answer is obtained is limited by the precision to which physical displacements, angles, voltages, currents, etc., can be measured. A precision of 10^{-3} or possibly 10^{-4} is normal and 10^{-5} is obtainable under highly restricted conditions.

Some of the advantages of analog computers are that they are simple enough so that many can be built to solve special problems, that an engineer can keep close track on the physical significance of the computation, and that a broad pattern of solutions representing many different combinations of conditions can be obtained in a short time. They are best for problems where engineering accuracy is sufficient and are the simplest type of computer for problems in which the precision of the computer is greater than the accuracy of the input data. They can handle almost all dynamic problems because the information needed is the presence of instability or the design of devices to prevent instability and not the numerical solutions.

The limitations of analog computers are that the equations for the original system or the transfer functions of the elements or components of the original system must be known, that a considerable amount of time must be allowed for setup of the computation schedule and analysis of results, and that the computing elements themselves, even though they may possess amazing versatility, have physical limitations. Some problems are not complicated enough to justify treatment by such an elaborate means. First cost is a consideration but the first cost of analog computers is low among computers. Analog computers are fundamentally calculus machines and are therefore, of course, inappropriate for basically arithmetical calculations such as census taking or cost accounting.

Some Large-Scale Computers and Their Applications

Large-scale general-purpose computers include the differential analyzer at the Massachusetts Institute of Technology, various d-c and a-c network analyzers, and similar analog computers operated by the Westinghouse Electric Corporation and the California Institute of Technology.

The MIT differential analyzer is used primarily for evaluating solutions of ordinary differential equations.[25] It is a mechanical device and the values of the variables involved are represented by positions of rotating shafts or by shaft rates. It has been used for solutions to problems in many branches of engineering and science and has stimulated activity in the fields of mathematical effort where processes of analysis are inadequate.

Network analyzers *have been used to solve* quickly the many and various problems concerned with the operation of power systems.[44] They are practical, adjustable miniature power systems. They can be used to analyze results during the progress of a system study and therefore play an active part in system planning as well as checking the performance of completed systems.

The electrical analog computers at Cal Tech and at Westinghouse (the Anacom) were constructed after a two year survey.[30] Various types of auxiliary equipment has been designed for use with these computers and the computers themselves are made up of many specially developed components.[45] A block diagram of a typical computer setup for solving a problem could consist of three blocks.[37] The first would be the steady-state or transient forcing functions. Electrical voltages are generated and applied to the analog which are equivalent to the forces applied to the actual physical system. The second block is the electrical analog of the system studied. Many analogs of different problems are already known and methods have been developed for systematically determining new ones. The third block is the measuring equipment, which includes oscillographic apparatus for transient problems. The exact type of measuring equipment will vary with each individual analysis. These computers have been used to solve a wide variety of problems including magnetic amplifier studies,[81] nonlinear mechanics and servomechanisms investigation,[95] transient vibration problems,[70] regulator problems,[29] the study of a steel mill drive,[35] and the study of a lubrication system.[45]

D-C Electronic Analog Computers and Their Applications

Computations, which are too extensive to be undertaken manually and not so elaborate as to justify using the facilities of a computation laboratory, can be handled easily and adequately by small, compact d-c electronic analog computers. Such computers can be constructed or can be purchased from manufacturers like Reeves,[28] Goodyear,[31, 39] Philbrick, and Boeing. They are easy to operate, easy to maintain and service, and relatively inexpensive. The parameters can be changed easily and the time required for setup is short in comparison with other types of computers. Additional units can be added easily to extend the capacity of the machine and make it more versatile. Such a machine permits convenient and economical testing by straightforward techniques and is an effective tool in the hands of those who understand its capabilities, advantages, and limitations.[33]

In general, the following steps should be followed in handling a problem:[33]

(1) Obtain a complete statement of the problem. This would include the equations to be solved, the initial conditions, the parameters to be varied, any available solutions or checks, numerical values of the parameters, and the estimated ranges of the variables and their derivatives if possible. Since this computer is an operational analog, a given system can be studied only if its response equations are known.

(2) Determine tentative scale factors for each variable and set up the transformation equations. These equations express the relationship between the problem variables and the computer variables.

(3) Choose the time scale and write the transformation equation for the independent variable (time). The computer time may be equal to, slower than, or faster than real time. Do not forget to transform initial conditions, limiting levels, etc., to computer variables.

(4) Establish the machine equations and draw a computer block diagram.

(5) Interconnect the computing elements by patch cords to perform the operations required.

(6) Set potentiometers, initial values, limiting levels, and function generators according to the block diagram. Set or check recorder calibration.

(7) Make the computing devices operative and thereby force the voltages in the machine to vary in the manner prescribed by the machine equations. The voltage variations with time are recorded and constitute the solutions of the problem. The machine is stopped at a time chosen by the operator. The maximum allowable computing time is usually determined by the limitations of the computing elements. Check operation for consistency by means of standard built-in test signals.

(8) Reset the machine for the next run with changed coefficients, initial conditions, etc.

(9) Obtain all the data required.

(10) Reduce the data and analyze and report the test results.

- TO BE CONTINUED -

Analog Computers and their Application to Heat Transfer and Fluid Flow — Part 2

(The Bibliography will be published in a forthcoming issue; Part 1 was published in the November, 1954, issue)

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A d-c electronic analog computer should include devices for multiplying a machine variable by a positive or negative coefficient, for generating the sum of two or more machine variables, for generating the product of two machine variables, for generating arbitrary functions of machine variables,[33] and for generating the time integral or the time derivative of a machine variable. The three basic elements which are interconnected to perform many of these functions are resistors, capacitors, and d-c amplifiers. The d-c amplifier is really the heart of this computer. Interconnections are made easily by means of patch cords and front panel jacks. Potentiometers can be used to set constant coefficients or can be wound to represent special functions. Multiplication or division can be effected by servomotor positioned potentiometers and trigonometric functions can be obtained by means of resolvers. The computer would also include regulated power supplies to furnish the voltages needed for the operation of the electronic components, means for recording or measuring the d-c voltage in the machine, and control circuits for starting the computation with the correct initial condition settings and for stopping the machine after the computation is completed.

D-C electronic analog computers have been used to solve linear differential equations with constant coefficients,[42] linear ordinary differential equations with variable coefficients,[33] nonlinear ordinary differential equations,[33] and sets of linearly independent simultaneous equations.[42] They have been used to study separately excited generators,[42] to study variable displacement hydraulic systems,[42] for trajectory calculations,[33] to solve aircraft flight equations,[33] and for the analysis and synthesis of servomechanisms.[33]

Special Purpose Computers

Because analog computers are reliable in operation, capable of continuous service over extended periods, and comparatively small and inexpensive, many have been constructed for special problems. One computer was constructed to determine the yield of radioactive isotopes produced by a pile or other source of radiation.[63] This computer could be used for any problems involving similar equations. Another was built to rapidly evaluate a in the equation cosh 2 a S/ cosh 2 a T = A, where S, T, and A were known.[87] Others have been constructed for the solution of phase equilibria in flash vaporization of mixtures of hydrocarbons,[96] for solving secular equations,[102] for the solution of partial differential equations,[86] for analyzing wave equation boundary value problems,[103] and for multicomponent fractionation calculations.[78]

Application of Analog Computers to Heat Transfer Problems

Analog computers of various types have been used to obtain solutions to the many problems concerned with the transfer of heat. Those problems specified by ordinary or partial differential equations or involving unsteady state heat transfer can be solved by means of general purpose electrical computers. The general purpose computer at the California Institute of Technology has been used to solve the ordinary differential equations concerned with the temperature rise in rotating electric machines during variable load cycles, to find the steady-state temperature distribution in a gas turbine rotor, and to solve various partial differential equations.[129] Electric circuit models for partial differential equations have been described by Kron.[84] The Heat and Mass Flow Analyzer (HMFA) at Columbia University is designed primarily for solving problems of unsteady-state heat conduction in solids with definite radiation and convection boundary resistances.[133] The HMFA is a continuation in this country by Victor Paschkis of work done in Europe on a method first devised by C. L. Beuken. It has been used to provide solutions to many problems -- including those involved in regenerator operation,[132] solidification of metals,[132] determination of economical insulation thickness,[109] the influence of through metal on heat loss from insulated walls,[135] and the setting up of charts and graphs on heat conduction problems.[115] Temperature patterns have been determined by geometrical analog methods.[119] This method consists of setting up an electrically conductive flat sheet to represent the heat transfer problem in question (current flow represents heat flow), applying the proper potentials to the edges of the sheet, and finding potential (temperature) patterns by means of a probe. Special electrical analog techniques have been used to analyze heat exchanger performance.[123] Special electrical analog computers have been constructed for particular thermodynamic calculations,[111] for analyzing a heating system,[137] and for studying the thermal behavior of houses.[127] Also, hydraulic[117] and air flow[110] analogy techniques have been used to study heat transfer problems.

When a solid mass changes temperature as a result of the exchange of heat between itself and its surroundings, there are variable temperature gradients in the mass, a series of isothermal surfaces, a nonuniform changing field within it, related to the time rate of heat gained or lost by the mass, and a time to attain steady state. This is known as the unsteady state of heat transfer and may be associated with heat transfer through the mass into one face and out from another.[133]

Mathematical calculations of the effect of any imposed conditions on a given mass defined by shape, size, and the physical properties of its materials, have depended upon the solution of Fourier's differential equations for these conditions. Those cases that can thus be solved with acceptable simplicity are relatively few in number and exclude most of those of industrial importance. The difficulties imposed by the mathematical approach are partially overcome by graphical methods. These methods are based on the replacing of each differential equation by an equation of finite differences, a process sometimes called step integration. Graphical methods, however, are tedious and of limited application. Experimental solutions of industrial problems of unsteady state heat transfer, depending upon inserted thermocouples or other thermometric devices and upon some means of measuring the rate of heat transfer, are difficult, expensive, and often impossible under service conditions.

In addition to these mathematical, graphical, and experimental methods[133] of solving unsteady heat transfer problems, there is the electrical analogy method. The analogies between the flow of heat and the flow of electricity[130] and electrical models for the solution of heat problems[126] have long been known. Early models were based on a geometrical similarity between the body subjected to heat flow and the model body. The application of general purpose computers to heat transfer problems is based on the identity in form of the fundamental equations of heat flow and the flow of electricity. The electrical analog bears no geometrical similarity to the body being investigated. Thus, a single electrical general purpose computer can be used for a wide variety of heat transfer problems.

The general form for the differential equation for heat conduction in solids can be written as:[112]

$$\frac{\partial}{\partial x} k(x,y,z,T)\frac{\partial T}{\partial x} + \frac{\partial}{\partial y} k(x,y,z,T)\frac{\partial T}{\partial y} + \frac{\partial}{\partial z} k(x,y,z,T)\frac{\partial T}{\partial z} + q(x,y,z,T,t) = cd(x,y,z,T)\frac{\partial T}{\partial t} \quad \text{--------(1)}$$

where t = time.
T = temperature at x, y, z at time t.
q = rate of heat supply per unit volume.
k = thermal conductivity = $\frac{1}{R_t}$ = Btu/(hr) (sq.ft) (deg^F/ft).
d = density = mass per unit volume.
c = specific heat = heat capacity per unit mass = Btu/(lb) (deg^F).
cd = C_t = Btu/^{o}F (cu.ft).

If q = o and k, d, and c are constants, equation (1) takes the form

$$\frac{\partial T}{\partial t} = a\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) \quad \text{-----(2)}$$

where $a = \frac{k}{cd}$ = thermal diffusivity = $\frac{1}{R_t C_t}$ = $\frac{\text{sq.ft}}{\text{hr}}$

The study of a thermal problem by the electrical analogy method involves the following steps:[134]

(1) Set up the analogous conditions by calculation. To calculate the circuit for a distributed irregular region or medium of any shape, consider that the region is composed of discrete parts or sections. The size and shape of these sections are governed by the configuration of the region, the boundary conditions, and the required accuracy of solution. Each section is represented by one lump in the electrical network. The electrical values for the lumps are calculated on the basis of the geometrical dimensions of the section represented; capacitance is proportional to volume and resistance is proportional to the ratio of the thickness per cross sectional area.[132] Radiation and convection surfaces for the region can be represented by boundary resistances.

(2) Build the R-C circuit to represent the heat transfer problem.

(3) Subject the circuit to the appropriate analogous initial and boundary conditions. Voltages would be applied to the circuit to represent temperatures and currents to represent flow. Transient or intermittent boundary conditions such as sudden applications of heat or changes in temperature can be reproduced. The resistances and capacitances can be adjusted to represent changes in k, c, d, or the surface boundary conditions.

(4) Measure the electrical quantities, such as voltage and current,[20] at the points at which the temperatures and heat flows are to be measured in the region or medium under investigation.

(5) Convert the results of the electrical investigation into heat units by calculation.

The fundamental similarity between the defining equations for the flow of heat within a rigid body and that of charge in an electric circuit are shown in Figure 2 on page 31. The solution of heat transfer problems by this electrical analogy method is based on two principles.[134] The first principle is the mathematical identity of the equations for heat flow and for certain electrical circuits and is exact. The second principle is the replacing of a circuit with evenly distributed properties by one with lumped properties and is approximate. Essentially this second step involves the replacing of each partial differential equation by an equation of finite differences.

To represent a one-dimensional heat-conduction problem by an electrical circuit the following procedure would apply.[133] Such a problem might arise in the case of heat flow across an infinite slab or through a rod insulated at the sides or in the case of a steam heated insulated pipe. Express the various quantities R_t, C_t, q, T, x_t, and t_t (defined in Figure 2 on page 31) for the thermal circuit in any desired consistent system of units. Choose a consistent system of units (not necessarily the same) for R_e, C_e, I, V, x_e, and t_e in the analogous electrical circuit. Then make $x_e = x_t$ and divide the thermal circuit into elements of dx_t length and the corresponding R_e, C_e cable into an equal number of elements of length dx_e. Give every element dx_e the same number of units of electrical resistance (R_e) and electrical capacitance (C_e), as the corresponding element dx_t has units of thermal resistance (R_t) and thermal capacitance (C_t). It is not necessary that the thermal path be of uniform cross section or that the elements dx_t be equal. Then by equations (3), (4), (5), and (6), or (7) and (8) of Figure 2, page 31, all the readings for V and I taken in the electrical circuit at the points defined by x_e and t_e will be numerically equal to values of T and q in the thermal circuit at points defined by x_t and t_t for $x_e = x_t$ and $t_e = t_t$.

Since R_e and C_e occur only as a product, the result will not be changed if R_e and C_e are changed individually. Refer to Figure 2, page 31. In equation (7), V can be replaced by nR_eC_e or $n(mR_e)C_e/m$, and t_e can be replaced by nt_e without altering the form of the solution. Since in equation (3) dV is replaced by k dV and R_e is replaced by nmR_e, I is therefore replaced by I(k/nm). Since in equation (5) dV is replaced by k dV and C_e is replaced by C_e/m, Q_e is therefore replaced by $Q_e(k/m)$. By suitably choosing k, m, and n, the electrical analog may be operated at convenient voltages and transient time intervals and may be built with feasible magnitudes of resistance and capacitance.[133]

The methods used for one-dimensional problems can also be applied to three-dimensional problems. Three-dimensional problems of course require a much greater number of resistors, capacitors, and other electrical equipment. Analogous circuits for three-dimensional elements [128] are shown in Figure 3, page 32. The connection of the resistances into a grid in accordance with the respective positions of each element and the connection of the bottom terminal of all capacitors to a common ground form the analogous circuit representing the entire body. The choice of coordinate systems depends upon the shape of the body and the boundary conditions. In many problems there is a certain degree of symmetry which can be advantageously used to reduce the number of components required in the network; e.g., in a cylindrical problem with axial symmetry it is necessary to use only a two-dimensional network in the z and r coordinates. Coefficients of surface heat transfer can be represented by resistances and transient boundary conditions can be approximated.

Any problem in physics or engineering which can be specified by partial differential equations, such as occur in heat flow, fluid flow, or stress problems, can be approximated by electrical networks. By extending the methods described above networks can be constructed for many more complex equations, such as those for transient heat flow.[129]

The possibility of changing the time scale is of paramount importance for the practicability of the method. The time ratio $TR = t_e/t_t = aC_eR_e$ where t_e is the time in the electrical circuit, t_t is the time in the thermal circuit, a is the thermal diffusivity, C_e is the electrical capacitance, and R_e is the electrical resistance. With a low TR, a heat process whose actual time may be hours or days could take a few minutes in the analog. With a high TR, a heat process whose actual time is fractions of a second could take several minutes in the analog. By changing the time scale, times are achieved which permit a practical experimental run and allow easy reading of the instruments.[133]

From the standpoint of time, electrical analog heat flow computers may be divided into three groups: long-time computers whose runs last from several minutes to several hours, intermediate-time computers whose runs last from fractions of a second to several seconds, and short-time computers whose runs last fractions of a second. The RC time constant for these computers has to be considered from various aspects: namely, cost of equipment, leakage, instrumentation, and manipulation. Because of the infinite variety of the possible designs for each type it is difficult to compare costs. As far as leakage goes, the short-and intermediate-time computers permit the use of small, extremely high quality capacitors and require low resistances which result in favorable useful to leakage resistance ratios. In short- or intermediate-time computers oscilloscopes or oscillographs are needed and the resultant accuracy is 2 - 5%. Long-time computers can use multiple point or slower recording instruments or instruments which print every two seconds. The attainable accuracy is 1/3 to 1/5%. When varying boundary conditions are specified, the short- and intermediate-time computers require special input circuits for each different boundary condition but the long-time computers can make use of *continuous* or *stepwise* manual control. For voltage dependent parameters (such as arise from the temperature dependence of thermal conductivity or specific heat) short- and intermediate-time computers require electronic circuits or cam drive equipment. In long-time computers the changes can be effected in steps from observation of instrument readings by manual switching or automatic relay operation. Some problems involve physical happenings such as two materials in and out of thermal contact. If alternation is at regular intervals depending on time only, no difficulties are encountered in the short-time apparatus. However, if the intervals are irregular or depend upon observations during the run of the test, only long-time computers are feasible. In conclusion, it may be said that the short-time computer is well suited for more qualitative analysis of problems with constant paramenters. The intermediate-time computer's field of greatest significance is in problems with simple operating conditions and constant parameters. The long-time computer is the most versatile apparatus and, so far, is indispensable for complex problems and for nonlinear parameters.[132]

The limitations of these methods are instrument errors, inaccuracies due to leakage and stray currents, inaccuracies due to lumping, and the need to know such physical constants as specific heat, density, thermal conductivity, and film conductance. However, this last is not too serious a problem because these properties are often known within limits (to 10% anyway). Besides, it is possible to determine these properties experimentally or operate the analog in reverse to establish the properties.[131] To counteract the inaccuracies due to leakage, use a small number of lumps. To counteract the inaccuracies due to lumping, use a large number of lumps. Therefore, it can be seen that for a given problem a practical compromise must be achieved. If the number of lumps is constant during a test run, the error is independent of the length of the run. The selection of nonuniform cross sections or lumps does not appear to influence accuracy. Therefore, the lumps can be chosen to give a greater number of readings in that part of the body to be investigated which is of greater interest.[134]

Application of Analog Computers to Fluid Flow Problems

Analog computers of various types have been used to obtain solutions to many problems concerned with the flow of fluids. The Westinghouse Mechanical Transients Analyzer was used to determine flow and pressure conditions in a penstock as a function of flow at the gate.[147] The general-purpose computer at the California Institute of Technology was used to solve various partial differential equations concerned with fluid flow.[129] Even before 1943 an electrical device was used for the analysis of the complex problems of reservoir and well behavior.[140]

Electrical analogy techniques have been advantageously employed in the analysis of many diverse hydraulic systems.[150] An equivalent circuit for any hydraulic system may be readily derived from two-terminal network analogies for each hydraulic component. A table of such analogies given in the cited reference is shown in Figure 4, page 33. The equations for system behavior may then be written and solved according to standard electric network analysis procedures. These methods can be most profitably applied in the field of automatic control, particularly servomechanisms.

Network calculators, such as the one at the Illinois Institute of Technology, have been used to solve the increasingly complex problems of calculating gas flows and pressure drops in gas distribution systems.[143] It is necessary to develop a relationship between analogous electrical and gas flow equations and then choose and adjust the electrical components to duplicate the gas distribution system. Also, an electrical network analyzer, comprised of special tungsten filament lamps whose nonlinear resistance characteristics closely approximate the fluid flow resistance of pipelines, has been constructed for solving the simultaneous head-loss equations for a pipeline network.[148]

Two-dimensional compressible fluid flow problems have been solved on adjustable resistance d-c calculating boards.[152] The convenient analogy employed is that between the equations of two-dimensional fluid flow and of conduction of electric currents in a plate for which the conductance is a function of the voltage gradient or of the current density. Since present day d-c boards are built for the analysis of short circuits in power systems, it is planned to build a new d-c board consisting of more units of higher accuracy which will be more suitable for field problems.

The Hydraulics Division of the Civil Engineering Department at the Massachusetts Institute of Technology is studying the application of analog computers to hydraulic engineering problems so that predictions can be made faster and more accurately.[153] Such problems would include river and reservoir behavior, surge tank behavior, and the many complex problems that arise in connection with the performance of hydroelectric plants. The application of analog computers to penstock, surge tank, and water hammer studies has been described.[151] This reference also sets up an electrical-hydraulic analogy in which the water hammer waves and surges of the hydraulic engineer become the traveling waves, electrical surges, and switching transients of the electrical engineer as shown in Figure 5, page 34.

Late in 1948, the Weather Bureau, U. S. Department of Commerce, developed an electronic device for stream flow routing that has proved to be highly effective in the preparation of river stage forecasts.[145] It was originally designed for routing flows from point to point along a stream, but subsequent studies indicated that the equipment is equally applicable to the direct routing of effective rainfall (runoff) over relatively large basins.

- END -

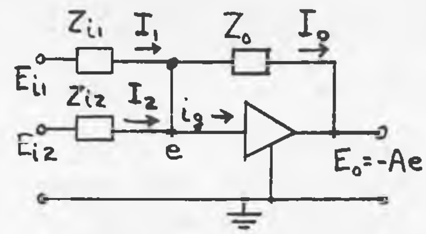
OPERATION	CIRCUIT	OUTPUT

1. General

$$L\,{di}/{dt} + Ri + \frac{1}{C}\int_0^t i\,dt = e(t)$$

$$Ls\bar{i} + R\bar{i} + \frac{\bar{i}}{Cs} = \bar{e}$$

$$\frac{\bar{e}(s)}{\bar{i}(s)} = \bar{Z}(s) = Ls + R + \frac{1}{Cs}$$



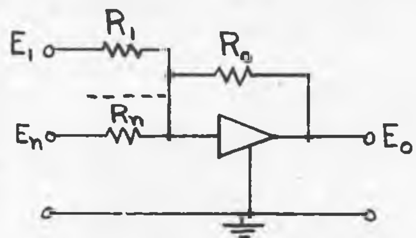
Assume $A \to \infty$, $i_g = 0$

$$E_o(s) = -\sum_{K=1}^{n} E_{iK}(s)\,\frac{Z_o(s)}{Z_i(s)}$$

2. a) Addition

b) Scale change on each input variable.

c) Sign change on all input variables



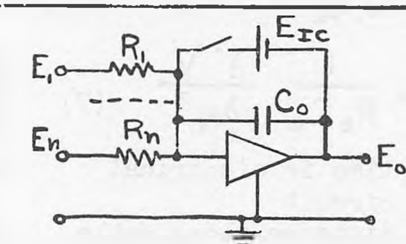
$$E_o = -\left[\frac{R_o}{R_1}E_1 + \ldots + \frac{R_o}{R_n}E_n\right]$$

3. Integration

$$E_o = -\frac{\int E_1/R_1\,dt}{C_o} - \ldots - \frac{\int E_n/R_n\,dt}{C_o}$$

or

$$\frac{E_1}{R_1} + \ldots + \frac{E_n}{R_n} = -C_o\frac{d}{dt}E_o$$



$$E_o = -\frac{1}{sC_o}\left[\frac{E_1}{R_1} + \ldots + \frac{E_n}{R_n}\right] - E_{IC}$$

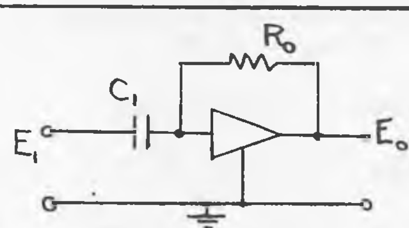
E_{IC} = Initial Condition

4. Differentiation

$$E_1 = -\frac{\int E_o/R_o\,dt}{C_1}$$

or

$$C_1\frac{d}{dt}E_1 = E_o/R_o$$



$$E_o = sR_oC_1E_1$$

5.

$$\frac{d^2y}{dt^2} + A\frac{dy}{dt} + By = f(t)$$

$y = E_o$ $\quad f(t) = E_1$

$$s^2Y = F(s) - AsY - BY$$

$$A = \frac{1}{R_1C_1} \qquad \frac{R_4}{R_3} = 1 \qquad B = \frac{A}{R_2C_2}$$

$$\frac{R_6}{R_5} = 1/B \qquad \frac{R_8}{R_7} = 1$$

$$\frac{R_{12}}{R_9} = \frac{R_{12}}{R_{10}} = \frac{R_{12}}{R_{11}} = 1$$

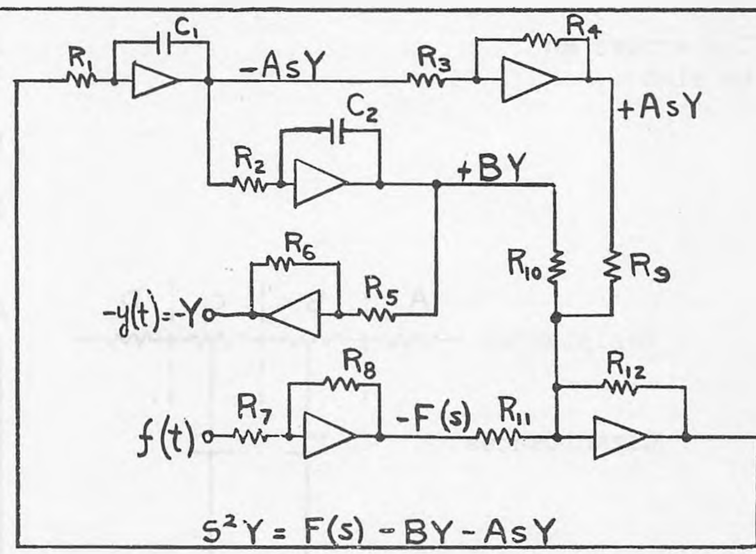


Figure 1 — Operational Amplifier Circuits

		ELECTRICAL	THERMAL
1.	Conservation of scalar quantity	Charge (Coulombs)	Heat (Btu)
2.	Scalar point function	Electric Potential (Volts)	Temperature (Degrees)
3.		Ohm's Law	Fourier's Law
4.	Resistance concept	$R_e = dV/I$ (3) R_e = electrical resistance I = current through R_e dV = difference in potential across R_e	$R_t = dT/q$ (4) R_t = thermal resistance q = heat flow through R_t dT = difference in temperature across R_t
5.	Capacity concept	$C_e = Q_e/dV$ (5) C_e = electrical capacity Q_e = charge stored in C_e dV = rise in electrical potential of C_e due to Q_e	$C_t = Q_t/dT$ (6) C_t = thermal capacity Q_t = heat stored in C_t dT = rise in temperature of C_t due to Q_t
6.	One dimensional form of heat conduction equation for solids	$\frac{\partial V}{\partial t_e} = \frac{1}{R_e C_e} \frac{\partial^2 V}{\partial x_e^2}$ (7) t_e = time in electrical circuit x_e = distance along cable V = electrical potential at x_e at time t_e	$\frac{\partial T}{\partial t_t} = \frac{1}{R_t C_t} \frac{\partial^2 T}{\partial x_t^2}$ (8) t_t = time in thermal circuit x_t = distance along flow path T = temperature at x_t at time t_t
7.	Flow	Amperes = Coulombs/sec	Btu/min
8.	Capacity	Farads = Coulombs/volt	Btu/degF
9.	Resistance	Ohms = Volts/Coulomb/Sec	degF/Btu/min
10.	Heat flow across an infinite slab		A - surface of slab B - first layer of slab material C - second layer of slab material D - surface of slab

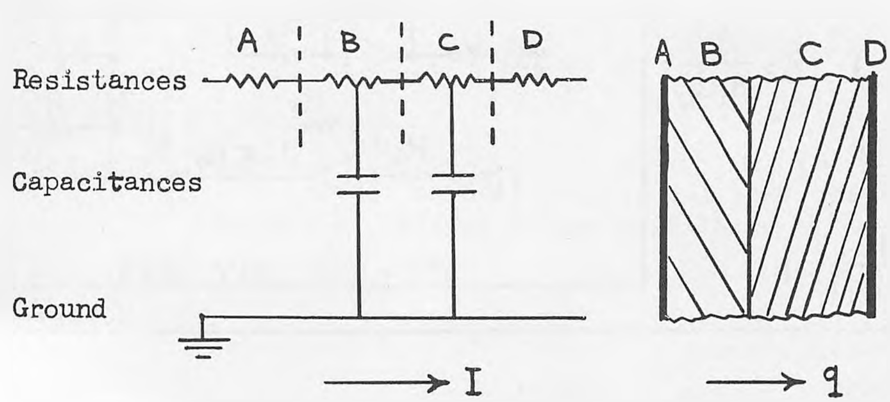
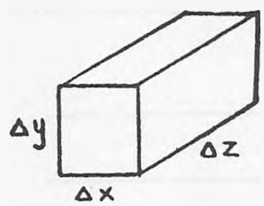


Figure 2 -- Comparison of Electrical and Thermal Relations



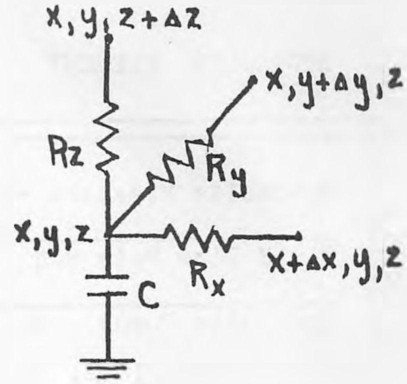
CARTESIAN COORDINATES

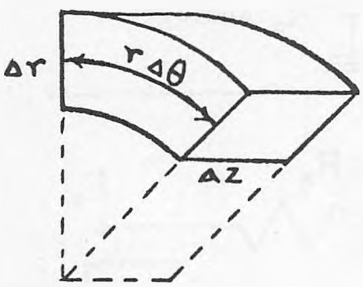
$$C = \Delta x \, \Delta y \, \Delta z \, c d$$

$$R_x = \frac{\Delta x}{\Delta y \, \Delta z \, k}$$

$$R_y = \frac{\Delta y}{\Delta x \, \Delta z \, k}$$

$$R_z = \frac{\Delta z}{\Delta x \, \Delta y \, k}$$



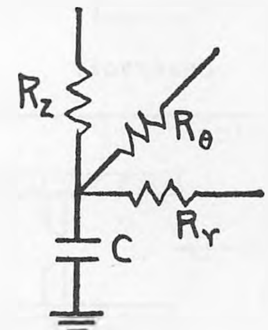


$$C = r \Delta\theta \, \Delta r \, \Delta z \, c d$$

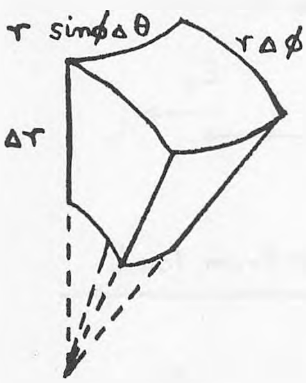
$$R_r = \frac{\Delta r}{r \Delta\theta \, \Delta z \, k}$$

$$R_\theta = \frac{r \Delta\theta}{\Delta r \, \Delta z \, k}$$

$$R_z = \frac{\Delta z}{r \Delta\theta \, \Delta r \, k}$$



CYLINDRICAL COORDINATES

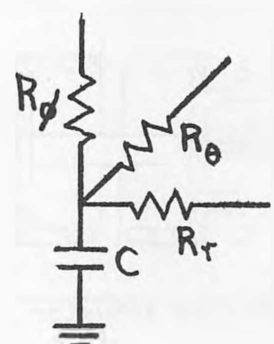


$$C = r^2 \sin\phi \, \Delta r \, \Delta\theta \, \Delta\phi \, c d$$

$$R_r = \frac{\Delta r}{r^2 \sin\phi \, \Delta\theta \, \Delta\phi \, k}$$

$$R_\theta = \frac{r \sin\phi \, \Delta\theta}{r \, \Delta r \, \Delta\phi \, k}$$

$$R_\phi = \frac{r \Delta\phi}{r \sin\phi \, \Delta r \, \Delta\theta \, k}$$



SPHERICAL COORDINATES

TRANSIENT HEAT FLOW EQUATION

$$\nabla^2 T = 1/a \; \partial T/\partial t + f(t)$$

$1/a = cd/k$
$f(t)$ - arbitrary heat function applied to system
Required boundary conditions can be handled by suitable potentials.
c - Specific heat
d - Density
k - Thermal conductivity
T - Temperature

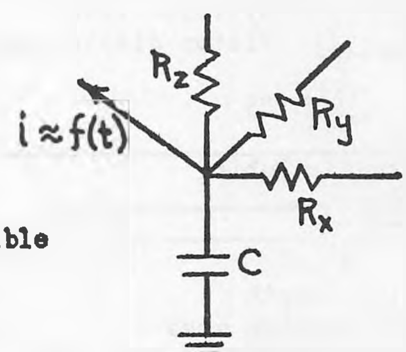


Figure 3 -- Analogous Electrical Circuits for Three-Dimensional Thermal Elements

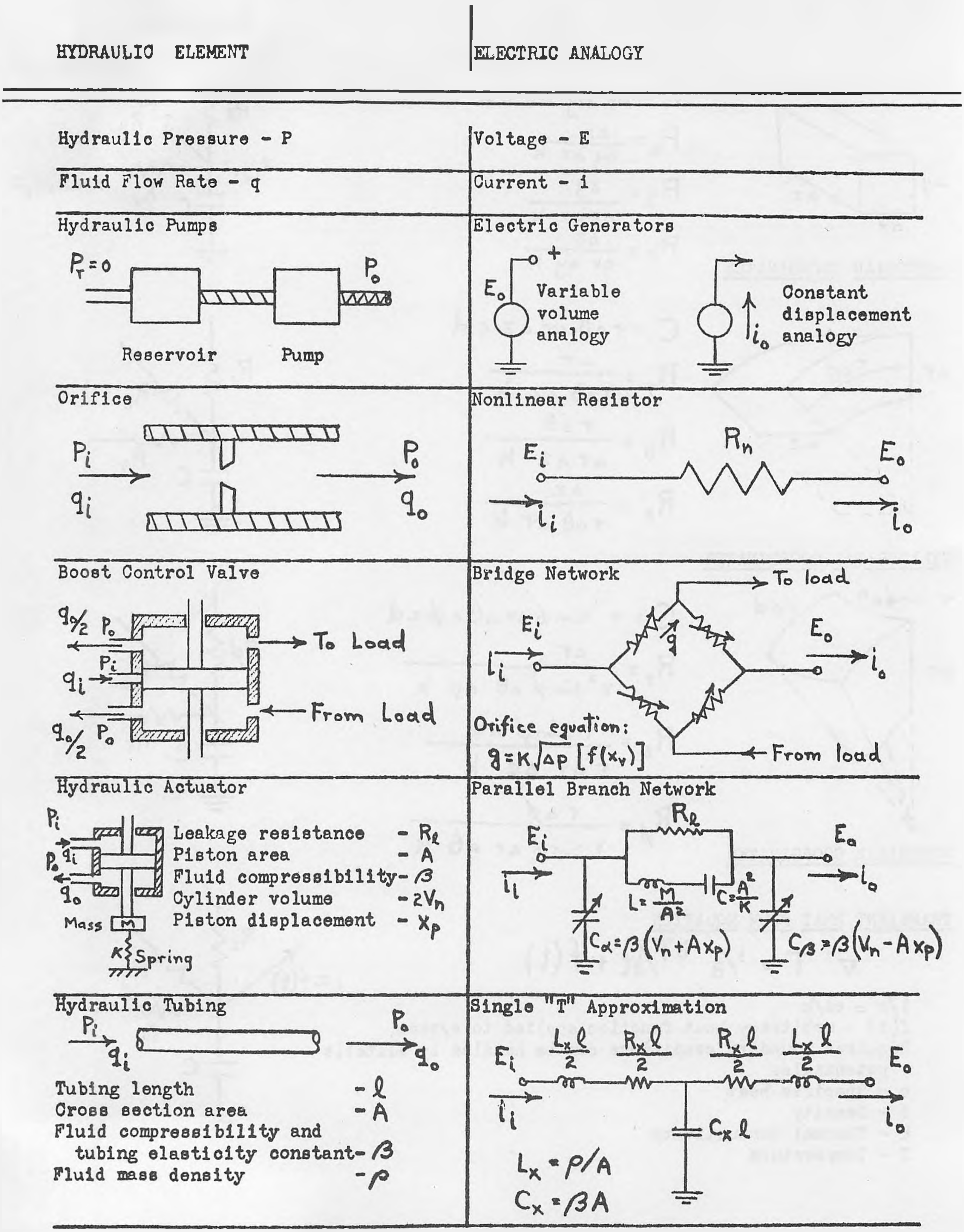
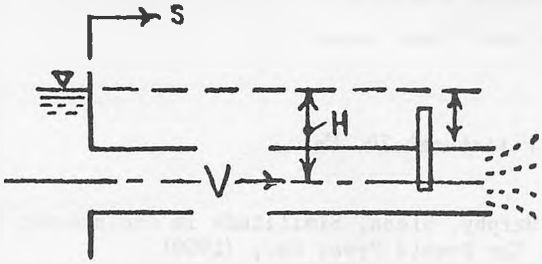
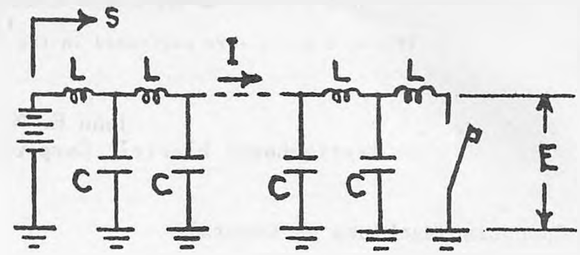


Figure 4 -- Electrical Analogies for Hydraulic Components

HYDRAULIC SYSTEM:
Uniform frictionless pipe line



ELECTRICAL SYSTEM:
Uniform lossless transmission line



1. **Inertia equation:**
$$-\frac{\partial H}{\partial s} = \frac{1}{g}\frac{\partial V}{\partial t}$$
Voltage drop:
$$-\frac{\partial E}{\partial s} = L\frac{\partial I}{\partial t}$$

2. **Continuity equation:**
$$-\frac{\partial V}{\partial s} = \frac{w}{E_b}\left(1+\frac{E_b}{E_y}\frac{D}{e}\right)\frac{\partial H}{\partial t}$$
Line charging:
$$-\frac{\partial I}{\partial s} = C\frac{\partial E}{\partial t}$$

3. **Wave equations:**
$$\begin{cases}\dfrac{\partial^2 H}{\partial t^2} = c^2\dfrac{\partial^2 H}{\partial s^2}\\[2ex] \dfrac{\partial^2 V}{\partial t^2} = c^2\dfrac{\partial^2 V}{\partial s^2}\end{cases}$$
Wave equations:
$$\begin{cases}\dfrac{\partial^2 E}{\partial t^2} = c^2\dfrac{\partial^2 E}{\partial s^2}\\[2ex] \dfrac{\partial^2 I}{\partial t^2} = c^2\dfrac{\partial^2 I}{\partial s^2}\end{cases}$$

4. **Propagation velocity:**
$$c = \sqrt{\frac{E_b/\rho}{1+E_b/E_y \; D/e}}$$
Propagation velocity:
$$c = \sqrt{1/LC}$$

5. **Surge impedance:**
$$Z_o = c/g$$
Surge impedance:
$$Z_o = \sqrt{L/C}$$

6. **Reflections:**

Open end: Pressure node: $\Delta H = 0$
Reflection factor: $r = -1$

Closed end: Velocity: $\Delta V = 0$
Reflection factor: $r = +1$

Reflections:

Grounded end: Voltage node: $\Delta E = 0$
Reflection factor: $r = -1$

Open end: Current node: $\Delta I = 0$
Reflection factor: $r = +1$

ANALOGY
Head H ⟷ Voltage E
Velocity V ⟷ Current I

Figure 5 -- Electrical-Hydraulic Analogy

Analog Computers and their Application to Heat Transfer and Fluid Flow—Part 3 (Concluding Part)

(Parts 1 and 2 were published in the November and December issues.)

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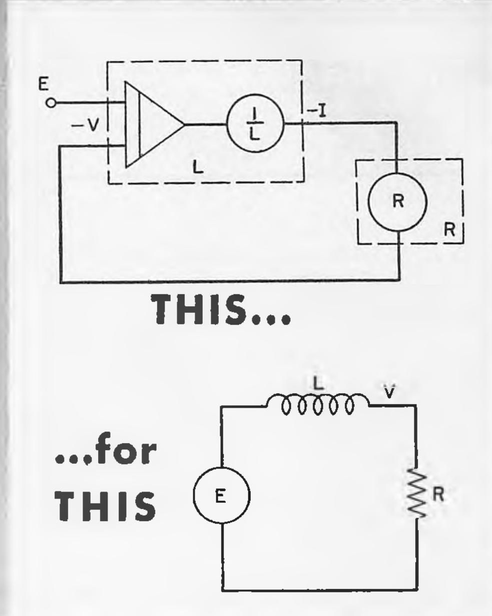
- END -

THE GIST: A computer model can predict many things about the performance of a control system; especially about the difficult-to-analyze dynamic behavior. Deriving the computer circuit, however, presents its own problems. And the results are not always easy to analyze.

This article describes a concept which simplifies simulation. The computer circuit is built from functional blocks, which are direct analogs of components in the system to be simulated. The system's dynamic equations need never be written. Manipulation of one of the computer's potentiometers affects a property of only one component in the system. Analysis becomes simpler.



Direct Simulation

BYPASSES MATHEMATICS, SIMPLIFIES ANALYSIS

VERNON L. LARROWE, University of Michigan

A SIMPLE R-L CIRCUIT and a computer circuit that directly simulates it are shown in Figure 1. The drawing suggests a direct correspondence between elements of the electrical network and elements of the analog computing circuit. The three types of linear circuit components—inductance, capacitance, and resistance—may be represented by analog elements as shown in Figure 2.

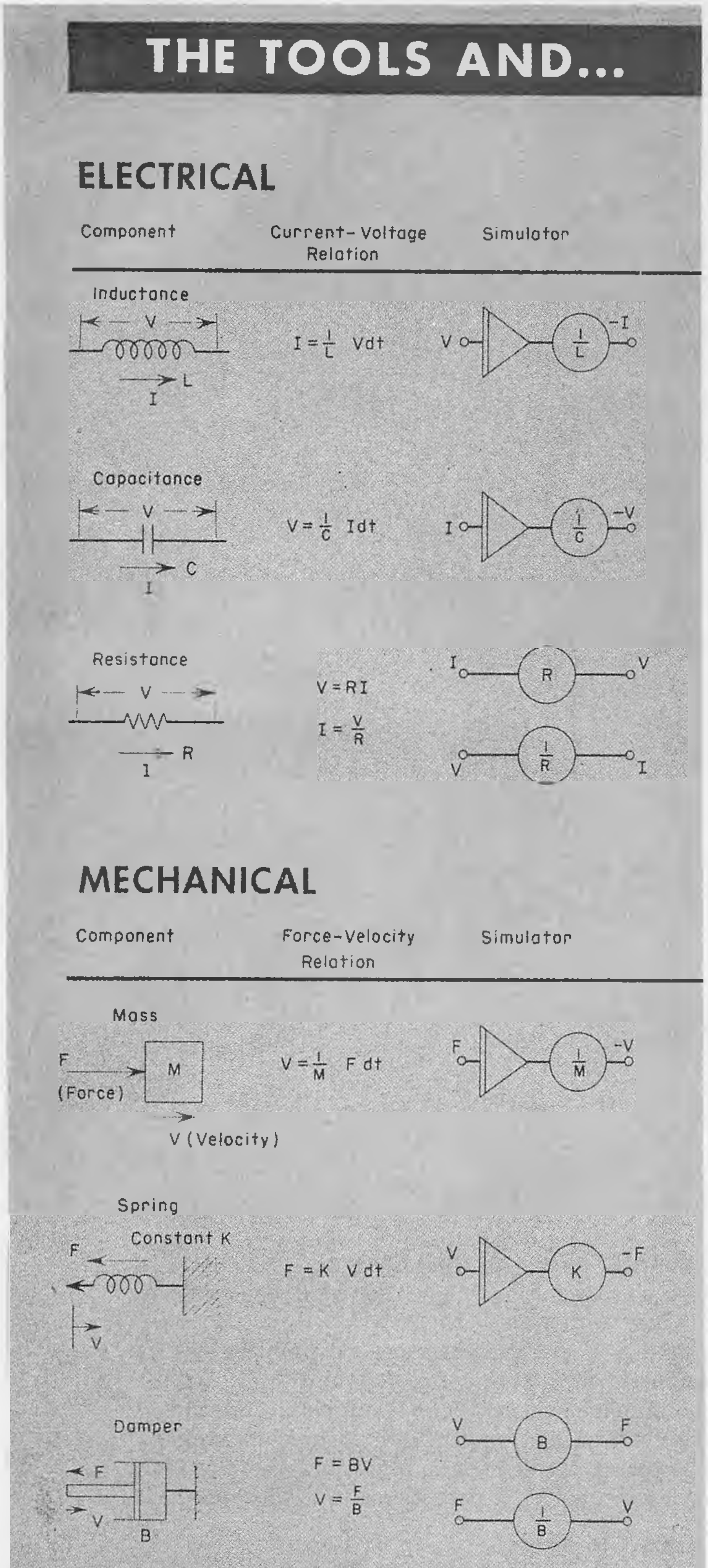
An inductance integrates the voltage across it to produce current. It may be represented by an electronic integrator connected to a potentiometer. The input to the electronic integrator represents the voltage across the simulated inductance. The output of the computing circuit is a voltage, which represents the resulting current in the inductance. The potentiometer set to $1/L$ may be placed in either the input or output circuit of the integrator, whichever is most convenient, as long as the value of L remains constant with time.

Similarly, a capacitance may be simulated by an electronic integrator connected in cascade with a potentiometer set to $1/C$. The input to this circuit represents current, and the output represents voltage. The voltage-current relationships in a resistance may be represented by a potentiometer as shown.

Using The Tools

Using Kirchoff's laws (1, The sum of the voltages around a closed loop is zero, and 2, The sum of the currents flowing to a junction is zero), these analog simulators of components may be assembled to simulate an entire electrical circuit as shown *in* Figure 3.

The method for its derivation is easy to trace. For example, the voltage across the coil L_1 in the circuit



Output of analog simulator is a voltage proportional to force, velocity, current, etc. Input is a voltage proportional to net force, net velocity, etc., acting on component. Fig. 2

is $E_1 - V_1$. Thus, feeding E_1 and $-V_1$ to the upper left integrator on the computer "road map" and multiplying the integrand by $1/L_1$ produces $-I_1$. The minus sign is the result of signal inversion by the electronic integrator. The voltage corresponding to $-I_1$, obtained in this manner, is added to a voltage corresponding to I_2, obtained elsewhere in the circuit and the resulting current is integrated in the simulator circuit for C to produce V_1. V_1 is inverted and used as one of the inputs to the L_1 simulator. Continuing the interconnection of elements in this manner produces all necessary input quantities for the component simulators and the only input needed is E_1.

It is unnecessary to write any differential equations in deriving the simulator circuit. The simulator has only as many energy-storage elements as the original circuit.

Mechanical Elements

Direct simulation techniques work also when deriving the analog computer road map for a network of mechanical elements. The three basic types of mechanical elements—masses, springs, and viscous damping devices—may be represented by corresponding analog computer circuits also shown in Figure 2.

A mass integrates the forces acting on it to produce velocity, and it may be represented by an electronic integrator and potentiometer. A spring may also be represented by an integrator and a potentiometer, with the relative velocity between its ends as the input, and the developed force as the output. Since the viscous damping device obeys a mechanical Ohm's law, relating velocity, force, and the damping coefficient, its operation may be simulated by a potentiometer as shown.

Figure 3 illustrates the method of assembling the basic analog elements to simulate a complete mechanical network. The most convenient method consists of first drawing the analog computer equivalents of the various elements, with their inputs and outputs labeled as shown, and then interconnecting them to form the complete diagram.

The resulting computer diagram is similar to the one for the simple electrical circuit, since the chosen mechanical network is an analog of the electrical network.

Conventional Methods

To illustrate the difficulties involved, Figure 4, 5, and 6 develop computer circuits for the same electrical network using the conventional differential-equation methods.

The differential equations that describe behavior of this circuit may be written in several different forms. Perhaps the most commonly used are the loop equations. These are written in terms of the mesh currents I_1 and I_2 and state that the sum of voltages around each loop is zero (see Figure 4).

Z_{11} and Z_{22} are the impedances of meshes 1 and 2 respectively. Z_{12} equals Z_{21} and is the impedance common to meshes 1 and 2.

A computer road map for the loop equations is derived in the conventional manner. The upper string of integrators deals with I_1 and the lower with I_2. The circuit requires four electronic integrators, although there are only two inductances and one capacitor, and hence three energy-storage elements, in the original electrical network.

Such a computer circuit usually gives erroneous results, because the extra integrator adds an extraneous root to the characteristic equation. It will be accurate only if the integrators are precisely matched.

For a simple circuit, such as the one under discussion, the computer road map can usually be inspected and rearranged to eliminate the extra integrator and give satisfactory results, but the entire process is time consuming and becomes very difficult as the complexity of the network to be simulated is increased.

Nodal Equations

The nodal equations for a network are statements that the sum of currents at any junction in the network is zero. The network used for the first example must be rearranged slightly, since expression in nodal form uses driving currents instead of driving voltages. In Figure 5, the current passing from the driving source through I_1 is designated as I_1, and the nodal points 1 and 2 are designated as shown.

Y_{11} and Y_{22} are the admittances connected to nodes 1 and 2 respectively. Y_{12} equals Y_{21} and is the admittance between nodes 1 and 2.

In Figures 4 and 5, p is the differential operator $\frac{d}{dt}$. Therefore, in Figure 4

$$pI = \dot{I} \text{ and } \frac{1}{p} I = \int I dt$$

and in Figure 5

$$pV = \dot{V} \text{ and } \frac{1}{p} V = \int V dt$$

The nodal equations may be mapped in a computer circuit as shown in Figure 5. Here again, four integrators are needed and the same difficulty with extraneous roots arises as in the case of the loop equations.

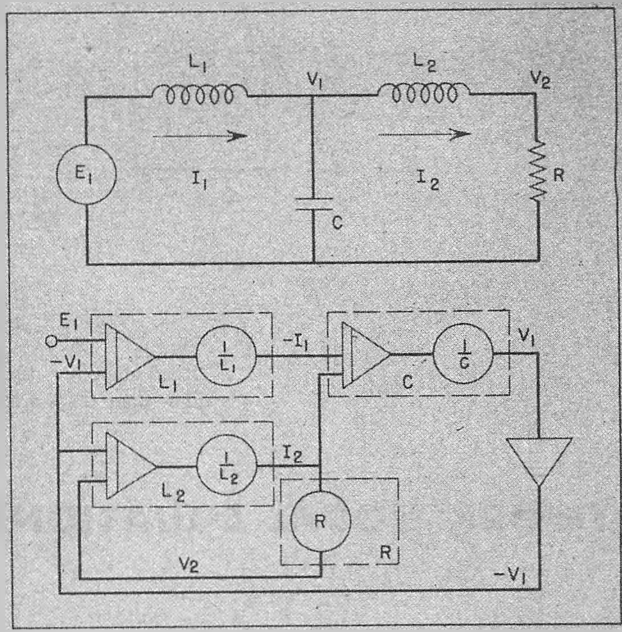
Branch Currents and Nodal Voltages

A third method of deriving the circuit equations uses nodal voltages and branch currents. If these voltages and currents are defined as in Figure 6, the equations may be written by inspection.

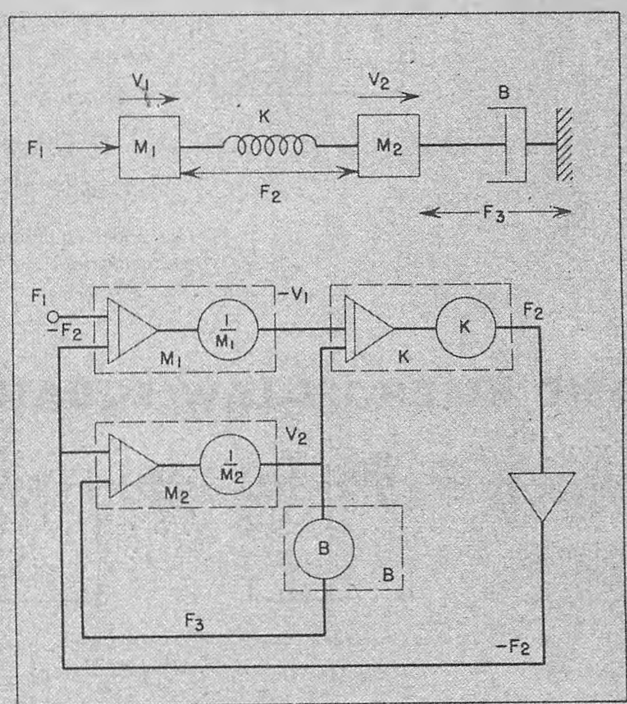
The computer road map for solving these equations consists of only three integrators and one inverting amplifier. This circuit will give satisfactory results since the number of integrators is correct. It is less expensive to assemble since it requires less equipment.

...WHAT THEY DO

ELECTRICAL



MECHANICAL

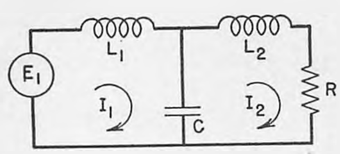


Direct simulation of analogous electrical and mechanical systems. Note that the same computer circuit merely assumes different analog quantities. Fig. 3

CONVENTIONAL METHODS...

NETWORK LOOP EQUATIONS

Circuit



General form:

$$E_1 = Z_{11}I_1 + Z_{12}I_2$$
$$0 = Z_{21}I_1 + Z_{22}I_2$$

Differential operator form

$$E_1 = (pL_1 + \frac{1}{pC})I_1 - \frac{1}{pC}I_2$$
$$0 = -\frac{1}{pC}I_1 + (pL_2 + R + \frac{1}{pC})I_2$$

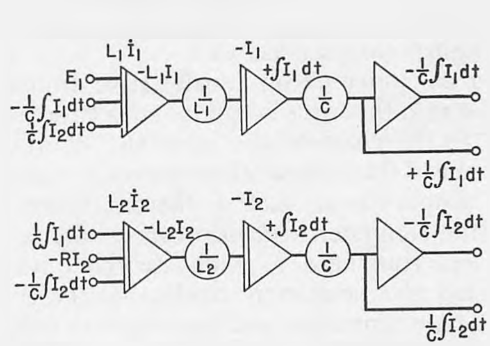
Differential equations

$$E_1 = L_1\dot{I}_1 + \frac{1}{C}\int I_1 dt - \frac{1}{C}\int I_2 dt$$
$$0 = -\frac{1}{C}\int I_1 dt + L_2\dot{I}_2 + RI_2 + \frac{1}{C}\int I_2 dt$$

Or

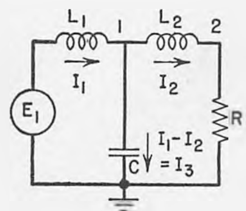
$$L_1\dot{I}_1 = E_1 - \frac{1}{C}\int I_1 dt + \frac{1}{C}\int I_2 dt$$
$$L_2\dot{I}_2 = \frac{1}{C}\int I_1 dt - RI_2 - \frac{1}{C}\int I_2 dt$$

Computer Diagram



"Road-mapping" these equations produces a computer circuit with one integrator more than there are energy-storage elements in the original circuit. Fig. 4

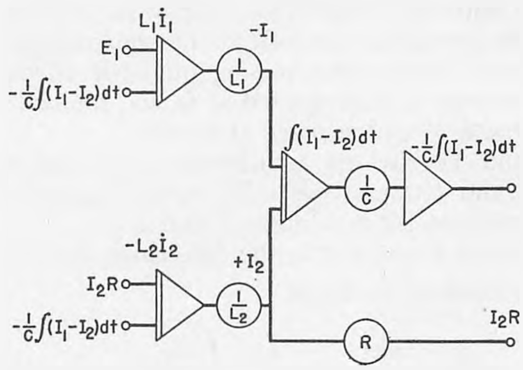
NETWORK NODAL EQUATIONS



$$L_1\dot{I}_1 = E_1 - V_1$$
$$L_2\dot{I}_2 = V_1 - V_2$$
$$I_3 = I_1 - I_2$$
$$V_1 = \frac{1}{C}\int I_3 dt = \frac{1}{C}\int (I_1 - I_2) dt$$
$$V_2 = I_2 R$$

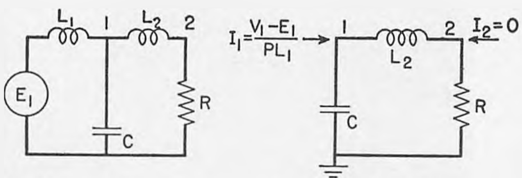
Combined form

$$L_1\dot{I}_1 = E_1 - \frac{1}{C}\int (I_1 - I_2) dt$$
$$L_2\dot{I}_2 = \frac{1}{C}\int (I_1 - I_2) dt - I_2 R$$



The nodal equations also require one integrator too many, which introduces an extraneous root and may cause instability. Fig. 5

BASIC KIRCHOFF-LAW EQUATIONS



Nodal equations

$$I_1 = Y_{11}V_1 + Y_{12}V_2$$
$$I_2 = 0 = Y_{21}V_1 + Y_{22}V_2$$

In differential operator form

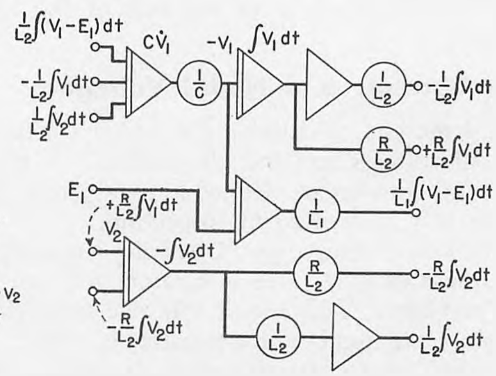
$$\frac{V_1 - E_1}{PL_1} = (PC + \frac{1}{PL_2})V_1 - \frac{1}{PL_2}V_2$$
$$0 = -\frac{1}{PL_2}V_1 + (\frac{1}{PL_2} + \frac{1}{R}).V_2$$

Differential equations

$$\int \frac{(V_1 - E_1)}{L_1} dt = C\dot{V}_1 + \frac{1}{L_2}\int V_1 dt - \frac{1}{L_2}\int V_2 dt$$
$$0 = -\frac{1}{L_2}\int V dt + \frac{1}{L_2}\int V_2 dt + \frac{V_2}{R}$$

Solving 1st equation for $C\dot{V}_1$ & 2nd for V_2

$$C\dot{V} = \frac{1}{L_1}\int (V_1 - E_1) dt - \frac{1}{L_2}\int V dt + \frac{1}{L_2}\int V_2 dt$$
$$V_2 = \frac{R}{L_2}\int V_1 dt - \frac{R}{L_2}\int V_2 dt$$



Nodal voltages and branch currents produce a circuit with the right number of integrators. Direct simulation saves even writing the equations. Fig. 6

Junctions of Like Elements

For networks that include junctions of three or more elements of the same kind, direct simulation becomes more difficult. Figure 8 shows a simple example of a network containing a junction of three inductances. If an attempt is made toward direct simulation involving the principles previously discussed, with an integrator for each inductance, and an integrator for the capacitor, some of the necessary input quantities for these elements cannot be obtained. This difficulty may be overcome by a special analog circuit for simulating the junction of the inductances as shown in Figure 7. The sum of the currents flowing to the junction is zero, and so the sum of their derivatives is zero.

Then $$\frac{V_1 - V_0}{L_1} + \frac{V_2 - V_0}{L_2} + \frac{V_3 - V_0}{L_3} = 0$$

or $$\frac{V_1}{L_1} + \frac{V_2}{L_2} + \frac{V_3}{L_3} = V_0\left(\frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3}\right)$$

Thus, the voltage at the junction, V_o is not dependent upon time and may be expressed in terms of the other voltages and the values of the inductances.

Solution of this equation in the first form by an analog computer gives a circuit having one potentiometer corresponding to each inductance and involving two more amplifiers than the number of inductances. This may be termed a "minimum-potentiometer" circuit since it uses a minimum number of potentiometers. Solution of the equation in the second form by the analog computer requires a circuit involving only one computing amplifier but requiring an additional potentiometer. This may be labeled a "minimum-amplifier" circuit. Choice of the proper circuit is discussed later.

The analog circuit for a junction of capacitances also derived is in Figure 7. The sum of currents flowing to the junction is set equal to zero. The currents are expressed in terms of time derivatives of the voltages across the capacitors. Integrating the resulting equation relates the voltage V_o to the voltages V_1, V_2, and V_3 without involving time as a variable.

These equations assume that the constant of integration, K, is zero. This assumption does not affect the use of the equations for an electrical network involving a junction of capacitors, since the initial value of V_o has no effect on the response of the rest of the circuit.

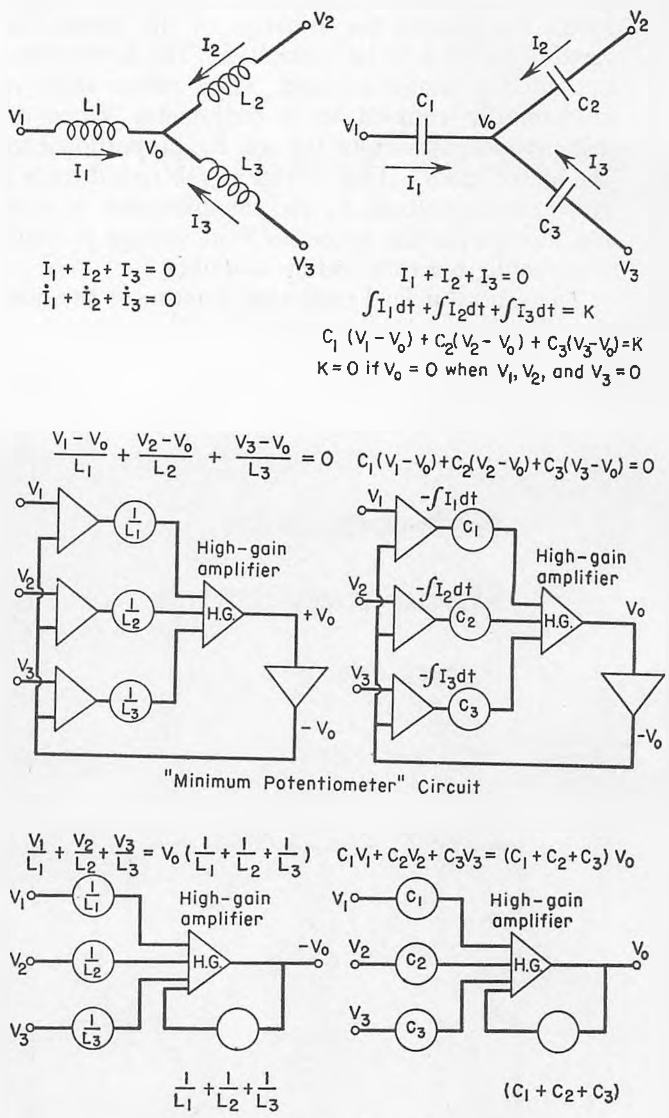
The junction of capacitors can be simulated by a minimum-potentiometer circuit or a minimum-amplifier circuit, with each circuit having the same advantages and disadvantages as the corresponding circuit for the junction of inductances.

The number of capacitances or inductances may be increased by adding more potentiometers and amplifiers to the minimum-potentiometer circuit, or more potentiometers and inputs to the single high-gain amplifier on the minimum-amplifier circuit.

Figure 8 illustrates the use of both types of computer circuit for solving a simple electrical network involving a junction of three inductances. Each computer circuit uses only three integrators. There is no integrator corresponding to L_1 since the current through L_1 is determined by the sum of the currents through L_2 and L_3. The behavior of the circuit can be completely described by a third order system of differential equations.

Either computer circuit gives valid solutions. Choice of which circuit to use depends upon the available equipment. The first circuit is preferable,

JUNCTIONS OF THREE OR MORE SIMILAR ELEMENTS



Number of elements may be increased by adding potentiometers and amplifiers to the "minimum-potentiometer" *circuit*, or potentiometers to the "*minimum-amplifier*" *circuit. Fig. 7* because it offers greater operating convenience with only one potentiometer for each circuit element. If the number of available computing components is limited, the second circuit should be used because it needs fewer amplifiers.

The second circuit is much less convenient to operate. For example, if it were desired to change the value of L_2, three potentiometers would have to be reset in the second circuit but only one in the first circuit.

The computer circuits for a junction of capacitors may be used in a similar manner. An example is not given here, because such junctions rarely occur in actual electrical networks.

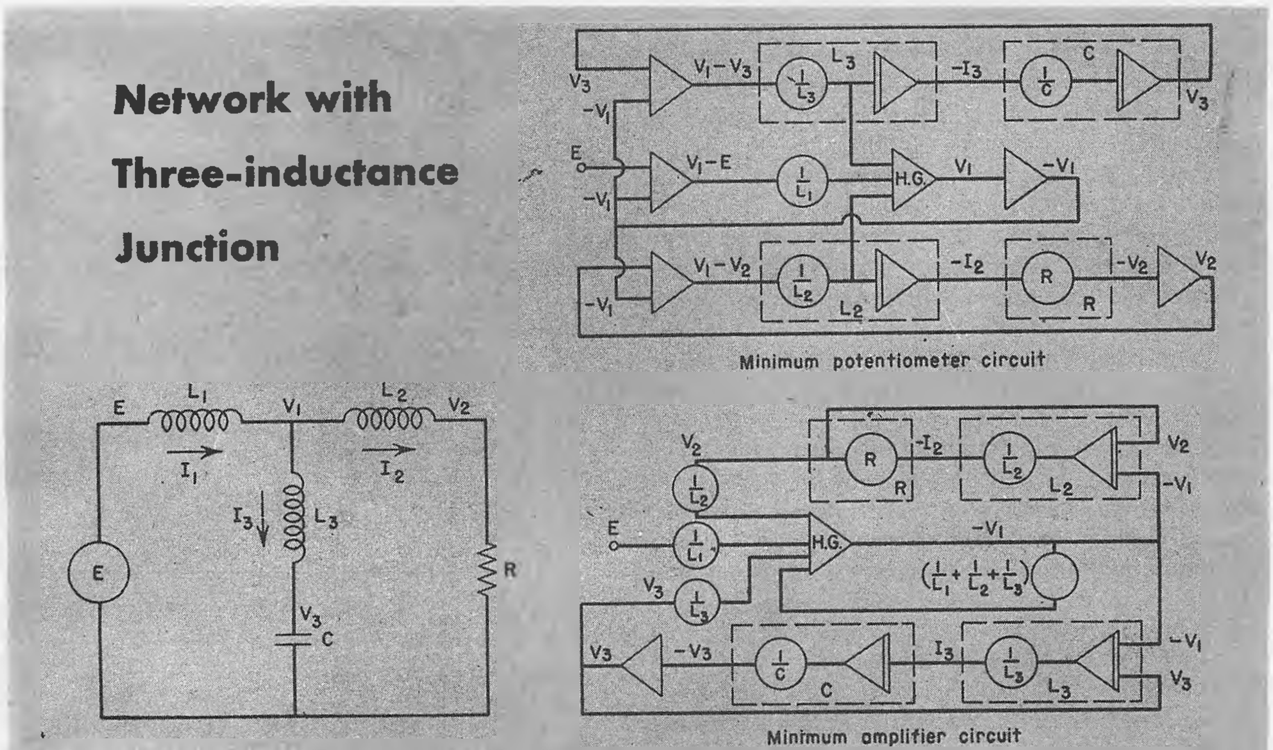
System Simulation

A common dc motor-generator speed control system is diagrammed in Figure 9. The electrical output of a dc generator, with shaft driven at constant speed, connects to the armature of the motor the speed of which is to be controlled. The field excitation for the motor is fixed. The motor shaft is mechanically coupled to a tachometer generator that produces an output voltage, E_t, proportional to the motor speed. This voltage is subtracted from a speed control voltage, E_s, and the difference, or error signal, controls the generator field voltage through a correcting network and an amplifier.

The generator field excitation determines the generator's output voltage, which in turn determines the motor's speed. Any difference between the tachometer voltage, E_t, and speed control voltage, E_s, makes the motor change speed to reduce the difference. Thus, the motor speed is determined by E_s. The correcting network can be simulated by the techniques already described.

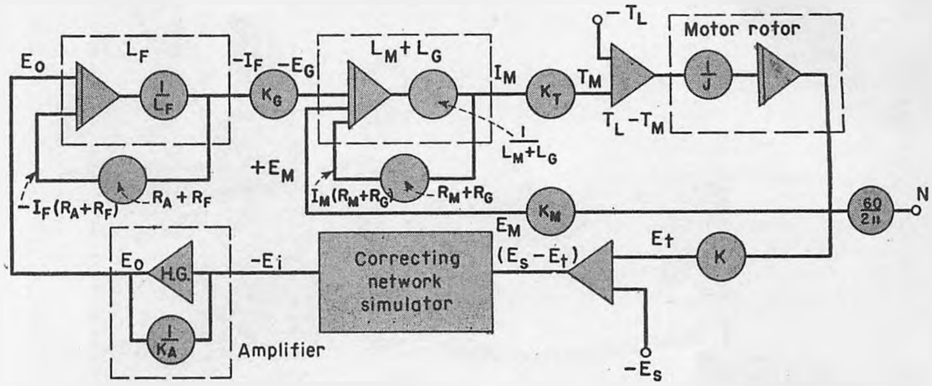
An analog computer circuit for directly simulating this system is shown in Figure 10. Starting in the upper left corner, the amplifier output voltage E_o, minus the voltage drops across the internal resistances of the amplifier and the generator field, is applied to a cascaded electronic integrator and potentiometer representing the field-inductance L_F. The output is an analog voltage representing the field current, $-I_F$. This is multiplied by a potentiometer set at the sum of the amplifier and generator field resistances to give the voltage drop across these elements. The field current analog voltage is also impressed on a potentiometer set at the generating constant K_G, of the generator. This gives an analog voltage corresponding to the emf of the generator.

In Figure 9, the electrical circuit between the generator and motor contains two series inductances, corresponding to the inductances of the generator and motor armatures, and two series resistances, representing the internal resistances of the motor and generator armatures.

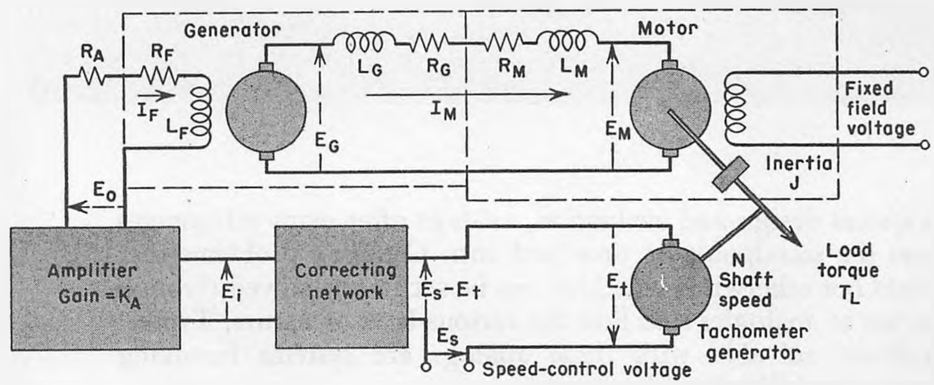


The "minimum-amplifier" circuit saves computing equipment, but the "minimum potentiometer" circuit is simpler to operate, having only one potentiometer for each component. Fig. 8

DIRECT SIMULATION OF A TYPICAL SPEED-CONTROL SYSTEM



Common dc motor-generator speed-control system demonstrates how easily direct simulation can be applied to more complex practical problems. Fig. 9



Computer diagram of the system of Figure 9. No equations have been written, and each component has computer elements independent of the others. The simulator circuit uses the minimum number of integrators. Fig. 10

The total voltage applied across the inductances alone is the generated voltage, E_G, less the motor's back emf, E_M, less the voltage drops across the resistors. Thus, feeding $-E_G$, E_M, and $L_M(R_M + R_G)$ into the integrator and potentiometer combination representing $(L_M + L_G)$ results in an analog voltage representing I_M at its output.

This quantity, I_M, is multiplied by a potentiometer set at $(R_M + R_G)$ to give one of the inputs to the integrator. It is also multiplied by the torque-constant of the motor, K_T, to give the motor's generated torque. The load torque, T_L, fed in externally, is subtracted from the generated torque to give the accelerating torque on the rotor. This accelerating torque is divided by the rotor's polar moment of inertia, J, and integrated with respect to time to give the rotor's angular velocity, ω.

The potentiometer set to the reciprocal of J and the integrator connected to it form an electronic analog of the rotor, with torque as the input and angular velocity as the output. The rotor angular velocity, ω, is multiplied by the proper constant K_M to give the motor's back emf, which is used as an input to the integrator for the $(L_M + L_G)$ circuit. It is also multiplied by the tachometer generator constant, K, to give the voltage generated by the tachometer generator. This voltage, E_t, is subtracted from the control voltage, E_s, in the operational amplifier as shown and the difference, or error voltage is passed to the correcting network simulator.

One of the main problems of a control system design is the correcting network. In actual practice, simulator circuits for various tentative network designs would be inserted in the space indicated, to be tested with the rest of the system. Since the network chosen can be simulated using the principles previously discussed, the correcting network simulator is not diagrammed. The output of the correcting network simulator is then amplified by the proper constant, K_A, to produce E_o. This is the input to the integrator representing L_F. Thus, the system is completely closed, and all needed internal quantities have been produced.

Direct simulation results in a minimum number of computing components, and so makes it relatively easy to simulate extremely complex networks.

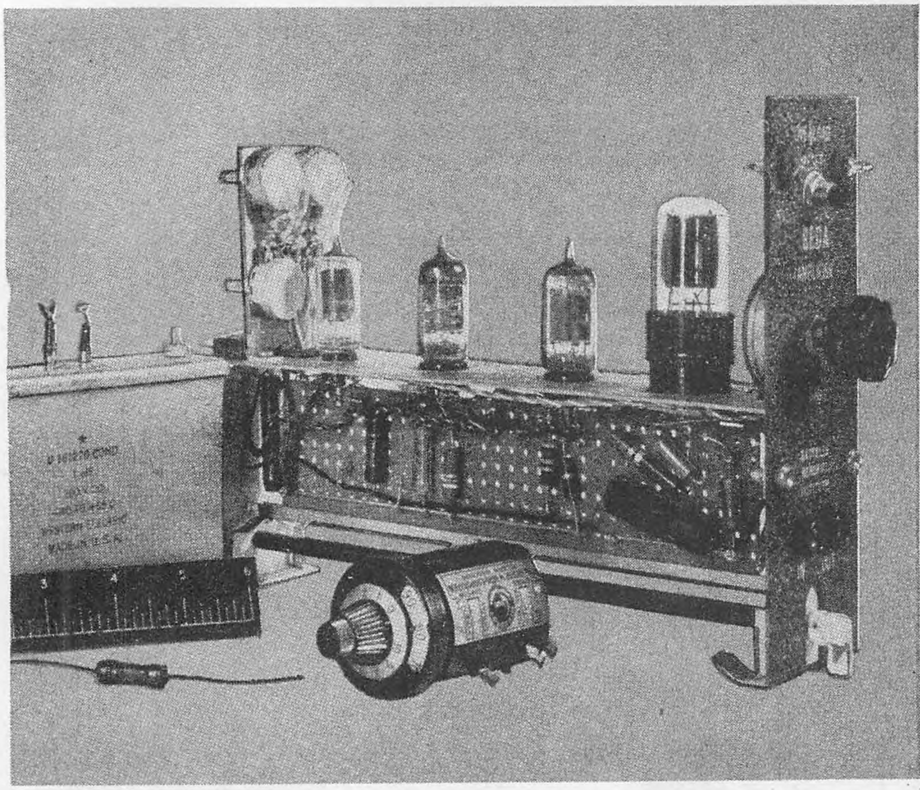


TABLE OF SYMBOLS

(in order of occurrence)

F_g—Gravitational force
M, M′—Masses
k_g—Gravitational constant
r—Distance
F_e—Electrical force
e, e′—Electrical charges
k_e—Dielectric constant
F_m—Magnetic force
m, m′—Magnetic charges
k_m—Magnetic constant
W_g—Gravitational work
W_e—Electrical work
W_m—Magnetic work
V_g—Gravitational potential
V_e—Electrical potential
V_m—Magnetic potential
X—Displacement
B—Mechanical damping
K—Spring constant
F—Force
L—Inductance
R—Resistance
C—Capacitance
Q—Electrical charge
E—Volts
n, n′—Scale factors
t, t_m—Time
$\nabla^2 \phi$—Laplacian of ϕ
T—Temperature
f(t)—Arbitrary function
θ, ϕ, z—Coordinates
Z_n—Impedance
Y_o—Displacement
G—Coulomb-friction factor

In system design and evaluation, analogs offer many advantages from the standpoint of time and cost. Complex problems that would not otherwise be tackled can be accurately solved through the use of analogies that link the various laws of nature. Typical problems solvable with these analogs are systems involving lumped and distributed constants.

Analogous Systems

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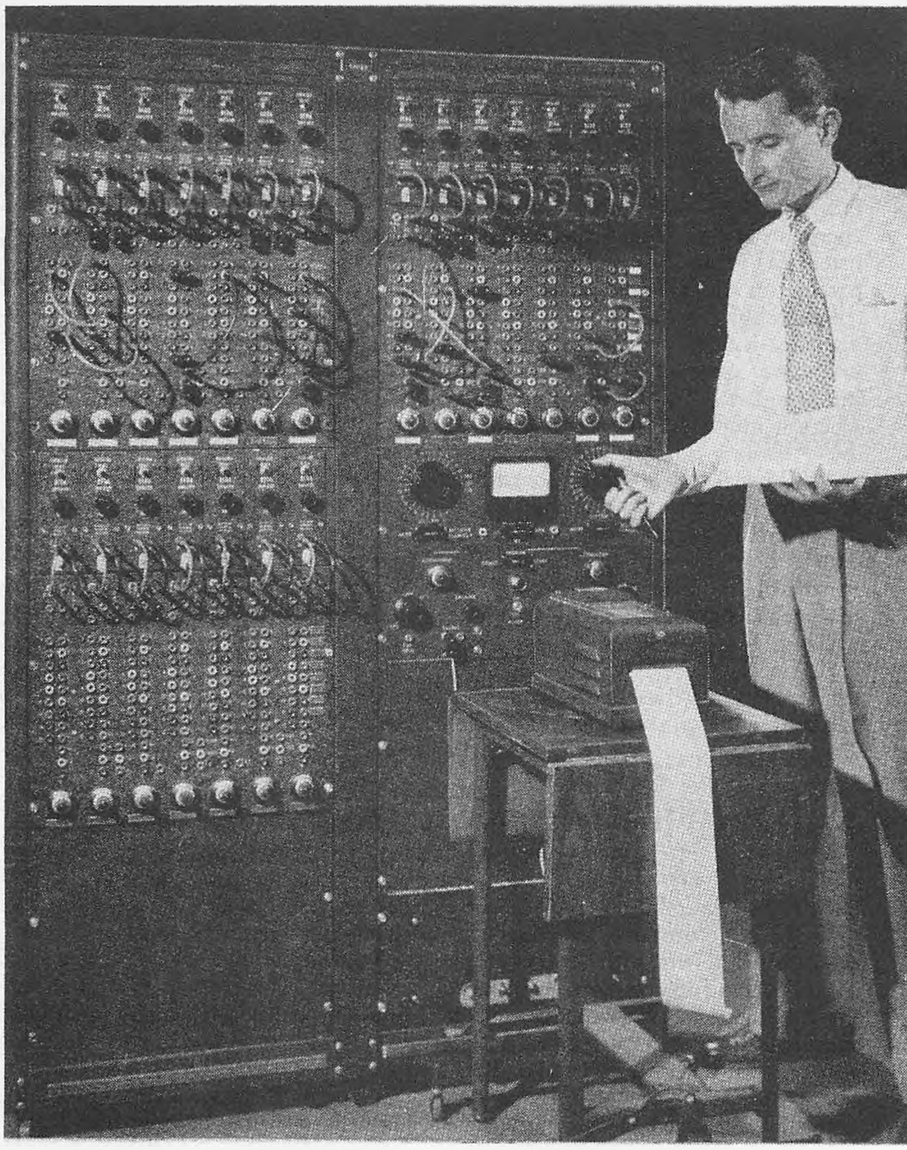
MANY PRESENT-DAY DESIGN PROBLEMS have become so complex that they are difficult if not impossible to solve by ordinary empirical procedures. A disproportionate amount of engineering time and money is therefore expended on extensive calculations, and in tests to prove the assumptions underlying these calculations. One answer to this problem is to make use of the analogies that relate the simple laws of science. These remarkable similarities are a powerful means of increasing the designer's productivity, and can result in better engineering designs at reduced cost.

Analogous engineering systems permit the designer to translate a given problem from one physical system in which design computations are difficult and test models expensive, to another physical system in which low-cost models with continuously variable parameters can be quickly produced and tested. Within the range of operating conditions in which known linear and other laws of design apply, data obtained in the analogous system are applicable in the *original* system. Thus, the dynamic performance of mechanical, acoustic, hydraulic, thermal, magnetic and even complex systems containing components in several of these physical fields, can often be reproduced with simple analogous electrical systems.

Electrical models can be produced at low cost in minimum time, and once assembled, can easily be modified with the many electrical components mass-produced to fill radio, television, industrial control and armed services requirements. It is easier to turn a variable resistance control through one-hundred settings than it is to machine



ELECTRONIC differential analyzer at Battelle Memorial Institute is used for solving differential equations that arise in the analysis of problems in many branches of engineering. It consists of 21 operational amplifiers, together with the necessary precision resistors, capacitors, and potentiometers required to solve equations of approximately the eighth order. The d-c amplifiers are mounted so that they can be removed individually for servicing. By means of suitable switching, the voltage from any point in the computer can be read on the recorder. The meter panel is incorporated into the computer to simplify balancing the amplifiers and checking the plate supply voltages.

• in • Engineering • Design

one-hundred similar mechanical components of varying dimension or mass. And more important, the computational and recording devices now available are designed to accept the ouput signals of voltage and current produced in electrical models, whereas complex sensing and conversion devices are required if equivalent measurements are to be made in mechanical or thermal systems. Many important mathematical manipulations, such as algebraic summation, trigonometric resolution, differentiation, integration and other data-processing steps, can be carried out rapidly with electrical analogs of the mathematical operations.

Analogous Laws of Nature

The creative scientists of past centuries showed great insight into analogies between basic laws of nature. Newton found that the *gravitational force* between two point masses is proportional to the product of the masses, and inversely proportional to the square of the distance between them. Coulomb concluded that the force between two point *electric* charges or between two isolated-point *magnetic* poles, is also proportional to the product of the two quantities and inversely proportional to the square of the distance between them. The striking analogy between these basic laws for three different physical systems is shown by the following relationships:

GRAVITATIONAL:

$$F_g = \frac{MM'}{k_g r^2} \tag{1a}$$

ELECTRICAL:

$$F_e = \frac{ee'}{k_e r^2} \tag{1b}$$

MAGNETIC:

$$F_m = \frac{mm'}{k_m r^2} \tag{1c}$$

Of course, the gravitational force between masses is always attractive, as is the force between unlike polarities; and the force between like electric charges or like magnetic poles is repulsive. Yet the mathematical expressions for these three laws are similar in form.

The existence of similar laws of *force* in static gravitational, electrical, and magnetic fields immediately establishes analogies between many other relations for these fields. For example, the *work* done in separating two adjacent bodies to an infinite distance is obtained by integrating the force over the path they follow:

GRAVITATIONAL: $$W_g = \int_{r=r_1}^{r=\infty} F_g dr$$

$$= \int_{r=r_1}^{r=\infty} \frac{MM'\,dr}{k_g r^2} = \frac{MM'}{k_g r_1} \qquad \text{(2a)}$$

ELECTRICAL:
(for charges of opposite polarity)

$$W_e = \int_{r=r_1}^{r=\infty} F_e dr$$

$$= \int_{r=r_1}^{r=\infty} \frac{ee'_2\,dr}{k_e r} = \frac{ee'}{k_e r_1} \qquad \text{(2b)}$$

MAGNETIC:
(for poles of opposite polarity)

$$W_m = \int_{r=r_1}^{r=\infty} F_m dr$$

$$= \int_{r=r_1}^{r=\infty} \frac{mm'_2 dr}{k_m r} = \frac{mm'}{k_m r_1} \qquad \text{(2c)}$$

Since the potential at any point in a field is a measure of the *work* done in bringing a unit magnitude from infinity to that point, the preceding analogies can be converted to similar expressions for potential and for expressions derived from potential as follows:

GRAVITATIONAL POTENTIAL:

$$V_g = \frac{M}{k_g r} \qquad \text{(3a)}$$

ELECTRICAL POTENTIAL:

$$V_e = \frac{e}{k_e r} \qquad \text{(3b)}$$

MAGNETIC POTENTIAL:

$$V_m = \frac{m}{k_m r} \qquad \text{(3c)}$$

These expressions hold true only for point magnitudes in the fields, but since they apply for each point magnitude, the effects of distributed quantities can be established by simple integration or superposition.

Analogous Engineering Systems

The above analogies apply only to the fields of static mechanics, electrostatics and magnetostatics. Analogous relations also exist between these and the related fields of heat flow, hydraulics, acoustics and other forms of energy transfer. From the analogies listed in Table I, it is evident that parameters in different physical systems occupy similar positions in the mathematical expressions of the natural laws. Several such examples appear in Table II. In those instances where it might be difficult to write all the equations relating the parameters of a physical system, it is sometimes possible to construct a direct analog of the original system in the model system, simply by replacing each component of the original system with its analogous component in the model.

Using the parameter analogs listed in Table II, it is possible to build up, step-by-step, electrical analogs for very complex physical systems. Each element of the original physical system is replaced by its corresponding element in the analog system. Furthermore, each interconnection, or possible interaction or interdependence in the original system, must be appropriately expressed in the analog system so that the dynamic performance characteristics of the two systems are similar. If the analog truly reproduces the original system, it is not necessary that the engineer be able to write explicit equations for either system (which might be difficult with very complex systems). Instead, it is only necessary that the analog be a true and valid model, and the applied forces and operating boundary conditions be known. The desired forces can then be applied for each design modification, and the system performance established from tests on the analog. The latter data are then translated directly back into terms appropriate to the original system.

Functionally-Analogous Physical Systems

It is not always necessary, and frequently not desirable, to replace every individual element or component of the original system with an equivalent element in the analog system. Instead, the analog models can often be greatly simplified and their usefulness broadened, if they reproduce only the functional operations of the original system. This is particularly appropriate if the performance of the original system can be formulated mathematically. If this is possible, the electrical model can reproduce the mathematical operations implicit in the mathematical formulation of the original system, without regard for the actual nature or elements of that system.

The differential analyzer is a good example. This computer is an electronic model whose carefully-designed amplifiers are adaptable for mathematical operations such as addition, subtraction, integration and differentiation. It can readily handle simultaneous integrodifferential equations (involving both integration and differentiation) of relatively high orders. Multiplication of two variables, arbitrary functions and nonlinear coefficients

Table I—Analogous Laws of Nature

Type of Law	Mechanical Translation	Mechanical Rotation	Thermal Heat-Flow	Hydraulic Mass Flow	Magnetic Circuits	Acoustic Systems	Electric Circuits
Force between point magnitudes	$F_g = \frac{MM^1}{K_g r^2}$				$F_m = \frac{mm^1}{K_m r^2}$		$F_e = \frac{ee^1}{K_e r^2}$
Static equilibrium of forces at a point	$\Sigma F_p = O$	$\Sigma T_p = O$	Law of conservation of energy	Law of conservation of mass	Flux lines are continuous		$\Sigma I_p = O$
Dynamic response to applied force	$F = Ma$	$T = J\alpha$				$p = M_a \ddot{x}$	$E = L\frac{dI}{dt}$
A = BC	$F = C^{-1}S$	$T = C_R^{-1}\theta$	Q = AV	Q = AV	$MMF = \Re\,\phi$		E = IR
$A = B^2C$	$\frac{1}{2}Mv^2$	$\frac{1}{2}J\omega^2$				$\frac{1}{2}M_a\dot{x}^2$	RI^2

Table II—Analogous Parameters in Different Physical Systems

Type of Parameter	Mechanical Translation	Mechanical Rotation	Thermal Heat-Flow	Hydraulic Fluid-Flow	Magnetic Fields and Circuits	Acoustic Systems	Electric	Electric	Electric	Electric
Applied Force	F	T	T	p	MMF $\mathfrak{F}$	Pressure p	E EMF, volts	I Current, amp	dE dt volts/sec	dI/dt amps sec
Inertial	M	J Polar Moment of Inertia				Acoustical Inertance M_a	L Inductance, henries	$\Gamma = L^{-1}$ Inverse Inductance	L Inductance, henries	$\Gamma = L^{-1}$ Inverse Inductance
Elastance	C_M Compliance	C_R Rotational Compliance	Thermal Capacity	Compressibility		Acoustic Capacitance C_a	$S = C^{-1}$ Elastance, darafs	C Capacitance, farads	$S = C^{-1}$ Elastance, darafs	C Capacitance, farads
Frictional Dissipative	B Translational Resistance	Rotational Resistance	Thermal Resistivity	Hydraulic Resistivity	Reluctance $\mathfrak{R}$	Acoustic Resistivity r_a	R Electrical Resistance, ohms	$G = R^{-1}$ Electrical Conductance, mhos	R Electrical Resistance, ohms	$G = R^{-1}$ Electrical Conductance, mhos
Momentum	$M\frac{ds}{dt}$	$J\frac{d\theta}{dt}$				$M_a c$	LI	ΓE	$L\frac{dI}{dt}$	$\Gamma\frac{dE}{dt}$
Kinetic Energy	$\frac{1}{2}M\left(\frac{ds}{dt}\right)^2$	$\frac{1}{2}J\left(\frac{d\theta}{dt}\right)^2$				$\frac{1}{2}M_a c^2$	$\frac{1}{2}LI^2$	$\frac{1}{2}\Gamma E^2$	$\frac{1}{2}L\left(\frac{dI}{dt}\right)^2$	$\frac{1}{2}L\left(\frac{dE}{dt}\right)^2$
Potential Energy	Ms					$M_a x$	LQ	$\Gamma\int Edt$	LI	ΓE
Power	Fv	$T\omega$	$Q_h T$	$Q_F p$	MMF x ϕ	pc	EI	EI	$\frac{dE}{dt}\frac{dI}{dt}$	$\frac{dE}{dt}\frac{dI}{dt}$
Force Impulse	$\int Fdt$	$\int Tdt$	$\int Tdt$	$\int pdt$	$\int \mathfrak{F}dt$	$\int pdt$	$\int Edt$	$\int Idt$	E	I
Displacement	s	θ				Acoustic Displacement x	$Q = \int Idt$ Electric Charge, coulombs	$\int Edt$ Voltage Impulse, volt/sec	I Electric Current, amp	E Voltage Drop, volts
Velocity	$\frac{ds}{dt} = v$	$\frac{d\theta}{dt} = \omega$	$Q_h \frac{Btu}{sec}$	$Q_F \frac{quantity}{sec}$	ϕ flux	$\frac{dx}{dt} = c$	$I = \frac{dQ}{dt}$ Current, $amp = \frac{coulombs}{sec}$	E Voltage, volts	dI dt Rate of Change of Current, amp/sec	dE dt Rate of Change of Voltage, volts/sec
Acceleration	$\frac{d^2s}{dt^2} = a$	$\alpha = \frac{d^2\theta}{dt^2}$				$\frac{d^2x}{dt^2}$	$\frac{dI}{dt} = \frac{d^2Q}{dt^2}$ amp/sec	$\frac{dE}{dt}$ volts sec	$\frac{d^2I}{dt^2}$ amp/sec²	$\frac{d^2E}{dt^2}$ volts/sec²

or parameters can be introduced with specially-designed analyzer elements. Thus, any physical system whose functions can be analyzed as a group of mathematical operations of these types can be simulated by the differential analyzer. When suitable reconnections are made between functional components and the controls properly adjusted, the same differential analyzer will handle successively a multitude of different physical systems or design problems.

Another way of approaching this subject is to recognize that different physical systems whose operations are describable by integrodifferential equations of the same form are of necessity functionally similar, and therefore representable in the same differential analyzer model. Thus, a single differential analyzer can be set up to handle three simultaneous second-order equations with constant coefficients, and used repeatedly, without changing internal connections, on problems from any engineering system whose performance can be described by such equations. Changes in the values of single elements or groups of elements in the engineering system can be reproduced simply by varying the appropriate potentiometers in the differential analyzer. Initial or boundary conditions are set as desired before the appropriate force functions are applied to the analyzer. Sometimes curve-drawing instruments or cameras photographing cathode-ray screens are used to record the applied force functions and the response of the systems.

Designers familiar with the differential or integral equations for their engineering systems find that even a simple differential analyzer can be very useful as a companion to the slide rule and calculating machines commonly used. Accuracies of one percent or better are obtainable in most cases and are quite adequate, considering the accuracy of the design assumptions and the designer's knowledge of engineering materials. Computations for typical system arrangements can be completed in a few minutes and do not require the lengthy manual calculations commonly used in engineering design. The references listed at the end of this article are just a few of the many authoritative sources of additional information on the equations for such physical systems, their analysis and application.

Analogous Systems as Components of Engineering Systems

The designer of complex systems can profit by recognizing and applying another basic principle of design. In an engineering system containing many functional elements or groups of components, certain chains of functions are difficult to design or produce while in the original physical system. In such instances, it may be advantageous to replace these operational functions in the original physical system with equivalent functional operations in an analogous system offering advantages of simplicity and reliability in design and operation. For example, a complex mechanical aircraft-control system can be miniaturized and improved if, at a suitable point, the mechanical system operation is converted into an equivalent function of an electrical or hydraulic system. This requires an adequate transducer or conversion device that faithfully translates the mechanical function into an appropriate analogue signal. The difficult transmission, computational or other design functions are then carried out in the analogous system. In a final step, the electrical or hydraulic response functions are translated back into the original physical system by a suitable transducer. This principle can be extended to industrial control systems and other applications, resulting in significant cost reductions, increases in operational speed and accuracy, and reductions in the over-all costs and size of equipment.

Electric Analogs for Lumped-Constant Problems

If a problem in dynamics is represented by a linear second-order differential equation with constant coefficients and an electric analogy is to be used, an electric circuit with the same general equation has to be found. In some instances, however, the electric analog is not the only solution and, as in some mechanical systems, an acoustic analog may have certain advantages. But, in general, the widespread knowledge of electric circuit theory and measurement techniques, coupled with the availability of high-quality variable electric components such as inductors, capacitors and resistors, makes the electric analogy most desirable.

To see in detail how an electric analog of a system is obtained, consider the mechanical system shown in Fig. 1(a). It consists of a mass M, of 38.6 lb connected to a fixed reference by a spring K, whose constant is 10 lb per inch. A driving force, $F(t) = 10 \cos \alpha t$, is applied to the mass M, and there is a damping force, B, equal to 5 lb per in. per second. The equation for displacement x in this system is:

$$M \frac{d^2x}{dt^2} + B \frac{dx}{dt} + Kx = F \cos \alpha t \tag{4}$$

or, dividing by M

$$\frac{d^2x}{dt^2} + \frac{B}{M}\frac{dx}{dt} + \frac{Kx}{M} = \frac{F}{M} \cos \alpha t \tag{5}$$

The analogous electrical system is shown in Fig. 1(b) in which inductance L, capacitance C and resistance R are connected in series with an alternating current source. Its equation is therefore:

$$L \frac{d^2Q}{dt_m^2} + R \frac{dQ}{dt_m} + \frac{Q}{C} = E \cos \omega t_m \tag{6}$$

where t_m is the time and Q the charge in the electrical system. Dividing by L

$$\frac{d^2Q}{dt_m^2} + \frac{R}{L}\frac{dQ}{dt_m} + \frac{Q}{CL} = \frac{E}{L} \cos \omega t_m \tag{7}$$

The most direct design procedure is to equate the corresponding coefficients in the two equations (1). In this case, charge Q becomes analogous to displacement x, inductance L to mass M, resistance R to damping B, capacitance C to compliance $1/K$, and voltage E to force F. Equating coefficients gives the following set of equations:

$$\frac{R}{L} = \frac{B}{M}; \frac{1}{CL} = \frac{K}{M}; \frac{E}{L} = \frac{F}{M} \tag{8}$$

There being four electrical parameters in these three equations, only one can be chosen arbitrarily. In this analog $t_m = t$ and $\omega = \alpha$. Therefore, velocity is

$$\frac{dx}{dt} = \frac{dQ}{dt} = I = \frac{E}{R}$$

and acceleration is

$$\frac{d^2x}{dt^2} = \frac{d^2Q}{dt^2} = \frac{dI}{dt} = \frac{E}{L}$$

Choosing E such that 100 v equals 10 lb,

$$L = \frac{EM}{F} = \frac{100 \times 38.6}{10 \times 386} = 1.0 \text{ henry}$$

($M = W/g$ where g is the acceleration due to gravity or 386 lb-in./sec^2)

$$C = \frac{M}{KL} = \frac{38.6}{386 \times 10 \times 1} = 0.01\mu$$
$$= 10{,}000\ m\mu$$

$$R = \frac{BL}{M} = \frac{5 \times 386 \times 1}{38.6} = 50 \text{ ohms}$$

This analogy is perfectly valid and useful, but it has several disadvantages. The capacitance value of 10,000 micro-

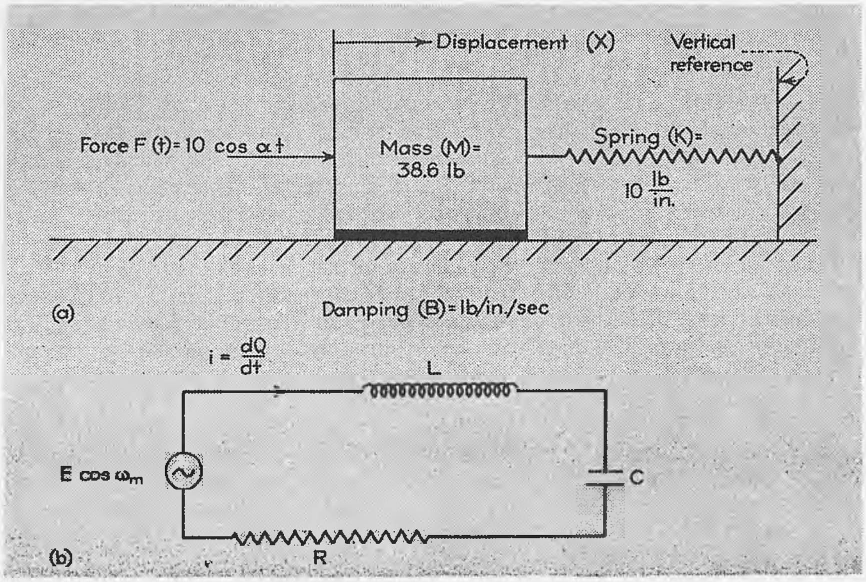


Fig. 1—Mechanical system (a) with mass M working against spring K and friction B is converted to analog (b). Q is analogous to x, L to M, R to B and E to F.

farads is high and therefore might be difficult to obtain in such a large size. The value of C could be reduced by increasing L, but this is already 1.0 henry, and increasing it would make it more difficult to obtain.

Another disadvantage is that the frequency of the electric circuit equals that of the mechanical circuit. If the mechanical frequency is very low, it might be convenient to speed up the solution. On the other hand, if the mechanical frequency were relatively high, say 2,000 cps, it would not be convenient to record it on available graphic recording instruments.

One solution is to use appropriate scale changes in time and displacement. Consider the mechanical system just discussed with the addition of two scale factors n_1 and n. To change the time scale, let $t = n_1 t_m$. Thus, if $n_1 = 10$, then one second in the mechanical system is only one-tenth of a second in the electrical system. The value of n can be arbitrarily chosen as a dimensional constant and used to change the displacement scale; ie, $x = nQ$. Thus:

$$\frac{dx}{dt} = \frac{n\,dQ}{dt} = \frac{n\,dQ}{dt_m}\,\frac{dt_m}{dt} = \frac{n}{n_1}\,\frac{dQ}{dt_m}$$

$$\frac{d^2x}{dt^2} = \frac{n}{n_1}\,\frac{d}{dt}\left(\frac{dQ}{dt_m}\right) =$$

$$\frac{n}{n_1}\,\frac{d^2Q}{dt_m^2}\,\frac{dt_m}{dt} = \frac{n}{n_1^2}\,\frac{d^2Q}{dt_m^2}$$

Eq (5) then becomes

$$\frac{n}{n_1^2}\,\frac{d^2Q}{dt_m^2} + \frac{n}{n_1}\,\frac{B}{M}\,\frac{dQ}{dt_m} +$$

$$\frac{KnQ}{M} = \frac{F}{M}\cos n\,\alpha t_m$$

or
$$\frac{d^2Q}{dt_m^2} + n_1\,\frac{B}{M}\,\frac{dQ}{dt_m} +$$

$$\frac{n_1^2 KQ}{M} = \frac{n_1^2}{n}\,\frac{F}{M}\cos n_1 \alpha t_m \qquad (9)$$

A comparison of coefficients in Eqs (9) and (7) gives:

$$\frac{R}{L} = \frac{n_1 B}{M}\,;\ \frac{1}{CL} = n_1^2\,\frac{K}{M}\,;$$

$$\frac{E}{L} = \frac{n_1^2}{n}\,\frac{F}{M}\,;\ \omega = n_1 \alpha \qquad (10)$$

There are now six parameters (R, L, C, E, ω, n) and only four equations so that two parameters can be chosen arbitrarily. This permits more flexibility in the choice of components, as can be seen below. Again letting 100 volts equal 10 lb, and making

$n_1 = 100$ and $n = 10{,}000$

$$L = \frac{nEM}{n_1^2 F} = \frac{10^4 \times 100 \times 38.6}{386 \times 10^4 \times 10} = 1.0 \text{ henry}$$

$$C = \frac{M}{n_1^2\,KL} = \frac{38.6}{386 \times 10^4 \times 10 \times 1.0} = 1.0\ m\mu$$

$$R = \frac{n_1 LB}{M} = \frac{100 \times 1.0 \times 386 \times 5}{38.6} = 5{,}000 \text{ ohms}$$

Thus, by means of the scale changes, the value of capacitance has been reduced to one microfarad, which is more workable than the value calculated above.

This analog method can be extended to mechanical systems having more degrees of freedom and to other systems such as acoustic and rotational-mechanical systems. The value of such a method becomes apparent when considering the problem of designing the spring in the foregoing example for optimum performance under varying applied forces. It is much easier and cheaper to vary the electrical parameters than their respective mechanical counterparts.

Electric Analogs for Distributed-Constant Systems

Many distributed-constant systems can be approximated by electrical networks. An example might be that of determining the forces produced in a long coiled spring subjected to cyclic or impact driving forces. The conventional mathematical solution of this problem, while possible, is long and involved, especially if such effects as clashing of adjacent coils is considered.

The analog of the system can be determined in the following way: Consider the spring to be composed of a series of lumped masses and of massless springs connected in series, and that there is damping associated with each mass. In the mechanical schematic of the spring shown in Fig. 2(a), each section of the spring (to be represented) is similar to the mechanical problem of Fig. 1(a). The analog for each section is therefore an inductance, capacitance and resistance element connected in series.

The electric network is shown in

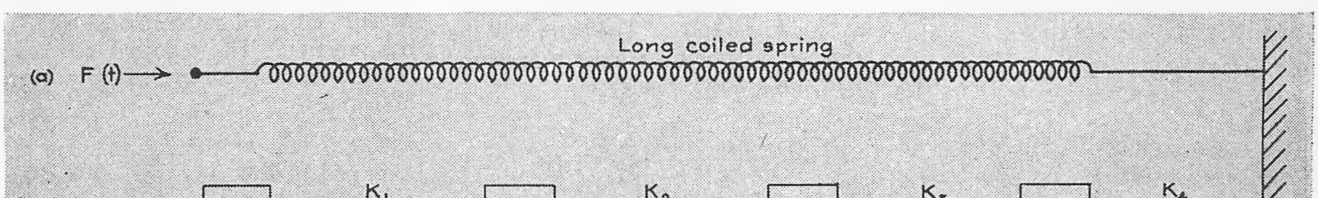


Fig. 2—Distributed-constant system such as long coiled spring (a) can be considered as a series of lumped masses (b) subjected to various forces. Conversion to analog (c) results in four sections, each containing an inductance, capacitance and resistance.

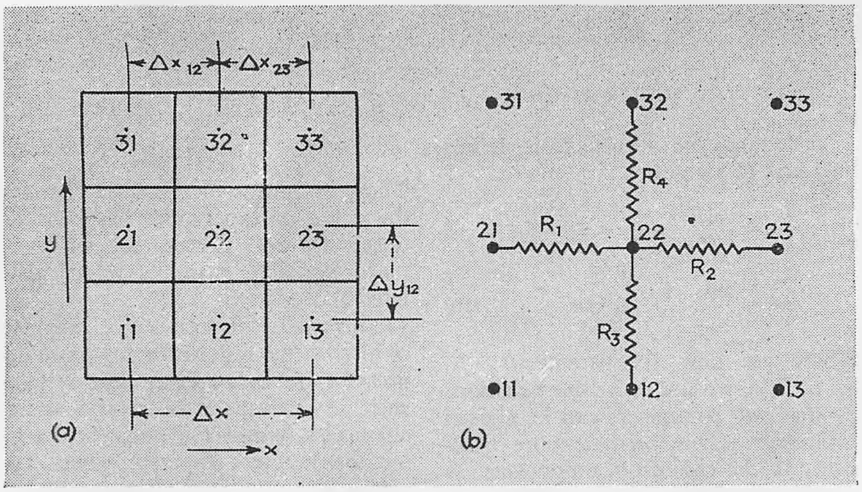


Fig. 3—Partial differential equation of Laplace's relationship and equivalent net (a) can be solved with analog network (b) composed of four resistances connected to common point. Sum of the currents at point 22 must be zero.

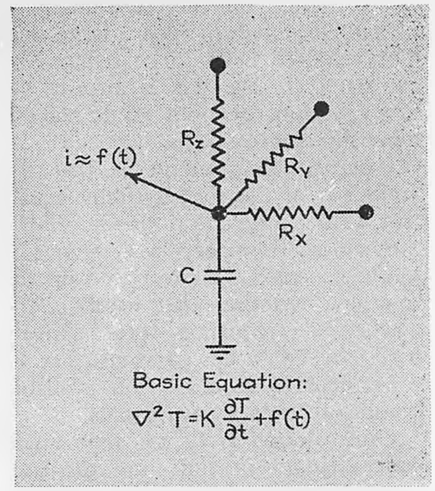


Fig. 4—Transient heat-flow equation involving partial differential with left side represented by three-dimensional network.

Fig. 2(b). Each succeeding section is coupled to the preceding one by a massless spring represented by C. The number of sections determines the accuracy of the solution, and it has been determined experimentally that ten or fewer sections are sufficiently accurate for most engineering problems. More than ten sections improve the accuracy, but also increase the overall expense.

Clashing of adjacent coils can be simulated in the electrical network by a nonlinear device consisting of diodes which limit the electrical displacement of each section when it reaches a value equal to that of its neighbor ([2]). The spring can thus be subjected to cyclic or impact forces of any type, and the effect of loading, clashing and resonance studied.

Analogs for Partial Differential Equations

Many problems in physics and engineering are specified by partial differential equations, e.g., heat flow, fluid flow and stress problems, and some of these can be approximated by electric networks. To see how a network is set up to represent a partial differential equation, consider Laplace's equation, $\nabla^2\phi = 0$. In cartesian coordinates, this becomes ([3])

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad (11)$$

Referring to the net in Fig. 3(a) and noting increments Δx, Δx_{12} and Δx_{23}

$$\frac{\partial \phi}{\partial x_{12}} \approx \frac{\phi_{22} - \phi_{21}}{\Delta x_{12}} \qquad (12)$$

$$\frac{\partial \phi}{\partial x_{23}} \approx \frac{\phi_{23} - \phi_{22}}{\Delta x_{23}} \qquad (13)$$

$$\frac{\partial^2 \phi}{\partial x^2} \approx \frac{\phi_{23} - \phi_{22}}{\Delta x\, \Delta x_{23}} - \frac{\phi_{22} - \phi_{21}}{\Delta x\, \Delta x_{12}} \qquad (14)$$

and similarly

$$\frac{\partial^2 \phi}{\partial y^2} \approx \frac{\phi_{32} - \phi_{22}}{\Delta y\, \Delta y_{23}} - \frac{\phi_{22} - \phi_{12}}{\Delta y\, \Delta y_{21}} \qquad (15)$$

Thus, by equating and substituting expressions

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}$$

$$= \frac{\phi_{23} - \phi_{22}}{\Delta x\, \Delta x_{23}} - \frac{\phi_{22} - \phi_{21}}{\Delta x\, \Delta x_{12}} +$$

$$\frac{\phi_{32} - \phi_{22}}{\Delta y\, \Delta y_{23}} - \frac{\phi_{22} - \phi_{12}}{\Delta y\, \Delta y_{21}} = 0 \qquad (16)$$

An electrical equation of the same general form is composed of four resistors connected to a common point as in Fig. 3(b). The sum of the currents at point 22 must be zero by Kirchoff's Law. Hence,

$$\frac{E_{23} - E_{22}}{R_2} - \frac{E_{22} - E_{21}}{R_1} +$$

$$\frac{E_{32} - E_{22}}{R_4} - \frac{E_{22} - E_{12}}{R_3} = 0 \qquad (17)$$

Eqs. (16) and (17) are analogous with

$$E_{23} - E_{22} \approx \phi_{23} - \phi_{22}$$
$$E_{22} - E_{21} \approx \phi_{22} - \phi_{21}$$
$$E_{32} - E_{22} \approx \phi_{32} - \phi_{22}$$
$$E_{22} - E_{12} \approx \phi_{22} - \phi_{12}$$

and if each term in Eq (16) is multiplied by $\Delta x\ \Delta y$

$$R_2 \approx \frac{\Delta x_{23}}{\Delta y}\ ;\ R_1 \approx \frac{\Delta x_{12}}{\Delta y}\ ;$$

$$R_4 \approx \frac{\Delta y_{32}}{\Delta x}\ ;\ R_3 \approx \frac{\Delta y_{21}}{\Delta x}$$

If alternating currents were used, the construction of inductance or capacitance grids would be handled in a similar manner.

By extending this method, networks can be constructed for more complex equations such as that for transient heat flow

$$\nabla^2 T = K \frac{\partial T}{\partial t} + f(t) \qquad (18)$$

Here T is temperature, K is thermal conductivity, and $f(t)$ is an arbitrary heat function applied to the system. For the three-dimensional case, the network is shown in Fig. 4, with the left side of Eq (18) represented by the three-dimensional resistance network. The current proportional to $\partial T/\partial t$ is obtained by the capacitor to ground because $i_c = C\ \partial E/\partial t$, while the heat source, $f(t)$, is represented by the externally-supplied current, $i \approx f(t)$, at the junction. The required boundary conditions can be handled by suitable potentials.

The values of the resistors are determined in the same way as for Laplace's equation:

$$R_x = \frac{\Delta x}{\Delta y\, \Delta z}\ ,\ R_y = \frac{\Delta y}{\Delta x\, \Delta z}\ ,$$

$$R_z = \frac{\Delta z}{\Delta x\, \Delta y}\ .$$

The value of $C = K\ (\Delta x\ \Delta y\ \Delta z)$.

In many problems it is better to work in cylindrical or spherical coordinates. From a study of the manner in which the values of R_x, R_y, R_z and C were obtained and a consideration of the geometry of the systems, it is seen that in cylindrical coordinates:

$$R_r = \frac{\Delta r}{r\, \Delta \theta\, \Delta z}\ ;\ R_\theta = \frac{r\, \Delta \theta}{\Delta r\, \Delta z}\ ;$$

$$R_z = \frac{\triangle z}{r \triangle \theta \triangle r} ; C = (r\triangle\theta\triangle r\triangle z)\ K$$

and in spherical coordinates:

$$R_r = \frac{\triangle r}{r^2 \sin\phi \triangle \theta \triangle \phi} ;$$

$$R_\theta = \frac{r \sin\phi \triangle \theta}{r \triangle r \triangle \phi} ;$$

$$R\phi = \frac{r}{r \sin\phi \triangle r \triangle \theta} ;$$

$$C = (r^2 \sin\phi \triangle r \triangle \theta \triangle \phi)\ K$$

In many problems of this type there is a certain degree of symmetry which can be advantageously used to reduce the number of components required in the network. For example, in a cylindrical problem with axial symmetry, it is necessary to use only a two-dimensional network in the z and r coordinates.

Operational Analogs

It is not always convenient to use the direct electric analog because of the difficulty of obtaining sufficiently high quality inductances and capacitances. In such cases the operational analog is often used.

In most of the analogous physical systems considered so far, there has been a one-to-one correspondence between the individual elements of one system and analogous elements in the other system. A distinct *type* of physical element is required in the analogous system for each distinct *type* of element in the original system. For example, in the electric analog of a mechanical system with voltage analogous to force, inductance is analogous to mass, resistance to viscous friction, and elastance (inverse capacitance) to stiffness. Such systems might be called *physical* analogs as opposed to *operational* analogs which will be discussed below.

Although these analogs have proven very useful in the analysis of many physical problems, they do have certain inherent limitations. It may not be possible, for example, to find a physically realizable element in one system that is analogous to a particular element in the other system. This is a very basic limitation. Mutual inductance and mutual mass, for example, are analogous elements in the voltage-force analog of simple electrical and mechanical systems. In some of the more complex mechanical systems, however, electrical transformers analogous to the mutual masses of the system are not physically realizable[4]. In most of these cases, an analogous electrical system can be found by a change in the dependent variable. A current-force analog of the system may be possible in which mutual capacitances are analogous to the mutual masses. Of course, the situation may also arise in which the current-force analog of a mechanical system is not physically realizable. Such situations tend to increase the complexity of establishing analogs and to reduce the over-all usefulness of the method.

It is a recognized fact that ideal elements in any physical system simply do not exist. Every spring has distributed mass, internal damping, hysteresis, and other properties, as well as its primary property of stiffness. Every mass has elastic properties, every inductance has resistance, and so on. Although the primary property of an element in one physical system may be ideally analogous to the primary property of an element in another system, secondary effects in the two supposedly analogous elements may depart from the ideal in radically different ways. This produces a consequent multiplication of the errors created by this non-ideal performance.

The most serious limitation of true analog systems is the difficulty encountered in constructing some types of elements so that they can be readily adjusted over a wide range of values. Adjustable elements of some types are relatively expensive, and the range of values over which they can be adjusted without introducing serious non-ideal characteristics may be quite limited. As a consequence, it may not be practical to analog a system over a wide range of variation of all the original elements. This leads to added expense and great inconvenience in the use of the analog. In electric analogs of mechanical systems, for example, variable capacitors and, in particular, high-quality continuously variable inductors are extremely difficult and expensive to construct. In addition, in-

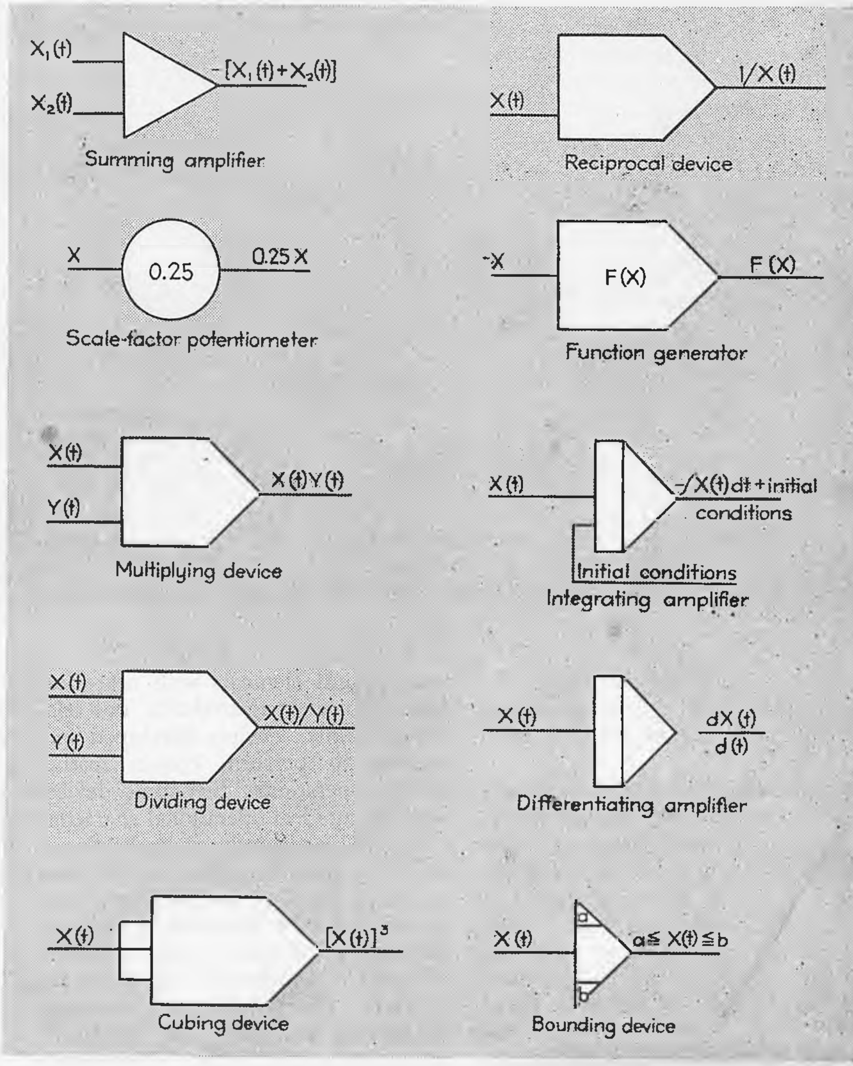
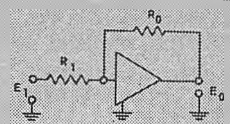
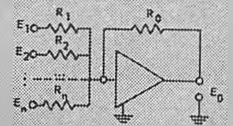
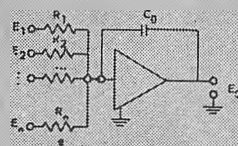
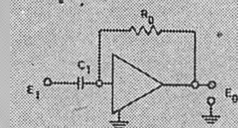
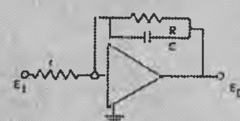
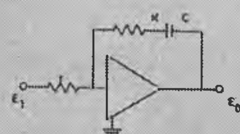
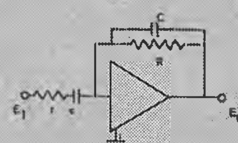
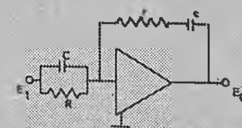


Fig. 5—Suggested symbols for various operational elements used in combination to produce commercially-available computers, such as differential analyzers.

Table III—Operational Amplifiers

Equation	Operation	Circuit	Output Function
$E_0 = -\frac{R_0}{R_1} E_0$	1. Scale and/or 2. Sign change		$E_0 = -\frac{R_0}{R_1} E_0$
$E_0 = -R_0\left[\frac{E_1}{R_1}+\frac{E_2}{R_2}+\dots\frac{E_n}{R_n}\right]$	1. Summing 2. Scale change on each input variable 3. Sign change on all input variables		$E_0 = -\left[\frac{R_0}{R_1}E_1+\frac{R_0}{R_2}E_2\dots+\frac{R_0}{R_n}E_n\right]$
$E_0 = -\frac{1}{\omega C_0}\left[\frac{\int E_1dt}{R_1}+\frac{\int E_2dt}{R_2}+\dots\frac{\int E_ndt}{R_n}\right]$	Integration		$E_0 = -\frac{1}{C_0}\left[\frac{E_1}{pR_1}+\dots\frac{E_n}{pR_n}\right]$
$E_0 = \omega R_0 C_1 \frac{dE_1}{dt}$	Differentiation		$E_0 = pE_1R_0C$
$\frac{dE_0}{dt} + K_1 E_0 = E_1$	Solution of first order linear differential equation		$E_0 = E_1\left[\frac{R}{r}\frac{1}{RCp+1}\right]$
$K_1E_1 + K_2\int E_1dt = E_0$	Ditto		$E_0 = E_1\left[\frac{RCp+1}{rCp}\right]$
$E_1 + K_1\frac{dE_0}{dt} + (K_2+K_3)E_0 + K_3\int E_0dt$	Solution of second order linear differential equation		$E_0 = E_1\left[\frac{Rcp}{(RCp+1)(rcp+1)}\right]$
$K_1\frac{dE_1}{dt} + (K_2+K_3)E_1 + K_3\int E_1dt = E_0$	Ditto		$E_0 = E_1\left[\frac{(RCp+1)(rcp+1)}{Rcp}\right]$

ductors are notoriously non-ideal. As a consequence, the use of inductors in analogs is to be avoided whenever possible.

Operationally, analogous systems overcome most of the limitations of the so-called physical analogs. An operational analog is an assembly of elements, each of which performs a certain mathematical operation. These elements are assembled so that the equations describing the performance of the assembly are of the same form as those of the system for which the analog is desired.

The elements or building blocks for operational systems may be either actual physical elements with near-ideal performance characteristics or relatively complex devices developed specifically to perform certain mathematical operations. Electronic devices with nearly ideal operational characteristics are particularly well developed and these units have been widely used to analog physical systems whose performance can be described by ordinary differential equations. Commercially-available assemblies of such units are generally referred to as "electronic differential analyzers" or "electronic analog computers". Fig. 5 shows suggested symbols for the various operational elements of these assemblies.

The performance characteristics of a very large number of physical systems can be described by ordinary differential equations. In practice, the performance of a great many systems can be defined with sufficient accuracy by an ordinary linear differential equation with constant coefficients. The familiar form of such an equation

$$a_0\frac{d^nx}{dt^n} + a_1\frac{d^{n-1}x}{dt^{n-1}} + \dots\dots\dots\dots$$

$$a_{n-1}\frac{dx}{dt} + a_nx = f(t) \qquad (19)$$

represents only a few basic types of mathematical operations. These are differentiation, multiplication (or division) by a constant, and summation (or subtraction). An operational analog for an equation of this type can be formed from an assembly of devices that perform only these three operations. By rearranging the equation, a similar analog can be formed using elements or devices that perform the operation of integration in place of differentiation.

Electronic operational elements with very nearly ideal characteristics have been developed by the use of feedback amplifiers in combination with static components. Fig. 6 is a schematic diagram of the general operational amplifier, and the relationship between the output and input voltages is given approximately by

$$E_O = -Z_O\left[\frac{E_1}{Z_1} + \frac{E_2}{Z_2} + \dots \frac{E_n}{Z_n}\right] \quad (20)$$

where z_n is operational impedance.

The degree of approximation in this relationship is determined primarily by the gain of the amplifier, A, and by the input-grid current. For the solution of most engineering problems on a real-time scale, an amplifier gain of 50,000 or more, and an input-grid current of the order of 10^{-9} amp or less are acceptable values. Since negative feedback is necessary for stability, the amplifiers usually have an odd number of stages, and the input-output function involves a change in sign. Amplifier circuits have been developed which do not impose this limitation, but they are not widely used.

A great variety of operational elements can be derived from Eq (20). For the solution of ordinary differential equations, however, the only basic operations are summing, scale and sign changing, and either integrating or differentiating with respect to time. These are given by the first four circuits shown in Table III. Actually, the sign-changing amplifier is a simplification of the summer so that there are only three basic operations, summing, integrating and differentiating.

Fundamentally, the gain characteristic of any differentiator is an increasing function with respect to frequency. Consequently, differentiators amplify noise and tend to be unstable, thereby discouraging their use with an analog operating on a real-time scale. Successful differentiators have been built, however, for high-speed repetitive computers.

The precision linear potentiometer is another element that is useful for varying the scale factors in the analog. These basic elements—summers, integrators and potentiometers—are the essential building blocks of a computer for the solution of ordinary linear differential equations with constant coefficients. An assembly of these basic elements, together with associated power supplies, auxiliary elements and hardware, constitutes an electronic differential analyzer.

The method of interconnecting these elements to form an analog of a physical system can be illustrated by setting up the analog of the mechanical system of Fig. 1(a). The differential equation for this system is

$$\frac{d^2x}{dt^2} = \frac{1}{M}\left(F \cos \alpha t - B\frac{dx}{dt} - Kx\right). \quad (21)$$

If it is assumed that d^2x/dt^2 is generated by an operational element, then $-dx/dt$ can be generated by operating on d^2x/dt^2 with an integrating amplifier. Similarly, x can be generated by a second integrating amplifier. Since $F \cos\alpha t$ is the known forcing function, d^2x/dt^2 can be generated by a summing amplifier with inputs $F \cos\alpha t$, dx/dt and x, each modified by a suitable scale factor. These factors are provided by three adjustable linear potentiometers, and constant multipliers are provided by operational amplifiers. The connection block diagram of this operational analog is given in Fig. 7, where amplifier #4 is

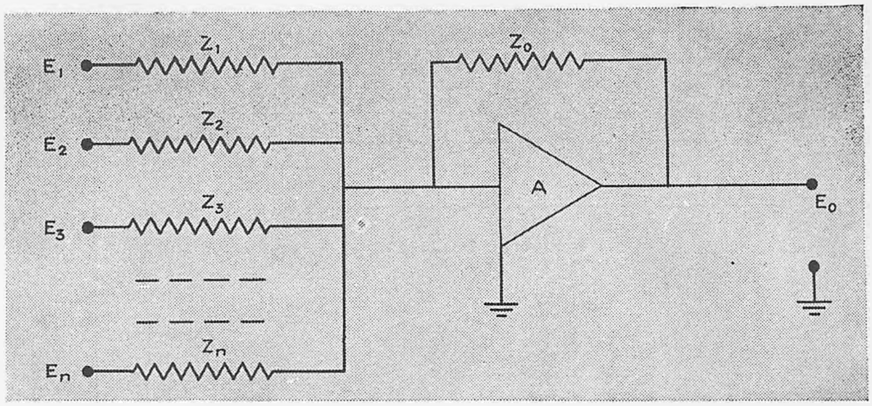


Fig. 6—Operational elements with near-ideal characteristics are generally combination of feedback amplifiers and static components of one kind or another.

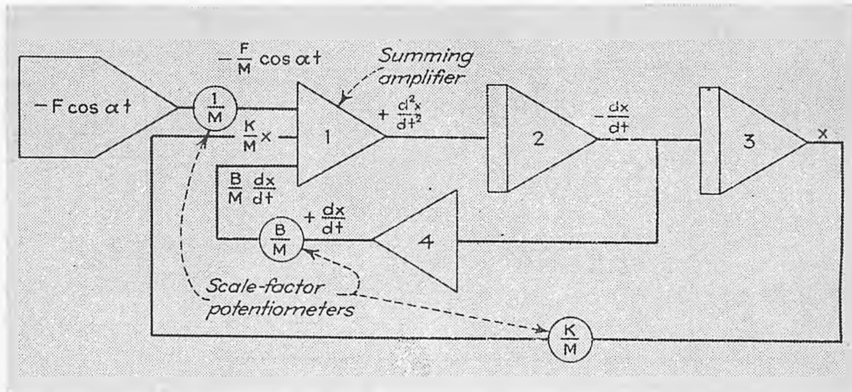


Fig. 7—Block diagram for operational integrating analog of mechanical system shown in Fig. 1(a). Inputs to summing amplifier #1 are modified by the potentiometers.

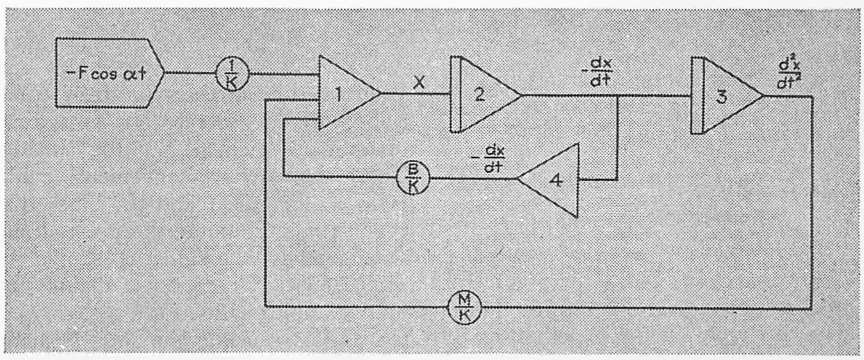


Fig. 8—Mechanical system of Fig. 1(a) as represented in form of an operational differentiating analog. Limitations of differentiators discourage use of this type.

(a) Problem Connection Tabulation

Eq(23): $L \frac{di}{dt} + Ri + U\frac{dx}{dt} = v_0(t)$

Amplifier	Variable	Operation	Type of Element
1	$\frac{di}{dt}$	$\frac{1}{L}\left[Ri + U\frac{dx}{dt} - V_0 t\right]$	Sum
2	$-i$	$\int \frac{di}{dt}\, dt$	Integrate
3	$+i$	$-i$	Sign Change

Eq(24): $M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Kx - Ui = 0$

Amplifier	Variable	Operation	Type of Element
4	$\frac{d^2x}{dt^2}$	$\frac{1}{M}\left[B\frac{dx}{dt} + Kx - Ui\right]$	Sum
5	$\frac{-dx}{dt}$	$\int \frac{dx^2}{dt^2}\, dt$	Integrate
6	$\frac{dx}{dt}$	$\frac{-dx}{dt}$	Sign Change
7	x	$-\int \frac{dx}{dt}$	Integrate

(b) Block Diagram

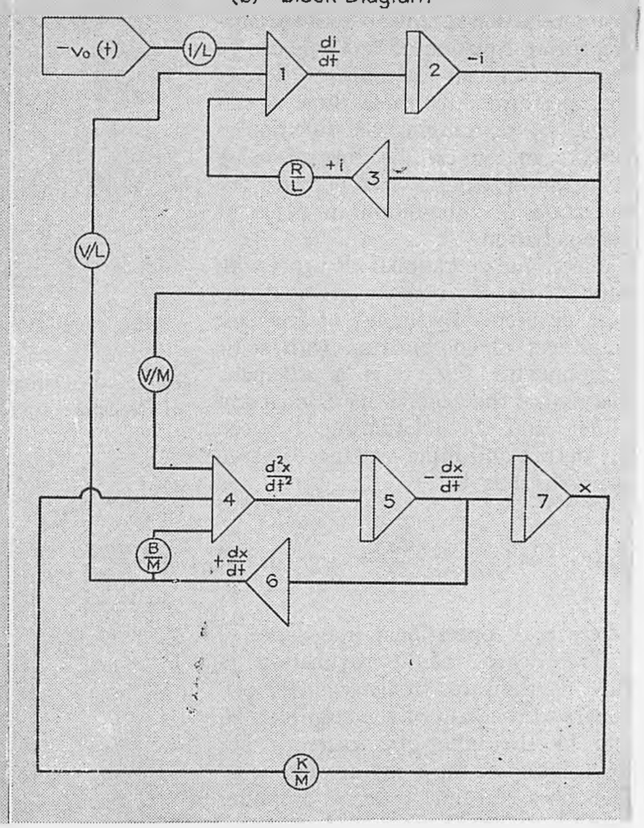


Fig. 9—Variables in simultaneous equations Eqs (23) and (24) are generated by separate operational elements as shown in connection tabulation (a) and block diagram (b). The more complex systems can also be analoged using similar systematic methods.

used only to change the sign of $-dx/dt$.

An analog using differentiators in place of the integrators can be formed in a similar manner by solving Eq (21) for x

$$x = \frac{1}{K}\left(F\cos\alpha t - B\frac{dx}{dt} - M\frac{d^2x}{dt^2}\right) \quad (22)$$

The connection block diagram for this analog is given in Fig. 8. However, this type analog is not normally used because of the above-mentioned limitations of differentiators.

It is usually more convenient to set up the analog so that each adjustable parameter in the original equation corresponds to a separate scale-factor potentiometer in the analog. This can be done simply in Fig. 7 by inserting a potentiometer corresponding to $1/M$ between amplifiers #1 and #2. The other potentiometers would then correspond to F, K and B, respectively.

A systematic method for setting up the connection block diagram can be demonstrated by means of an example. Assume that it is desired to analog the following set of simultaneous equations:

$$L\frac{di}{dt} + Ri + U\frac{dx}{dt} = v_o(t) \quad (23)$$

$$M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Kx - Ui = 0 \quad (24)$$

Each of the variables in these equations will be generated by a separate operational element in the analog. These elements are conveniently tabulated in Fig. 9(a) and the block diagram for a computer to solve these simultaneous equations is shown in Fig. 9(b). Operational analogs of more complex systems can be easily established by using similar systematic methods. The number of operational elements available determines the order of complexity of the problem which can be successfully analoged.

The amazing versatility of operational analogs offers many advantages in the solution of every-day dynamics problems. Since the only basic elements are amplifiers, capacitors, fixed resistors and potentiometers, suitable connection arrangements are set up so that any amplifier can be conveniently connected to perform any of the functions shown in Table III. Furthermore, the operational amplifiers can be easily interconnected to form the operational analog of a very wide variety of dynamic systems. Since the elements used are of standardized types, the size of the computer can be selected to correspond with the complexity of the problem to be solved. Special elements are easily added to accommodate more complex problems.

A relatively small number of operational elements will often accommodate a wide variety of common problems that are tedious to solve by manual methods. Although the solutions to many of these problems may be relatively straightforward, the complete evaluation of the system over a wide range of system parameters may require considerable computational work.

Consider, for example, the mechanical system of Fig. 1(a), but assume that the point of attachment of the spring, K, to the vertical reference is actually subjected to a horizontal displacement caused by vibration, $Y_o \cos \beta t$, with respect to the reference. Also assume that the force $F(t)$ is replaced by a suddenly applied acceleration a. Suppose that it is desired to select suitable values of M, B, and K such that the peak displacement of the mass M will never exceed a maximum value x_2, and such that it will reach a minimum value of x_1 within a given time interval T_1 over a range of values for a. It is also desired to realize the maximum possible isolation of the mass from the vibration $Y_0 \cos \beta t$. This is a somewhat simplified representation of an actual design problem. The optimum values of M, B and K, and the over-all performance of the system are not immediately apparent by inspection and, although it is a relatively simple system, its evaluation over a wide range of values of M, B and K would be tedious and time-consuming.

The differential equation for this system is

$$M \frac{d^2x}{dt^2} + B \frac{dx}{dt} + K(x - Y_O \cos \beta t) = a[u(t)] \qquad (25)$$

where $u(t)$ is a step function. The connection block diagram of an operational analog for this system appears in Fig. 10. This computer illustrates the usefulness of even a very few operational elements.

Perhaps the greatest advantage of such simple computers is the ease with which certain types of common nonlinearities can be incorporated. Very simple devices will accommodate such nonlinearities as limit stops, coulomb friction, hysteresis and backlash. These factors are present in every dynamic system to some extent and are extremely difficult to handle by normal analytical methods.

As an example, suppose that the system described by Eq (25) is further modified by a coulomb-friction factor to give

$$M \frac{d^2x}{dt^2} + B \frac{dx}{dt} + G \frac{dx}{dt} + K(x - Y_O \cos \beta t) = a[u(t)] \qquad (26)$$

where

$$G = \begin{cases} G_1 \text{ for } \frac{dx}{dt} > 0 \\ -G_1 \text{ for } \frac{dx}{dt} < 0 \end{cases}$$

In this equation, the function G is easily simulated by a differential relay and a high-gain saturating amplifier together with a suitable voltage source. The connection diagram of the analog for this system is shown in Fig. 11. Amplifier #5 can be a single high-gain stage used to reduce the dead zone of the relay for small values of dx/dt. It should saturate for large values of dx/dt to avoid damage to the relay. Simple diode circuits can also be used in place of the relay. The effects of coulomb friction on the performance of this system can be explored with this simple computer.

The simple mechanical system of Fig. 1(a) may be further complicated by hysteresis in the spring, limit stops and similar factors. These may be important in the physical system, but are usually ignored in the analysis since their investigation by manual methods would be economically impossible. The use of small operational analogs, however, permits these factors to be economically and conveniently investigated by simple, straightforward techniques.

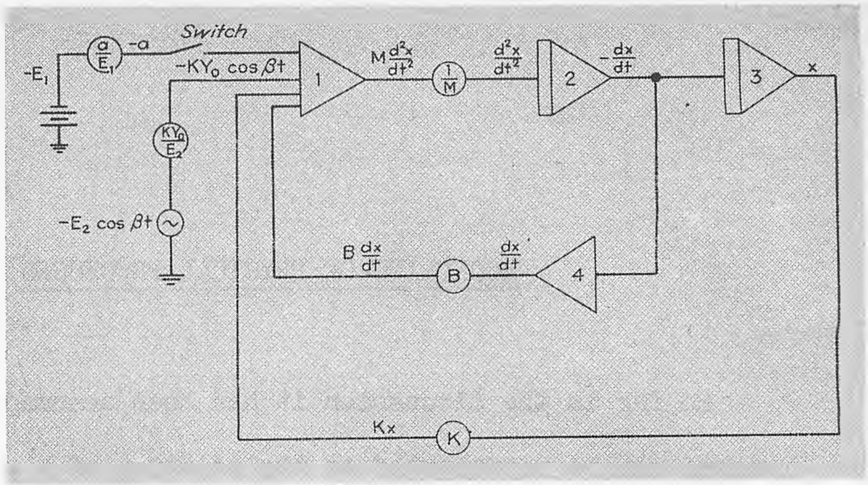


Fig. 10—Simple mechanical system of Fig. 1(a) modified by subjecting right end of spring to horizontal vibratory displacement. Optimum values of *M*, *B* and *K* as well as performance of system, are easily evaluated with analog shown.

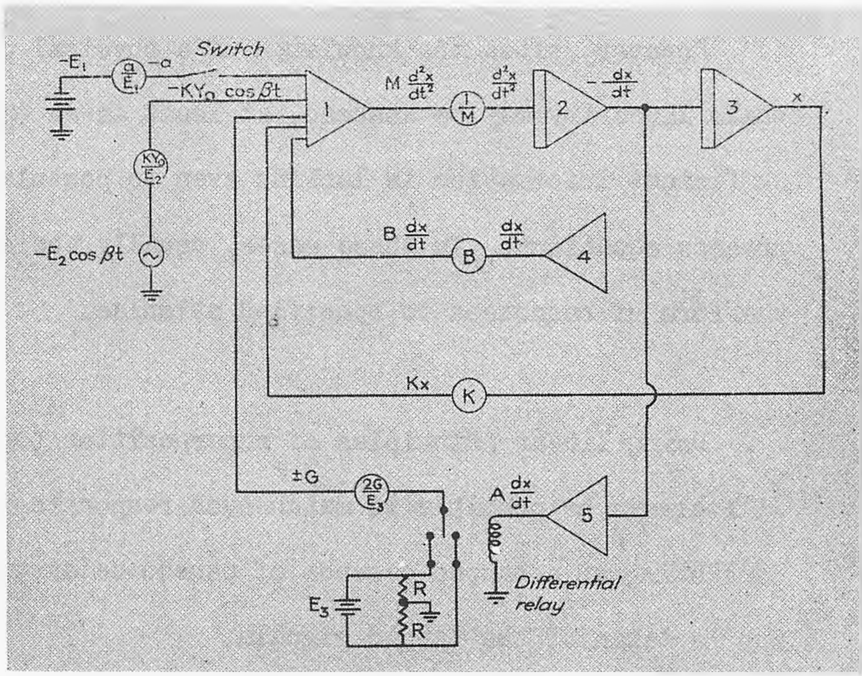


Fig. 11—Simulation of common physical nonlinearities such as coulomb friction is illustrated by addition of a differential relay to the analog circuit. Other analogous factors (*M*, *B* and *K*) in diagram are similar to those in Fig. 1(a).

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(2) McCann, G. D., Wilts, C. H. and Locanthi, B. N., "Application of the California Institute of Technology Electric Analog Computer to Non-Linear Mechanics and Servomechanisms," *Transactions* of the AIEE, Vol. 68, 1949, pp. 652-660.

(3) McCann, G. D., Jr. and Wilts, C. H., "Application of Electric-Analog Computers to Heat-Transfer and Fluid Flow Problems," *Journal of Applied Mechanics*, September, 1949, pp. 247-258.

(4) Fagin, Irving, "Electrical Analogs of Rigid Body Suspension Systems," Thesis for Master & Science Degree, The Ohio State University, 1948.

(5) Korn, Granino A., "Design of d-c Electronic Integrators," Electronics, Vol. 21, No. 5, May 1948, pp. 124-126.

(6) McDonald, D., "Analog Computers for Servo Problems," Review of Scientific Instruments, Vol. 21, No. 2, Feb. 1950, pp. 154-7.

MODELLING OF PHYSICAL PROCESSES

So far in the discussion it has been assumed that any physical process can be represented as the solution of a particular set of linear (or nonlinear) differential equations.

However, often the knowledge of a physical process is such that, while linearity may be assumed, at least as an approximation, sufficient information is lacking even to postulate a proper set of process equations. In these cases, usually the data available is in the form of responses to specified stimulae.

Using linear principles of superposition (or convolution), it will always be possible to match such responses in a structure consisting of a tapped cascade of psuedo-delays, where a weighted sum is taken of the tapped signals.

The papers to follow demonstrate the principles and applications of such techniques.

THE ELECTRO-ANALOGUE, AN APPARATUS FOR STUDYING REGULATING SYSTEMS

I. COMPONENTS AND FUNCTIONS

by J. M. L. JANSSEN and L. ENSING. 621-52:621.3.012.8:53.072.13

Mathematical operations such as adding, subtracting, multiplying, dividing, differentiating and integrating can be carried out with the aid of electrical circuit elements. On this principle it is possible to construct electrical models of, say, the processes of heat exchange or of mechanical processes. Such models are attractive because of the ease with which, by employing an oscilloscope, a sufficiently accurate picture can be obtained of the working of the process imitated, as also of the effect produced when various parameters are altered. Electrical models have proved their value particularly in the case of automatic controllers used to keep a certain working factor constant. An electro-analogue is a collection of instruments required for building up an electrical model for such cases and studying its behaviour. This first article will deal with the components required for building an electro-analogue. Details of the electrical circuits will be discussed in another article to follow.

In industry nowadays automatic controllers are being employed on a large scale for stabilizing certain quantities — temperature, voltage, current, resistance, acidity, rate of flow of a liquid or gas, etc. Some considerations were given to this subject in a previous article in this journal [1]), where it was pointed out that in a particular case, for designing the best regulating system or for the best manner of adjusting a particular automatic controller, the purely theoretical process of the calculation cannot be followed: the mathematical difficulties in taking into account all the pertinent factors would be far too great even in fairly simple cases. Neither is it advisable, in many cases, to follow entirely empirical methods, since this may lead to lengthy interruptions in the industrial process, with the resultant decline in the quantity and quality of the production.

As already mentioned in the article quoted, a solution can often be found by working with an electrical model of the process to be controlled and of the controller itself, thereby choosing a time scale such that the phenomena can be observed on the screen of an oscilloscope. Measurements of the actual process can be limited to the recording of the step-function response, i.e. the characteristic indicating how the quantity to be regulated responds as a function of time to a sudden change of the regulating quantity at the input of the process.

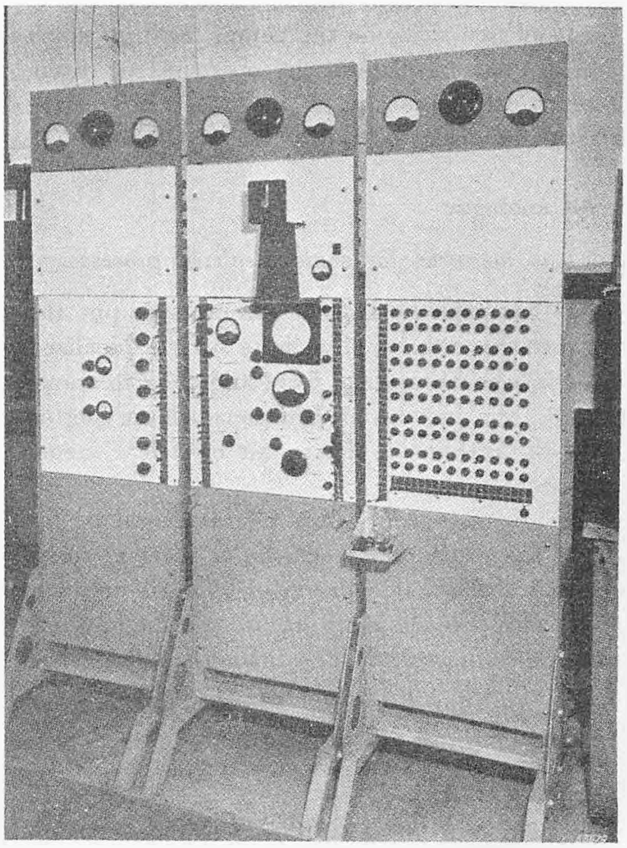

Fig. 1. Photograph showing the electro-analogue. The left-hand panel comprises mainly the model of the automatic controller, the middle one the oscilloscope and accessories, and the right-hand one the model of the process. At the top of each rack is a variable transformer for adjusting the A.C. voltage supply to the correct value, shown by the voltmeter on the right, and an ammeter (on the left) indicating the total current consumption of the panel.

[1]) H. J. Roosdorp, Philips Techn. Rev. 12, 221-227, 1950/51 (No. 8).

The electrical model of the process has to be of such a construction as to show the same step-function response.

Given an apparatus which is designed for building up such an analogous model of the process and which comprises, further, a model of the automatic controller in which a proportional, an integrating and a differentiating term can be made to play a part as needed, and which moreover comprises an oscilloscope, a square-wave generator and further accessories, then the effect of altering various parameters in the controller, of introducing disturbances at various points, etc. can be seen at a glance. Such an apparatus is the "electro-analogue", a specific design of which is illustrated in *fig. 1.*

In addition to solving concrete regulating problems the electro-analogue has proved its worth in research work in the field of regulating. The great variety of models — both linear and non-linear — that can be built with it and the clarity of the results have contributed much towards a broader insight into regulating problems.

We shall first describe the components of which the electro-analogue built by us consists, while the electrical execution of the most interesting parts will be discussed in another article.

Process analogue

Analogous networks for some idealized processes

There has been no lack of attempts to put the step-function response of a more or less idealized process into mathematical form and then to translate that form into an electric network serving as a "process analogue", i.e. as an electrical model of the process.

Many processes are mainly characterized by an inertia caused, for instance, by a thermal resistance and a thermal capacity. Obviously in such cases the process analogue can to a first approximation be composed of a resistance R and a capacitance C in series (*fig. 2a*). The equation for the step-function response of this circuit (fig. 2*b*) is:

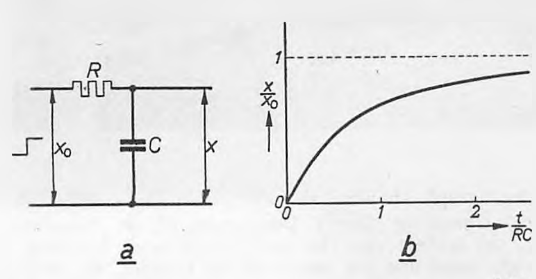

Fig. 2. *a*) Resistor R and capacitor C in series, as the simplest electrical model of a process with thermal resistance and thermal capacity. *b*) Step-function response curve of this network.

$$\frac{x}{x_0} = 1 - e^{-\frac{t}{RC}}.$$

where x_0 is the amplitude of the applied voltage step, x is the voltage across the capacitor and t is the time.

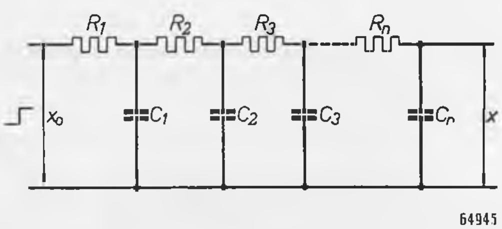

Fig. 3. Extension of the network in fig. 2*a* to a series of n R-C sections connected in cascade.

Usually, however, one has to do with a series of thermal (or other) resistances and capacities. A circuit according to *fig. 3* then gives a better approximation of the situation. In this network the sections R_2-C_2, R_3-C_3 etc. constitute a certain load on all the preceding sections, as is more or less the case in an actual process.

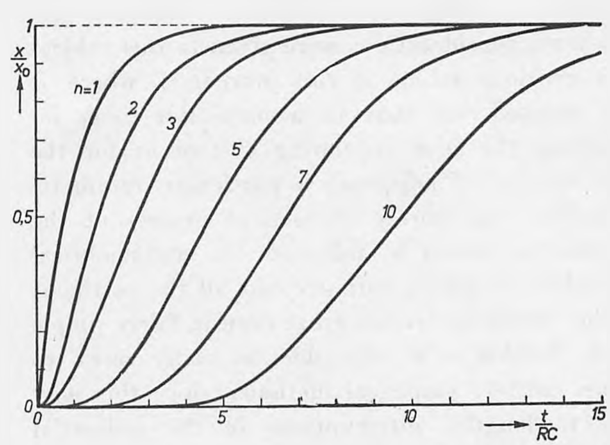

Fig. 4. Calculated step-function response curves for $n = 1$ to 10 sections of the network according to fig. 2 with equal resistors R and capacitors C in all sections, these sections being separated by interposed amplifying valves so that the one section does not constitute a load on the preceding sections.

Other hypothetical processes in combination with an automatic controller have been theoretically studied by mathematicians of the "Bataafsche Petroleum Maatschappij" [2]): one of these processes has its electrical analogue in a series of R-C sections where there is no such loading of the preceding sections by the following ones as just referred to.

[2]) P. Hazebroek and B. L. van der Waerden, The optimum adjustment of regulators, Trans. Amer. Soc. Mech. Engrs. 72, 317-322, 1950 (No. 3).

This case can be realized by separating the sections in the network of fig. 3 by means of amplifying valves (details will be given in a subsequent article). The calculated step-function response curves of such a network for different numbers of sections are given in *fig. 4*.

To verify the proper functioning of the electro-analogue this is provided with both kinds of *R-C* networks. The network without interposed valves consists of 20 "coupled", and that with valves of 40 "decoupled" identical sections, any number

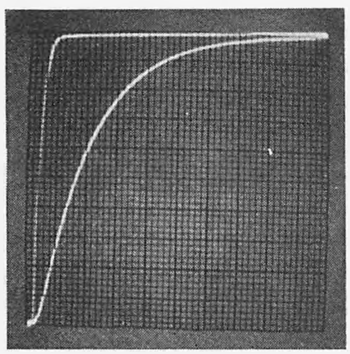

Fig. 5. Oscillograms of the step-function response curves of 10 sections of an *R-C* network; upper curve with decoupled sections, lower curve with the sections not decoupled. The upper curve agrees with the calculated curve for $n = 10$ in fig. 4, in which figure a widely different time scale has been used.

of which can be used as required. *Fig. 5* shows the oscillograms of the step-function responses of 10 sections of both networks.

These networks have rendered good service particularly for research purposes, but they do not form a universal process analogue with which any step-function response curve can be obtained. To arrive at such a universal process analogue, instead of adhering to an electrical analogy with the physical working of the actual process, an entirely different network has been devised by means of which any step-function response can be sufficiently approximated by a discontinuous line.

Universal process analogue

In the universal process analogue used by us an electric delay network [3]) is employed. This consists of a large number of sections, made up of capacitors and inductors, so arranged that voltages are transmitted, unattenuated, with a delay time τ which over a very wide frequency range is independent of the frequency. When a voltage V_0 having the shape of a step-function is applied to the input of the network then the output voltage V_1 of the first section likewise has the shape of a step-function but the step is delayed by a time τ (*fig. 6*) and is thus produced at $t_0 + \tau$; the voltage V_2 behind the second section, also with the shape of a step function, makes the step at $t_0 + 2\tau$, and so on.

If one now has a device with which any arbitrary proportions a_1, a_2, a_3 ... of the voltages V_0, V_1 V_2 ... can be added — which proportions it must be possible to adjust independently of each other between the limits $+a_{max}$ and $-a_{max}$ — then one can obtain a sum voltage V_s:

$$V_s = a_0 V_0 + a_1 V_1 + a_2 V_2 + \dots,$$

of echelon shape as shown in *fig. 7*, and with this

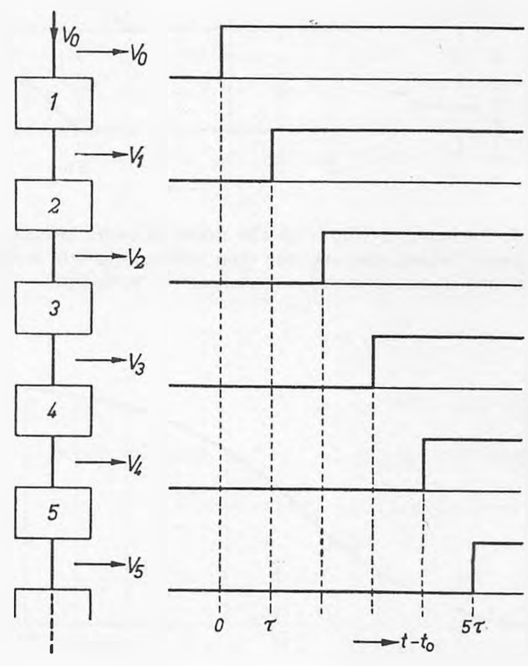

Fig. 6. Left: delay network with sections *1, 2, 3*, ... When a voltage step V_0 (see right-hand part of the diagram) is applied to the input then there arise at the outputs of the sections equally large pulses V_1, V_2, V_3 ... lagging a time τ, 2τ, 3τ, ... behind V_0.

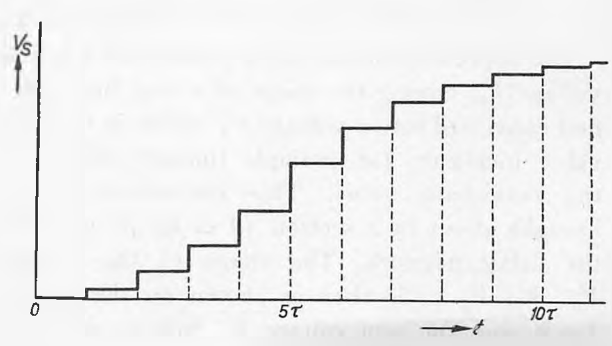

Fig. 7. When variable fractions a of the voltages V_0, V_1, V_2, V_3, ... of fig. 6 are added together, a sum voltage V_s of echelon shape is obtained, with which any curve can be approximated by varying the coefficients a.

[3]) British patent specification No. 517,516 in the name of A. D. Blumlein, H. E. Kallmann and W. S. Percival.

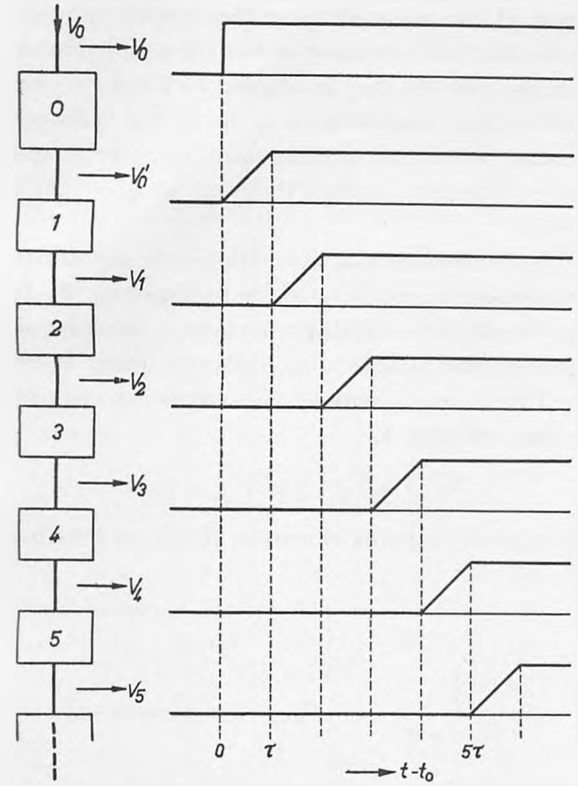

Fig. 8. As in fig. 6 but with the delay network preceded by a section *0* which converts the step voltage V_0 into a voltage V_0', which in the interval τ rises linearly with time.

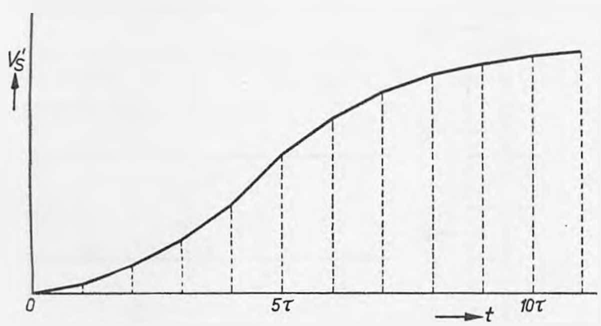

Fig. 9. Adding up variable fractions a of the voltages V_0', V_1, V_2, V_3, ... of fig. 8 yields a sum voltage V_s', the curve of which more closely approximates to a smooth line than the curve of fig. 7.

it is possible to approximate quite well the measured step-function response of any actual process.

The approximation is much better still when the voltage V_0, having the shape of a step function, is first converted into a voltage V_0' which in the interval τ increases, for example linearly, from 0 to the maximum value. This conversion can be brought about in a section (*0* in *fig. 8*) preceding *the delay network*. The shape of the voltages V_0', V_1, V_2 ... is then as shown on the right in fig. 8, and the sum voltage V_s' follows an almost smooth curve (*fig. 9*).

In our electro-analogue the method last outlined is applied. The delay network consists of 50 sections. There is no trace of any kinking in the curve of the sum voltage.

Oscilloscope and square-wave generator

For making the step-function response of the process analogue identical with that of the actual process it is necessary to be able to observe both these characteristics simultaneously. To that end the variation of the output voltage V_s' of the delay network is displayed on the screen of a cathode-ray oscilloscope, over which a transparent sheet of cellophane marked with a system of coordinates is placed, on which the measured step-function response of the actual process has been plotted. The curve of the process analogue is then matched as closely as possible to the plotted curve by turning the knobs with which the coefficients a_0, a_1 ... a_{50} are varied (see the right-hand rack in fig. 1).

For oscilloscopic observation it is advisable that the voltage V_0 does not make one single step but that this is repeated periodically, thus using a square-wave voltage (*fig. 10a*). This voltage is produced in a square-wave generator, in which the peaks of a sinusoidal voltage (50 c/s) taken from the mains are clipped off, leaving a practically pure square-wave voltage.

The output voltage V_s' of the process analogue now has a shape as represented in fig. 10*b*. The dotted part of this curve is made invisible by periodically blacking out the electron beam in the cathode-ray tube. The fully-drawn part is traced faithfully on the screen of the oscilloscope on account of the fact that in the respective half-cycles the time-base voltage is linear.

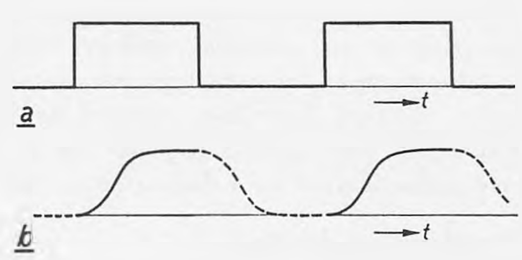

Fig. 10. *a*) Square-wave voltage at the input of the section *0* (fig. 8). *b*) Example of the variation of the sum voltage V_s' traced on the screen of an oscilloscope (the dotted part is blacked out).

Models of continuously acting automatic controllers

In the article quoted in footnote [1]) a distinction was made between discontinuously and continuously acting automatic controllers. We shall first consider only the latter type, leaving the other type of controllers to be dealt with in the last section of this article.

In the equation for the relation existing between the amount x by which the quantity to be regulated deviates from the desired value and the amount q by which the regulating quantity is consequently

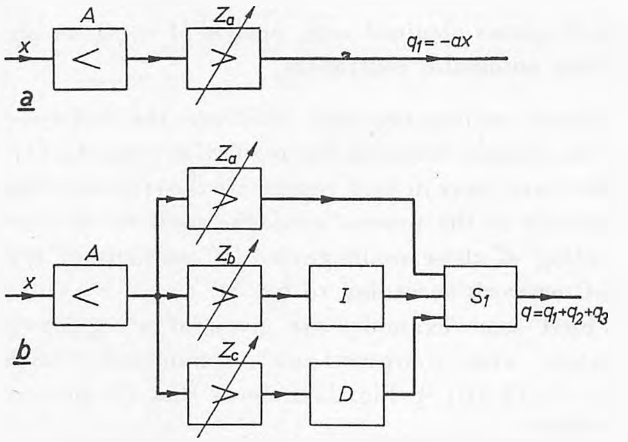

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Fig. 11. *a*) Block diagram of a model of an automatic controller with proportional action: amplifier *A* and variable attenuator Z_a. *b*) Block diagram of the model of an automatic controller with proportional, integral and derivative action. *A* is a common amplifier, *I* the integrator, *D* the differentiator. The coefficients *a*, *b* and *c* of the three components are adjusted by means of the variable attenuators Z_a, Z_b and Z_c respectively. S_1 is a collecting stage in which the three components are added together.

changed by the automatic controller, various terms may occur, e.g. a proportional, an integrating and a differentiating term [4]):

$$q = -ax - b\int x\,\mathrm{d}t - c\frac{\mathrm{d}x}{\mathrm{d}t} = q_1 + q_2 + q_3. \quad (1)$$

It must be possible to realize these three terms in the electrical model of the automatic controller and the coefficients *a*, *b* and *c* have to be variable in the model.

The proportional term, $q_1 = -a\,x$, is easily realized by means of an electronic amplifier. The coefficient *a* can be made variable by adding a variable attenuator (*fig. 11a*).

The integrating term, $q_2 = -b\int x\,\mathrm{d}t$, and the differentitating term, $q_3 = -c\,\mathrm{d}x/\mathrm{d}t$, are obtained with the aid of an integrating and a differentiating network respectively. Separate attenuators provide for the variability of the coefficients *b* and *c* (fig. 11*b*). Finally the three voltages, q_1, q_2 and q_3, are summated in a "collecting stage".

[4]) In order to get positive numerical values of *a*, *b* and *c* we have given these factors a minus sign, in deviation from the article quoted in footnote [1]). That *q* and *x* must have different signs follows from the fact that the automatic controller has to bring about a change of *q* counteracting a preceding variation of *x*.

The electro-analogue is provided with a second integrator and a second differentiator by means of which terms of the type $b_2\int\int x\,\mathrm{d}t\,\mathrm{d}t$ and $c_2\,\mathrm{d}^2x/\mathrm{d}t^2$ can also be added, though as a rule there is no need for this.

Form and location of the disturbances

When a process analogue has been built up in the manner described then this network is combined with the model of the automatic controller according to fig. 11 *b* to form a closed circuit (*fig. 12*). Contrary to fig. 11*b*, a common attenuator Z_0 is added with which the coefficients *a*, *b* and *c* are varied simultaneously.

The oscilloscope can be connected to various points of the circuit by means of a lead. When it is found that the situation is stable — this can always be reached by adjusting Z_0 for sufficient attenuation — then at all input and output terminals of the component parts of the regulating circuit the voltage will be zero. Upon a disturbance being introduced somewhere, then at the output of the process analogue and at that of the model of the automatic controller voltages will arise which are proportional to the quantities *x* or *q* respectively, and on the screen of the oscilloscope it can be seen how these voltages vary with different values of the coefficients *a*, *b* and *c*.

The disturbances occurring in a regulating system in practice can be distinguished both ac-

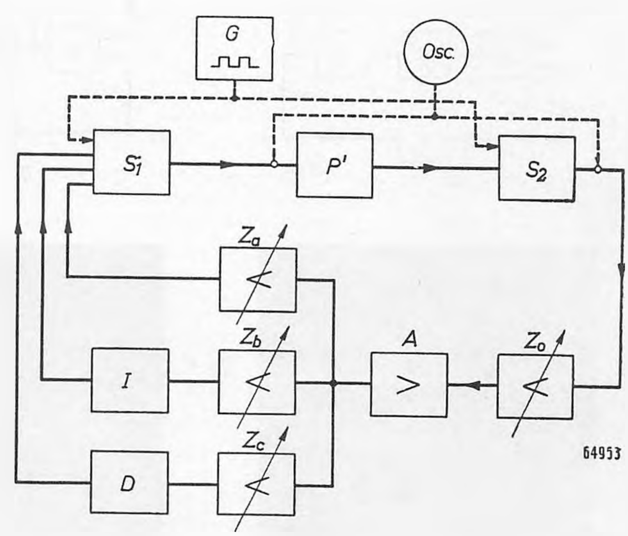

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Fig. 12. Block diagram of the model of a regulating circuit. *P'* process analogue, whose step-function response has been matched to that of an actual process. Z_0, *A*, Z_a, Z_b, Z_c, *I*, *D* and S_1 form the model of the automatic controller (fig. 11). *G* square-wave generator supplying a disturbance, which can be applied, for instance, to the input of the process (at S_1) or at its output (at the collecting stage S_2 provided for that purpose). The oscilloscope *Osc* can be connected to various points of the circuit. Z_0 is a common attenuator.

cording to their form (as function of the time) and according to the point at which they occur in the circuit.

In respect to their form there are continuous and step disturbances. Since the former may be regarded as being the limit of an infinitely large number of infinitesimal steps occurring in succession (cf. fig. 6), it is only necessary to investigate the response of the system to step disturbances. These can therefore be obtained from the square-wave generator employed for adjusting the process analogue.

According to the point at which they occur, one may distinguish between:

1. disturbances in the regulating unit, thus at the input of the process,
2. disturbances somewhere in the course of the process, and
3. disturbances in the detecting element, thus at the output of the process.

There may also be counted among these "disturbances" the changes resulting from the automatic controller being adjusted to a different setting. It is with these distinctions in mind that the square-wave generator is also fitted with a lead for connecting it either to the input of the process (i.e. to the input of the collecting stage S_1) or to the output of the process, for which purpose a second collecting stage, S_2, is provided. If the disturbances introduced at these two points are sufficiently neutralized by the automatic controller then the same will be the case with the disturbances sub (2) occurring somewhere in the course of the process.

Oscillograms obtained with models of continuously acting automatic controllers

Some oscillograms will illustrate the influence of the various terms in the regulation equation (1). The lower curve in fig. 5 represents the step-function response of the process analogue used for the recording of these oscillograms (10 sections of the *R-C* network according to fig. 3).

First some examples are given of a regulating system with proportional action only, with $a = 4$ (12 dB) [5]). Fig. *13-I* shows that the process

[5]) In the model x and q are both voltages and therefore $a = -q/x$ is a non-dimensional quantity. But also in the actual process a is non-dimensional if x is expressed as a percentage of the desired value of the quantity to be regulated and q as a percentage of the corresponding value of the regulating quantity.
The practical advantage of expressing a in decibels is that the total value of a is found by adding up the readings of the various attenuators calibrated in dB (Z_0 and Z_a in fig. 12) and the gain (of A in fig. 12) expressed in dB, which is easier than multiplying.

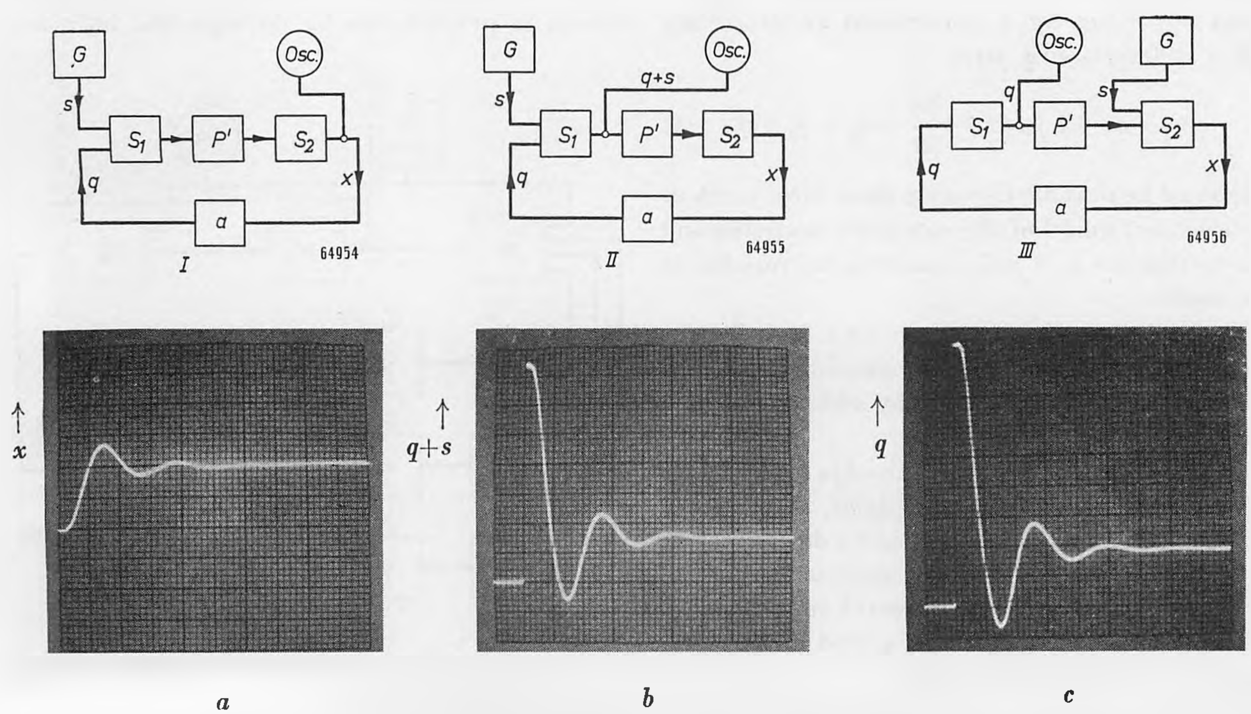


Fig. 13. Oscillograms obtained with the electro-analogue. The step-function response of the process analogue (10 not "decoupled" *R-C* sections) was as represented by the lower curve in fig. 5. The controller had only a proportional element ($a = 4$), as denoted by the rectangle a in the block diagrams *I*, *II* and *III*. In these diagrams it is also indicated at what points the generator *G* and the oscilloscope *Osc* were connected while the oscillograms underneath were being recorded.

analogue received, via the collecting stage S_1, the sum of the output voltage of the controller $q = -a\,x$ and the step-function voltage s from the square-wave generator, and that the output x from the process analogue was fed into the oscilloscope.

From the oscillograms it is seen that in course of time x and $q + s$ (thus q) become constant. The final values are related to the quantity a in the following way.

The gain of the process analogue (temporarily

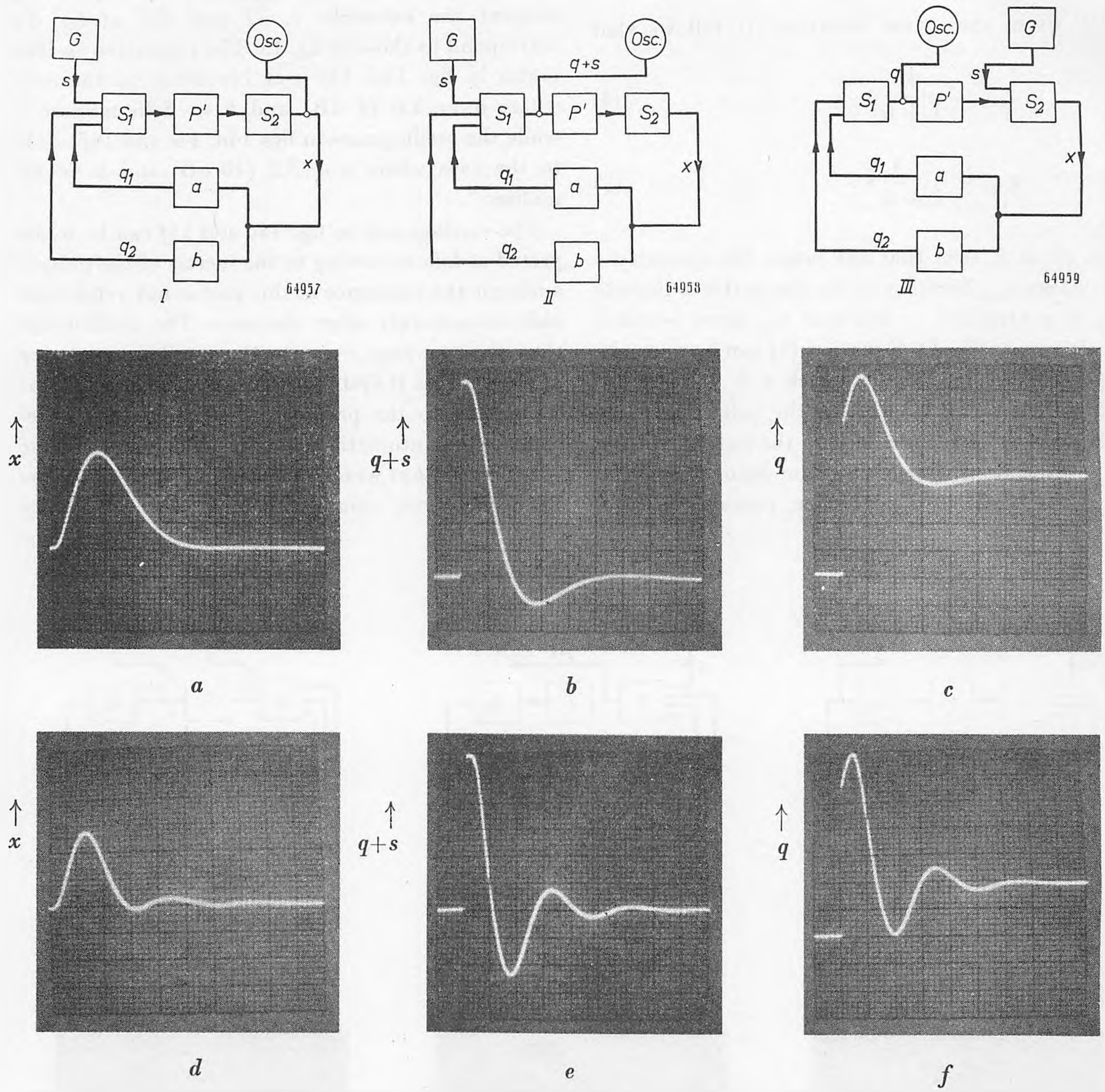


Fig. 14. As in fig. 13 but with an automatic controller having proportional and integral action. For the oscillograms (*a*), (*b*) and (*c*) the values of the coefficients were $a = 1.6$ and $b = 1.25$ millisec^{-1}, whilst for the oscillograms (*d*), (*e*) and (*f*) these were $a = 3.2$ and $b = 3.2$ millisec^{-1}.

Fig. 13*a* is the oscillogram obtained, on the same scale as fig. 5.

Fig. 13-*II* shows how $q + s$ is fed into the oscilloscope. In the corresponding oscillogram (fig. 13*b*) it is easy to distinguish the step-function voltage s from the square-wave generator, followed by the oscillation of q.

disconnected from the automatic controller) is so adjusted that a step disturbance with the amplitude s, introduced at the input, causes a change in the output voltage the final *of value which*, x_∞, *is* just equal to s. If, in the closed circuit, at the input of the process analogue we have in addition the output voltage q from the automatic controller with the final value q_∞, then in the stationary state we find:

$$x_\infty = q_\infty + s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In the automatic controller the equation

$$q_\infty = -a\,x_\infty \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

holds. From these two equations it follows that

$$x_\infty = \frac{1}{1+a}\,s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and

$$q_\infty = \frac{-a}{1+a}\,s \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

From (4) it is seen that the larger the quantity a the smaller x_∞ becomes — i.e. the better a disturbance is neutralized — but that x_∞ never becomes exactly zero (offset). Equation (5) can be numerically verified in fig. 13*b*, for which $a = 4$.

According to fig. 13-*III*, in the collecting stage S_2 the step voltage s is added to the output voltage of the process analogue, the sum being termed x. Since the automatic controller contains only a proportional term the oscillogram of q (fig. 13*c*) is of the same shape as that in fig. 13*b*.

When the automatic controller contains an integrating element (*fig. 14*) then, as already explained in the article quoted in footnote [1]), x is indeed reduced exactly to zero after a disturbance has occurred. With the exception of the integrating element the networks *I*, *II* and *III* of fig. 14 correspond to those of fig. 13. The respective oscillograms in figs 14*a*, 14*b* and 14*c* relate to the case where a = 1.6 (4 dB) and b = 1.25 millisec^{-1}, while the oscillograms in figs 14*d*, 14*e* and 14*f* relate to the case where a = 3.2 (10 dB) and b = 3.2 millisec^{-1}.

The oscillograms in figs 14*c* and 14*f* can be interpreted as follows: owing to the inertia of the process analogue the prescence of this part is not yet noticeable immediately after the step. The oscilloscope thus shows a step $-as$ (in the oscillogram a step upward). This is first followed by a linear rise, corresponding to the presence of the integrating element in the automatic controller, yielding a voltage $-b\int s\cdot dt = -b\,s\,t$ which increases linearly with time. Then, however, comes evidence of the presence

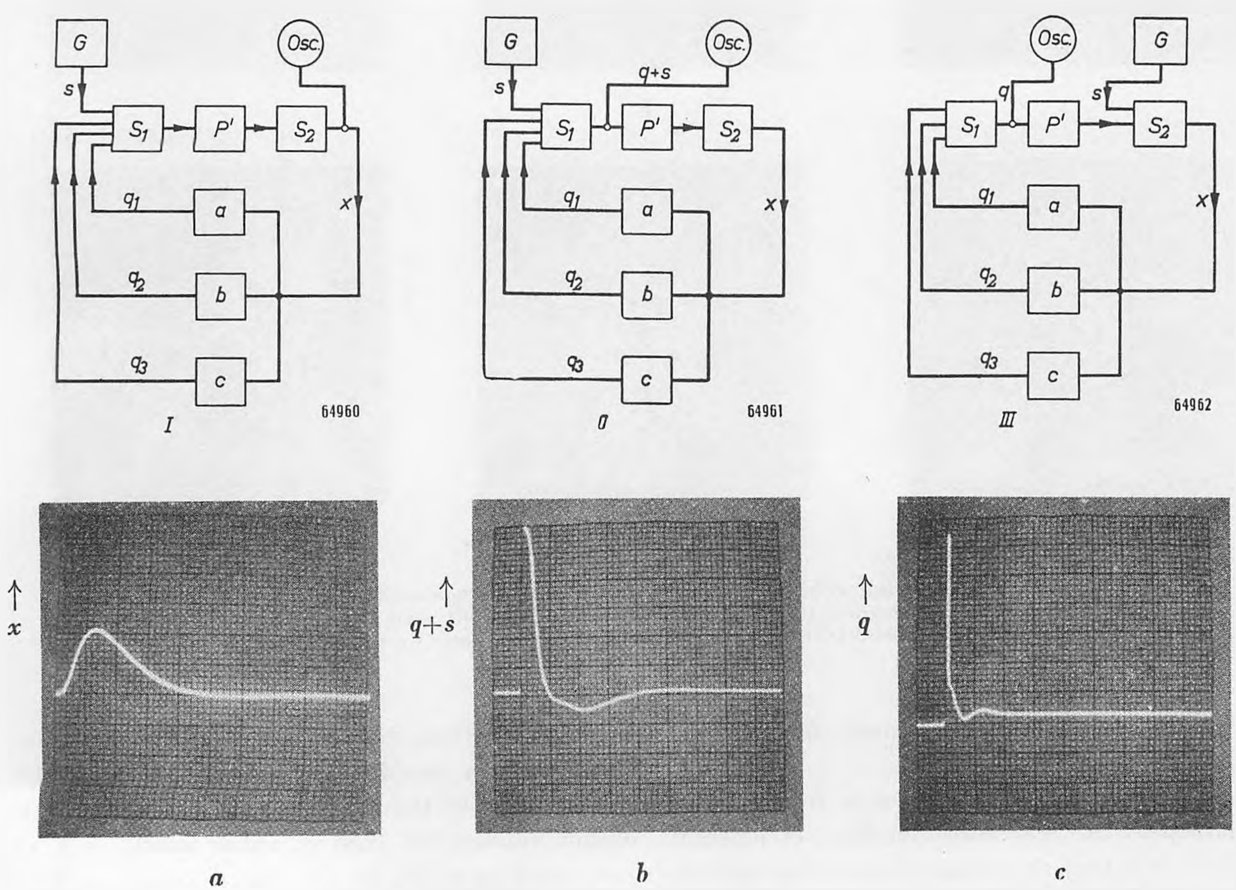


Fig. 15. As in fig. 14 but with also a derivative action in the regulator (a = 3.2, b = 3.2 millisec^{-1}, c = 1 millisec).

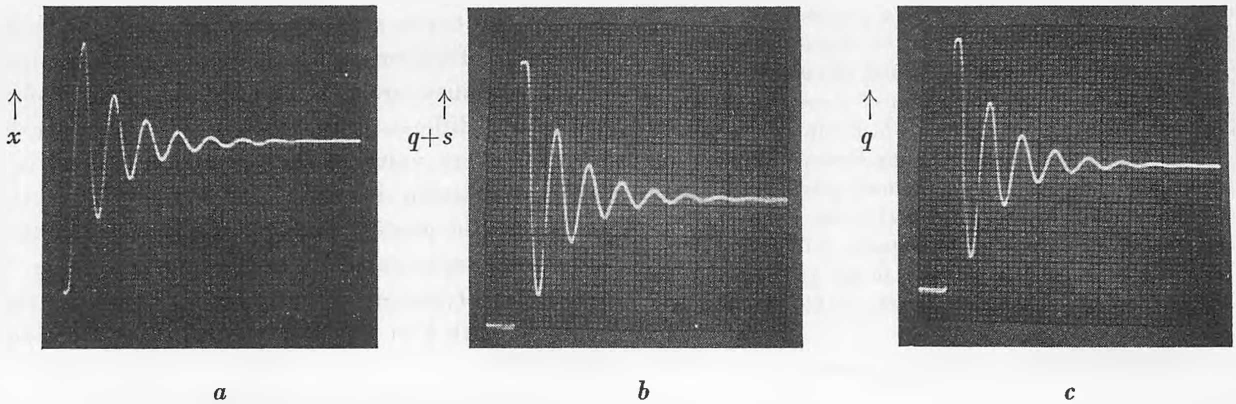


Fig. 16. As in fig. 13, but with the process analogue consisting of a network (10 "decoupled" *R-C* sections, with a step-function response as represented by the upper curve of fig. 5. Here the coefficient *a* of the proportional term was 1.

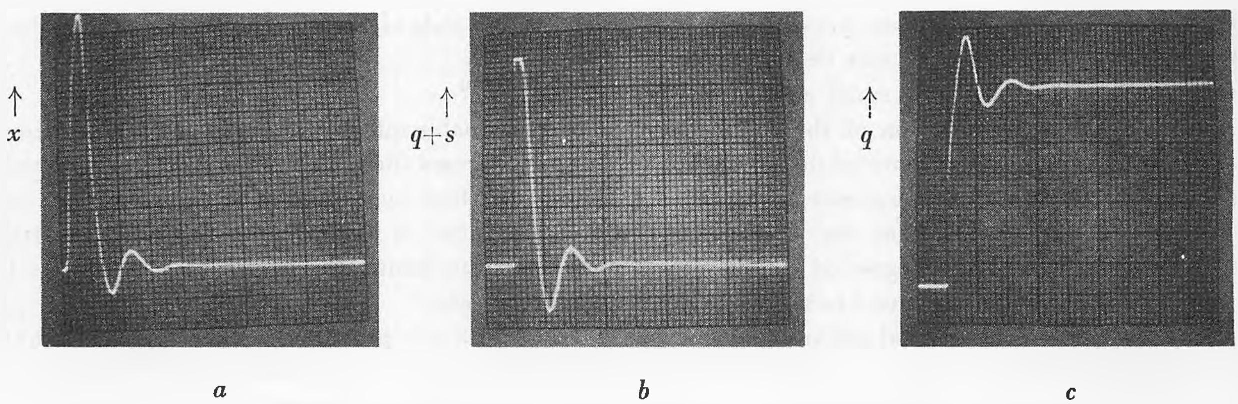


Fig. 17. As in fig. 14 but with a process analogue formed by a network with a step-function response as represented by the upper curve of fig. 5. $a = 0.5$,$b = 2$ millisec^{-1}.

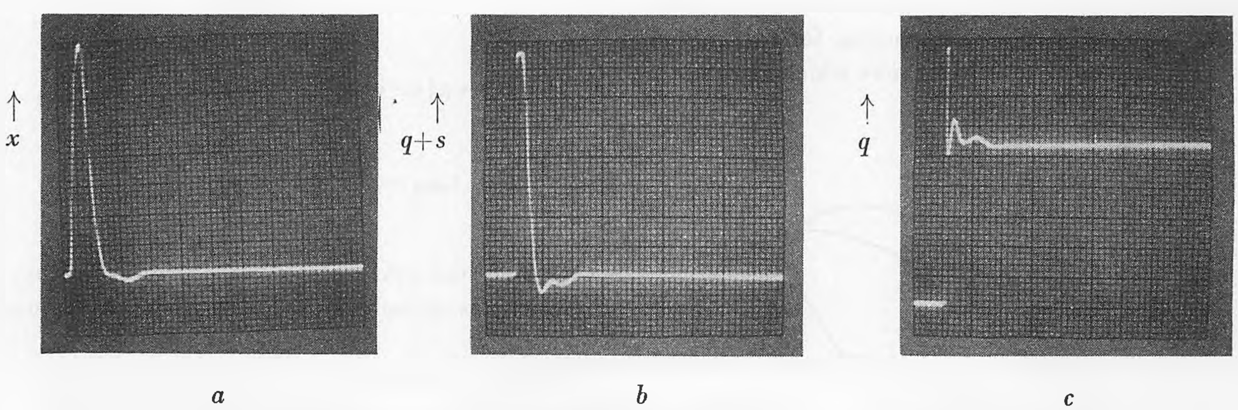


Fig. 18. As in fig. 15 but with a process analogue formed by a network with a step-function response as represented by the upper curve of fig. 5 and with coefficients $a = 0.5$, $b = 2.5$ millisec^{-1}, $c = 0.08$ millisec.

of the process analogue, which now also contributes a certain voltage to the input of the automatic controller. In the final state again equation (2) holds, but since now $x_\infty = 0$ we find $q_\infty = -s$, which means to say that in order to neutralize an interference at the output of the process a permanent change in the position of the regulating unit is necessary. Further the initial step q_0 made by q is equal to $-as$. The ratio of this step to the final value of q. thus q_0/q_∞, is therefore equal to a, as may be seen from figs 14*c* anf 14*f*.

Finally in *fig. 15* similar oscillograms are shown for the case where the automatic controller contains also a differentiating element ($a = 3.2$, $b = 3.2$ millisec^{-1}, $c = 1$ millisec). Apparently the differentiation has a stabilizing effect upon the regulation.

Similar curves, but recorded with a process analogue with a step-function response as shown by the upper curve in fig. 5, are represented in *figs 16, 17* and *18* on the same scale as figs 13, 14 and 15. A comparison of these two groups of oscillograms shows that a process which can be represented by *R-C* sections loading the preceding sections can as a rule be better regulated than a (hypothetical) process corresponding to "decoupled" *R-C* sections; in the former case the regulated quantity gives smaller amplitudes (cf., for example, figs 13*a* and 16*a*), and the position in fig. 16, with a equal to only 1, is decidedly less stable than that in fig. 13 with $a = 4$.

Criteria for the best regulating system

Stability

An idea of the stability can be obtained at once from the oscillograms (e.g. figs 13 and 16): the less the damping of any oscillations present, the less is the stability. The curves do not show, however, whether, in a stable case, any small change in the working conditions or variation of the coefficients a, b and c may already result in instability. (Mechanical wear of parts of the regulating unit, for instance, may well cause a variation of the coefficients.)

A better insight into the degree of stability can be obtained by measuring the stability area. A process with only proportional action ($q_1 = -a x$) may become unstable when a exceeds a certain value. If the automatic controller has only an integrating component ($q_2 = -b\int x\, \mathrm{d}t$) then stability may likewise be lost when b becomes too high. An automatic controller having both these functions has, in the case of a certain process, for each value of a a certain limit value of b above which instability occurs; the curve representing this limit value as a function of a forms, with the a-axis, the boundary of the stability area. If the automatic controller has also a differentiating action ($q_3 = -c\, \mathrm{d}x/\mathrm{d}t$) then for every value of the parameter c there is a different relation between the limit values of b and a. For the process analogue whose step-function response is represented by the lower curve in fig. 5, a number of curves determined with the electro-analogue with $b = \mathrm{f}(a)$ and $c =$ constant are given in *fig. 19*.

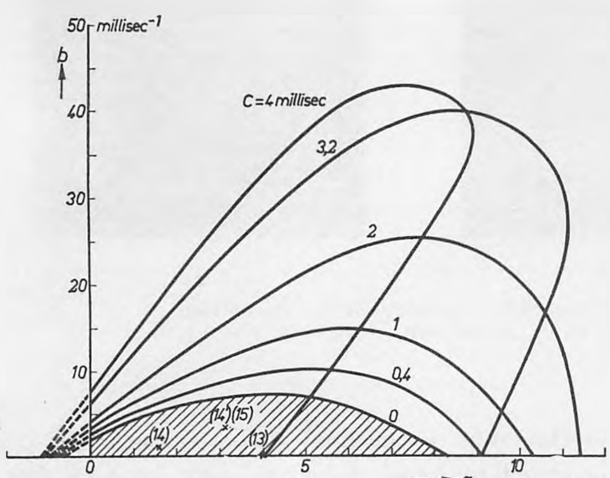


Fig. 19. Values of the coefficient b of the integrating term at which instability just arises, as function of the coefficient a of the proportional term, for constant values of the coefficient c of the differentiating term. This diagram applies for a process having a step function response as represented by the lower curve of fig. 5. The stability zone for $c = 0$ is hatched. The points (*13*), (*14*), (*14'*) and (*15*) correspond to the conditions under which the oscillograms of figures 13, 14*a*, *b*, *c*, 14*d*, *e*, *f* and 15 respectively were recorded.

If practical data are available concerning the variations likely to occur in the coefficients a, b and c, then with the aid of a number of graphs like those in fig. 19 it is possible to investigate whether there is any risk of the system becoming unstable.

Other criteria

Some valuable information is to be gathered from the oscillograms direct, as for instance the magnitude of the first maximum of x, the time that has to elapse after a disturbance before x remains within certain limits, the length of a cycle (if x is oscillatory, etc.

Other data are provided by the performance meter built into the electro-analogue, with which one can determine, as desired, the mean absolute value of x:

$$|x| = \frac{1}{T}\int_0^T |x|\cdot \mathrm{d}t\,,$$

or the r.m.s value of x:

$$x_{\mathrm{rms}} = \sqrt{\frac{1}{T}\int_0^T x^2\, \mathrm{d}t}\,.$$

where T is the cycle of the repetition frequency, for which, as already stated, the mains frequency is chosen.

It may be required of an automatic controller that one of these quantities shall be as small as possible. It depends upon the case under investigation whether they are useful as criteria and, if so, which of them forms the best criterion. That is why both possibilities have been provided for in our electro-analogue.

Recording frequency response curves

In the modern theory of regulated systems use is often made of frequency response curves. The behaviour of the system as a whole and also that of its parts is studied with sinusoidal disturbances. Conclusions can be drawn therefrom as to the stability and the quality of the regulation. Literature on this subject has been published in abundance in recent years.

For sinusoidal disturbances to be introduced in the model of the regulating system the electro-analogue has been provided with an *R-C* oscillator the frequency of which is variable. This oscillator can be connected, for instance, to the collecting stage S_1 of the closed regulating circuit (*fig. 20*). In this way sinusoidal voltages arise at all points of the circuit. By measuring the input and output

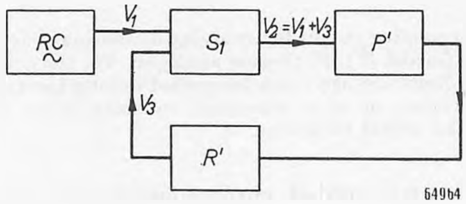


Fig. 20. Arrangement for taking Nyquist diagrams of the closed circuit of the process analogue P' and the model R' of the automatic controller. The (variable frequency) generator *RC* supplies a sinusoidal voltage V_1, which in the collecting stage S_1 is added to the output voltage V_3 from R'.

voltages of any element in the circuit one can determine the frequency response of that element. The same can be done for two or more elements connected in cascade, and even for the cascade connection of all the parts making up the circuit. In the latter case it is possible to derive from the measurements the Nyquist diagram, from which, as is known. conclusions can be drawn regarding the stability of the system [6]). Usually the measurements for the Nyquist diagram are taken after the circuit has been opened at some point. If, however, there is an integrator in the automatic controller (or the model) this is not possible, because when the circuit is cut open it becomes unstable: in fact the integral of an accidental step disturbance would continue to increase unrestricted, because in the opened circuit the disturbance is not neutralized. In that case the Nyquist criterion in its original form does not hold. In literature other forms of a wider scope are indicated [7]), but it would lead us too far afield to go into these here.

Models of discontinuously acting automatic controllers

Automatic controller with two-step action

The simplest form of a discontinuously acting automatic controller is that in which the regulating unit can have only two positions (as is the case, for instance, with a thermostat), so that the regulating quantity can assume only two values, Q_1 or Q_2. *Fig. 21a* represents the case — already discussed in the article quoted in footnote [1]) — where $Q = Q_1$ so long as the quantity X to be regulated is greater than a critical value X_{cr}, and $Q = Q_2$ so long as $X < X_{cr}$; usually it is so arranged that X_{cr} is just the desired value X_0. Between Q_1 and Q_2 is a value Q_0 at which — if Q could assume that value — under normal working conditions X would have just the desired value X_0. Instead of that, the regulating unit is alternately in the positions Q_1 and Q_2. If it is desired to change over to a different value of X_0 then Q_0 has to be given a different value too. If, however, Q_0 becomes greater than Q_2 or less than Q_1 then the automatic controller no longer functions.

In practice the transition from Q_1 to Q_2 will not usually take place at exactly the same value of X as the change-over from Q_2 to Q_1: owing to backlash in the mechanical parts of the regulating unit for instance, or owing to the difference between the current at which a relay is closed and that at which it is opened, the transitions take place at different values of X (X_1 and X_2) either side of X_0. Instead of fig. 21*a* we then get fig. 21*b*. Denoting the deviations of Q and X from the equilibrium values again by q and x, then $q = Q - Q_0$ and $x = X - X_0$; the regulating cycle $q = f(x)$ for an automatic controller with two-step action having backlash then has the shape shown in fig. 21*c*.

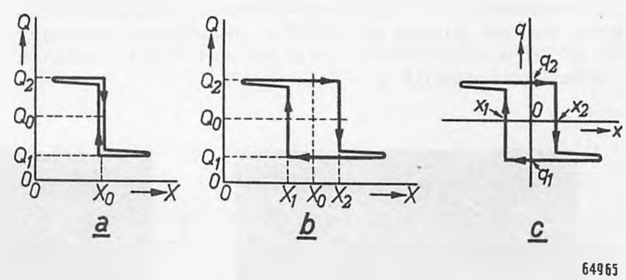


Fig. 21. Automatic controllers with two-step action, (*a*) without, (*b*) and (*c*) with backlash. In case (*a*) the regulating unit takes up the position corresponding to $Q = Q_1 < Q_0$ when X is greater than the desired value X_0, and $Q = Q_2 > Q_0$ when $X < X_0$ (Q_0 is the — impossible — position in which under normal working conditions X would assume the value X_0). *b*): The same but with a backlash X_1-X_2. In (*c*) the quantities $q = Q - Q_0$ and $x = X - X_0$ have been plotted against each other for the case (*b*).

The presence of backlash naturally reduces the response sensitivity and the accuracy of the regulation, but there is also a good side to it. In the case of an automatic controller without any backlash the regulating unit takes up the other position as soon as there is the least difference between X and X_{cr}, whereas in the case of a controller with backlash this does not take place until that difference reaches a certain value. In the latter case, therefore, the regulating unit is less subject to mechanical wear, and that is why sometimes some backlash is purposely given to the regulating unit.

It appears to be possible to build up electric circuits which show a relation between the output and the input voltages similar to that existing between q and x according to the *hysteresis loop* of fig. 21*c*. Such a circuit can serve as an electrical model for an automatic controller with two-step action. If, for instance, a sinusoidal voltage (x in

[6]) H. Nyquist, Regeneration theory, Bell Syst. techn. J. **11,** 126-147, 1932. See also B. D. H. Tellegen, Philips Techn. Rev. **2,** 292, 1937.
[7]) H. Bode, Network analysis and feedback amplifier design (Chapter VIII), New York 1945, and F. Strecker, Die elektrische Selbsterregung, Stuttgart 1947.

fig. 22) is applied to this circuit then a square-wave output voltage (q) is obtained.

In the electro-analogue containing such a circuit the latter can be combined with the process analogue to form a closed regulating circuit (*fig. 23*), in which also an amplifier, variable attenuators and a collecting stage are included. The limit $x_2 = -x_1$ is governed in the circuit by a direct voltage; thus by changing this voltage it is possible to vary the width of the loop. Via the collecting stage a direct voltage s can be added, corresponding to a variation of Q_0 or to bringing about a variation of Q_0.

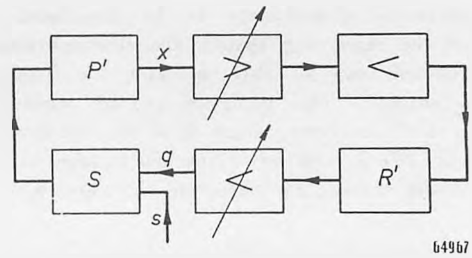


Fig. 23. Regulating circuit for studying discontinuously acting controlers (model R'). P' process analogue. Via the collecting stage S a direct voltage s can be applied to form the analogue of a disturbance or of a transition to some other desired value in the actual controller.

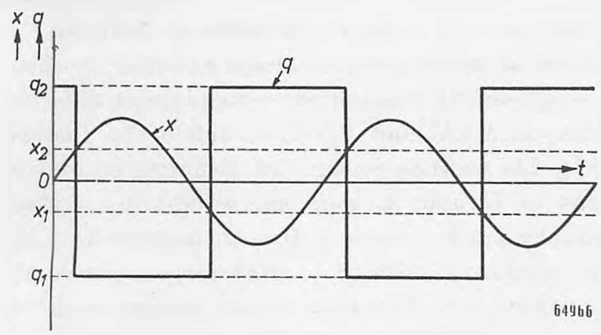


Fig. 22. When a sinusoidal voltage x (with amplitude greater than the limits $x_2 = -x_1$ of the backlash) is applied to an electric network forming the model of an automatic controller with two-step action there arises at the output of the network a square-wave voltage q.

In *figs 24a, b* and *c* oscillograms of q and x are represented respectively for $s = 0$, $s = +5$ V and $s = -5$ V, where changes takes place in the ratio of the intervals in which q has the value q_1 or q_2 respectively.

Automatic controller with three-step action

By giving the regulating unit a third position, Q_0, in between the positions Q_1 and Q_2 (*fig. 25*) one has the advantage that the system is at rest so long as the value of X is between X_1 and X_2.

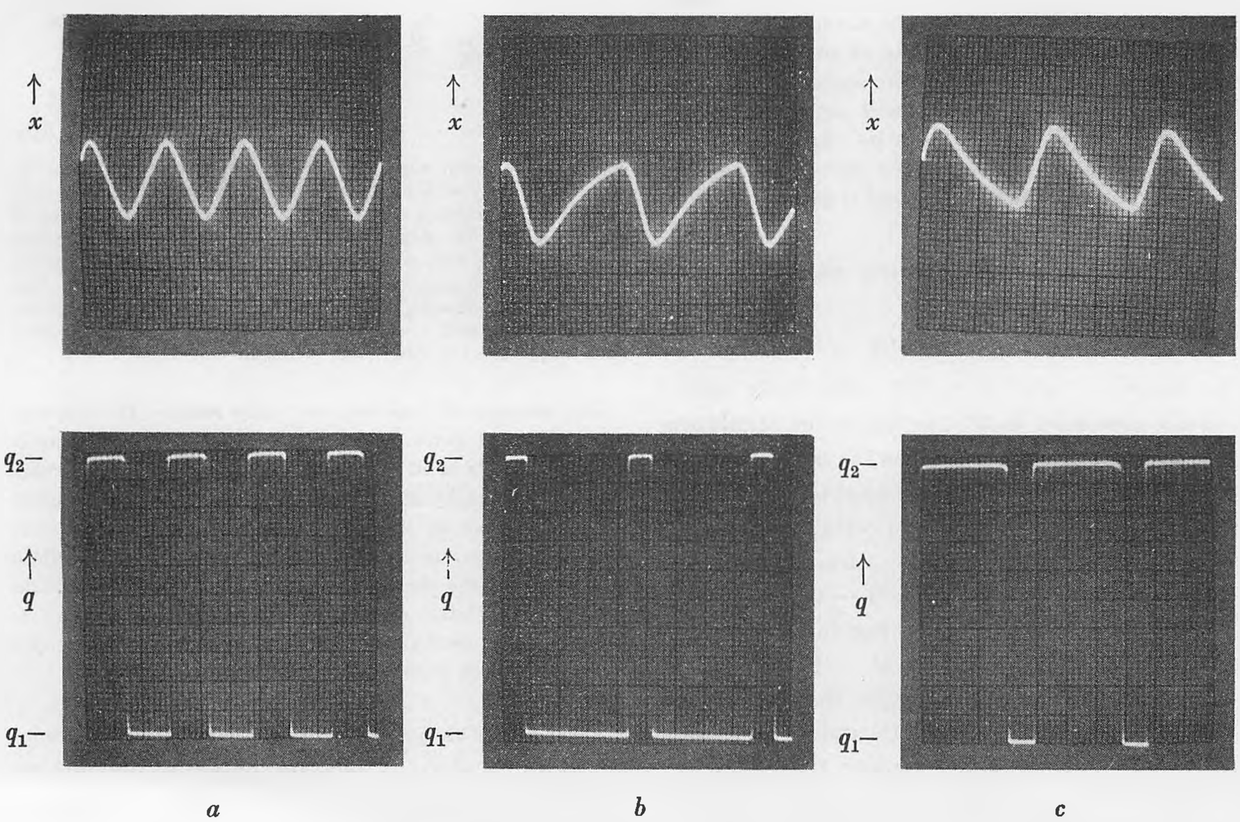


Fig. 24. Oscillograms obtained with a circuit according to fig. 23 with an automatic controller with two-step action. In the case (a) $s = 0$ and thus Q_0 lies halfway between Q_1 and Q_2 (fig. 21b). The intervals during which $q = q_1$ and q_2 respectively are of equal length. In the case (b) s is positive and in case (c) equally negative; the intervals during which $q = q_1$ and q_2 respectively are here unequal.

so that it suffers less from wear than a system lacking a state of rest. If considerable deviations from the normal working conditions arise only at long intervals then the controller and the regulating unit can remain at rest for a long time.

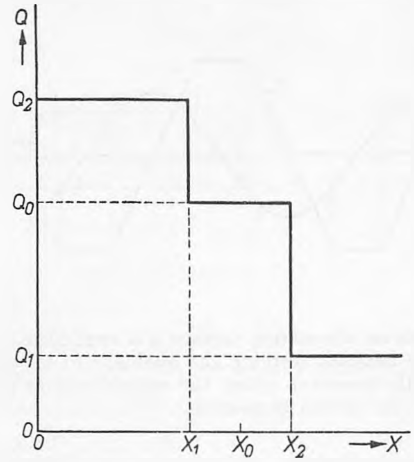
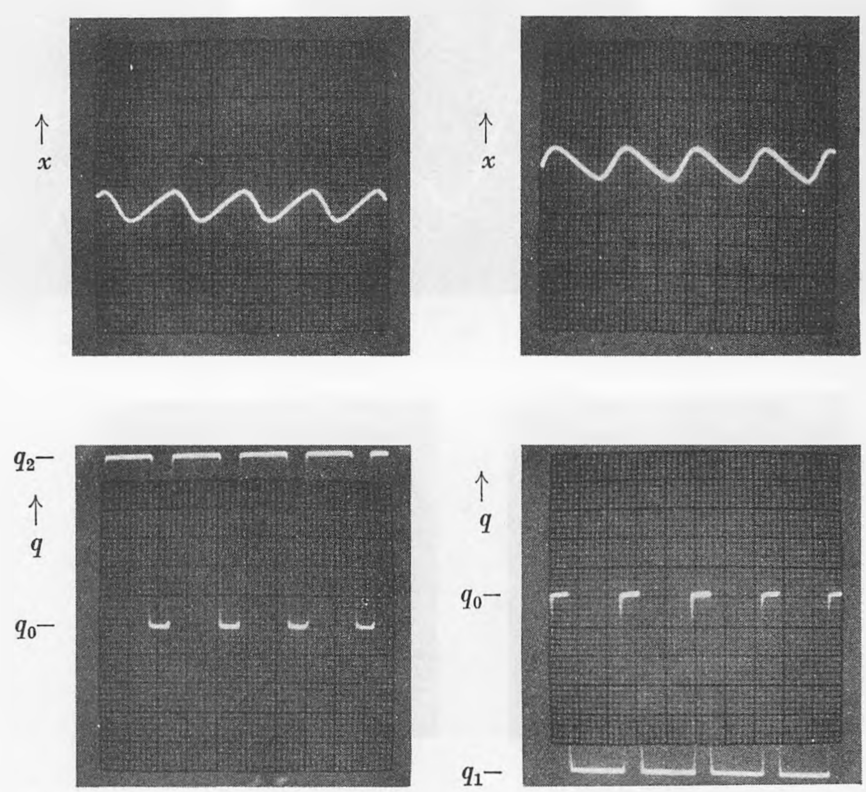


Fig. 25. Q as a function of X for an automatic controller with three step-action: in addition to the positions Q_1 and Q_2 also the intermediate position Q_0 is now possible. The regulating unit will be in the position Q_0 so long as X_0 remains between X_1 and X_2.

If, for some reason or other, X exceeds the value X_2 then the controlling unit causes the regulating unit to move to and fro between the position of rest Q_0 and the lower position Q_1, thereby correcting the too high value of X. The functioning can then be regarded as being similar to that of an automatic controller with two-step action (positions Q_0 and Q_1). If, on the other hand, X drops below X_1 then the controlling unit will fluctuate between the position Q_0 and the higher position Q_2, so that apparently we again have to do with an automatic controller with two-step action, but this time with the positions Q_0 and Q_1. Since the differences $Q_2 - Q_0$ and $Q_0 - Q_1$ are smaller than the difference $Q_2 - Q_1$ in the case of a controller with two-step action, also the periodic fluctuation of X caused by the changes of the controlling unit is smaller. This can also be regarded as an important advantage of an automatic controller with three-step action over one with two-step action. The fluctuation referred to can be reduced still further by increasing the number of positions of the controlling unit. The higher the number of positions the more closely the curve of echelon shape representing the relation between Q and X approaches a straight line, and the more the working of the automatic controller approaches that of a continuous controller with proportional action.

The functioning of an automatic controller with



a *b*

Fig. 26. Oscillograms obtained with a circuit according to fig. 23 with an automatic controller with three-step action, (*a*) with positive, (*b*) with equally negative disturbance voltage *s*.

three-step action is illustrated by the oscillograms in *fig. 26*. In this case, by applying a positive or a negative voltage s to the input of the process analogue, the rest level Q_0 was shifted in such a way that the controller came into action and moved to and fro between the positions Q_0 and Q_2, respectively between the positions Q_0 and Q_1.

Multi-speed floating control

Instead of the regulating unit being so made that it can only take up certain positions, it can be arranged so as to run with a certain, constant, speed dq/dt in the one direction or in the other or remain at rest according to the value of x. For instance:

$$dq/dt = -p \text{ when } x > x_2,$$

$$dq/dt = 0 \text{ when } x_2 > x > x_1,$$

and $$dq/dt = +p \text{ when } x < z_1.$$

This is made technically possible, for instance, by having a controlling unit with three-step action driving a servomotor which operates the regulating unit and turns at a constant speed in one direction or the other or remains at rest according to the position of the controlling unit. The motor then adjusts the regulating unit by an amount q varying linearly with the time: $q = \int p \, dt = p\,t + \text{const.}$

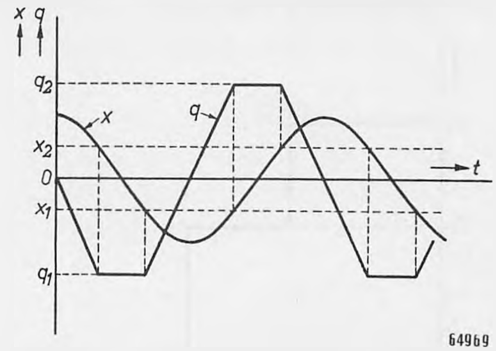


Fig. 27. When an alternating voltage x is applied to the input of an electric network forming the analogue of an automatic controller with three-step action the output voltage q assumes the shape of the drawn trapezium.

Compared with the automatic controller with three-step action, the multi-speed floating control has the advantage that in the event of permanent change from the normal (average) working conditions the motor provides for a permanent readjustment of the regulating unit.

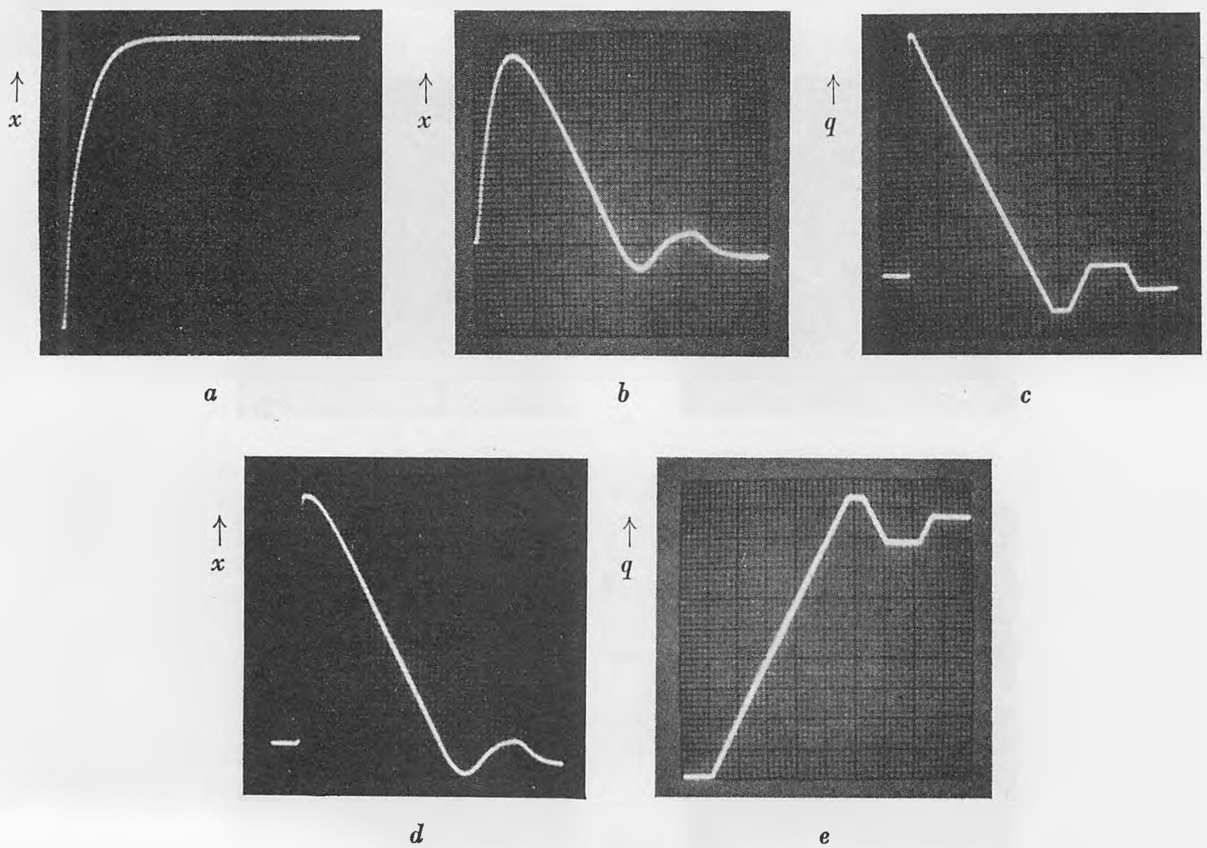


Fig. 28. *a*) Step-function response of a process analogue which together with an automatic controller with three-step action formed a regulating circuit from which the oscillograms (*b*) - (*e*) have been recorded. (*b*) and (*c*) represent the variation of x and q respectively in the case of a step disturbance at the input of the process, (*d*) and (*e*) the same quantities when a disturbance is applied at the output.

In an electrical model the function of the motor can be performed by an integrating network. When the output voltage ($-p$, $+p$ or 0) from a model of an automatic controller with three-step action is applied to an integrating network then at the output of that network there arises a voltage q which either increases or decreases linearly with the time or remains constant, thus in analogy with the readjustment of the regulating unit.

If, for instance, a sinusoidal voltage (with amplitude greater than x_2) is applied to the input of the automatic controller then it is easy to see (*fig. 27*) that at the output of the integrator there will be a trapezoidal voltage.

Fig. 28 shows some oscillograms recorded with a regulating circuit consisting of a process analogue, an automatic controller with three-step action and an integrator. Fig. 28*a* is the step-function response of the process analogue employed. A step disturbance at the input of the process analogue (corresponding to a sudden readjustment of the regulating unit) produces the variation x in the regulated quantity as illustrated in fig. 28*b*, and this in turn causes the voltage q at the input of the process analogue to change according to fig. 28*c*. The consequences of a step disturbance at the output of the process analogue are shown in figs 28*d* and *e*, respectively at the input and at the output of the process. In both these cases in the final position x comes to rest at a value between x_1 and x_2.

The examples that have been given here should be sufficient to show how valuable an aid the electro-analogue is for studying regulating processes. Details of the electric circuits of various parts such as the universal process analogue, the collecting stages, integrators and differentiators have purposely been omitted, since they will form the subject of a second article.

Summary. An electro-analogue is an apparatus with which electrical models of regulating devices can be built up and studied. The electro-analogue discussed here comprises two special models and one universal model of the process to be regulated, the latter model consisting of an electric network — mainly a delay network — the step-function response of which, displayed on the screen of an oscilloscope, can be given a shape identical to that of any process to be imitated. Further the electro-analogue comprises electrical models of automatic controllers with continuous action and of certain controllers with discontinuous action. In the model of the continuously acting controller a proportional term, a single and a double integrating term and a single and a double differentiating term can be realized.

After the model of the process has been given the right characteristic it is combined with one of the models of the automatic controller to form a closed regulating circuit. By applying a step disturbance at a suitable point it is possible to study with the aid of oscillograms the behaviour of the circuit at different values of the parameters. In particular it can be investigated under what conditions instability arises and how high the quality of the regulation is. Oscillograms are given for various cases, both with continuously and with discontinuously acting controllers.

A description of the electric circuits of the principal parts will be given in a subsequent article.

TIME DELAY ELEMENTS

The delays and pseudo-delays discussed in the previous articles presumed the existence of purely analog structures of satisfactory quality. However, these elements suffer from the inability to readily adjust their time scale.

The need for such adjustments arises in computing assemblages involving transport lags, for example, where the time delay depends on the transport length divided by the transport velocity. If this velocity should vary during the process, then a variable time delay is involved, but one in which no signal information is lost, as would occur if merely the tapping position of a continuous line were adjusted by servos.

A time delay method which operates at high speeds and is at once the strict analog of a transport lag or a sampling servomechanism is disclosed in the next pages.

SAMPLING DELAY LINES

The following short note, reprinted from NATURE, represents a significant contribution by J.M.L.Janssen to the art of time delay representation in analog devices. The same mechanism was shortly after independently conceived by G.A.Philbrick, and dubbed the "bucket-brigade" delay. An early Philbrick form is diagrammed following the Janssen exposition.

(*Reprinted from Nature, Vol.* 169, *p.* 148, *January* 26, 1952)

Discontinuous Low-frequency Delay Line with Continuously Variable Delay

CONVENTIONAL low-frequency delay lines, built from capacitors and iron-cored inductors, show both linear and non-linear distortion. The linear distortion gives rise to dispersion, since the time delay depends on frequency. It is only possible to realize a time delay that is approximately constant over a limited frequency range[1-3]. The non-linear distortion is due to the iron-cored inductors. Only for signals that are sufficiently small is this distortion negligible. So the iron limits the allowable energy-level.

In some applications these limitations are not tolerable. If, moreover, a very long time delay is wanted, then either the value for the inductances or the value for the characteristic impedance becomes impracticable.

In order to avoid these difficulties, the delay circuit shown in Fig. 1 is suggested. The blocks B are buffer amplifiers with the following idealized properties: (1) amplification is unity; (2) input impedance is infinite; (3) output impedance is zero; (4) available output power is infinite.

The switches S are actuated by electrical signals. To explain the operation of the delay line, it is assumed that at a certain moment all switches are open. The voltages of the capacitors C_{n-1}, C_n, C_{n+1}, etc., are E_{n-1}, E_n, E_{n+1}, etc., respectively. Since the input impedance of the buffer amplifiers is infinite, these voltages are constant as a function of time.

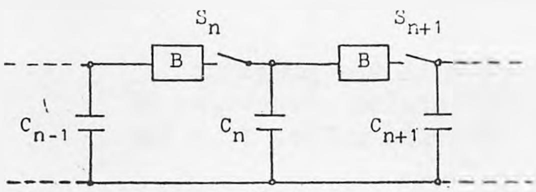


Fig. 1

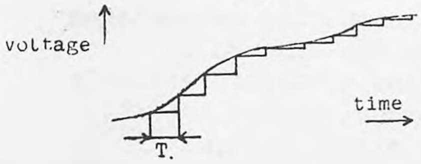


Fig. 2

Now the switch S_{n+1} is closed for a short time. Then immediately the voltage of the capacitor C_{n+1} becomes E_n. If the next switch S_n is closed for a short time, the voltage of the capacitor C^n becomes E_{n-1}, and so on. So if a single switching pulse moves from the right to the left along all the switches, then all voltages move one step to the right. If a second pulse moves from the right to the left, all voltages move one step to the right again, and so on. This process may be continued by moving an endless train of equally spaced pulses with constant velocity from the right to the left. It can be shown that the system also behaves as a delay line if the train of pulses moves in the opposite direction. However, the most simple system is obtained when the pulses travel in neither direction, using what might be called a 'standing wave' of switching pulses instead of a travelling one. This condition can be brought about by actuating alternately all even and all odd switches. For example, all even switches may be actuated at times $t=0$, T, $2T$ . . ., and all odd switches at times $t = T/2$, $3T/2$, $5T/2$, . . . Thus, two switching signals are required with period T, shifted in time with respect to each other over an interval $T/2$. As a matter of fact, this 'standing wave' of switching pulses cannot be distinguished from a travelling train of pulses in either direction if in the latter case the distance between successive pulses is equal to twice the distance between successive switches[4].

Essentially the sections of the delay line are clamping circuits. Fig. 2 shows the voltage at the input of the first section (solid line) and the 'staircase voltage' at the output of the first section (capacitor C_1). The voltage at the output of the second section (capacitor C_2) has the same shape as that at the output of the first section, but has an extra time delay $T/2$. The delay line as a whole acts as a filter with clamping[5].

It is obvious that the time delay per section can be varied continuously by varying the period of the switching pulses. This property is especially useful when the delay line forms part of a computer for the evaluations of auto-correlation functions. If the frequency is suddenly made zero, in other words, if the switching pulses are stopped, then a certain amount of information remains stored within the line. So the circuit may be used as a memory circuit in a computer. Another possible application is the synthesis of a given transient wave-form[6].

J. M. L. JANSSEN

Royal Dutch/Shell Laboratory,
Delft. Oct. 25.

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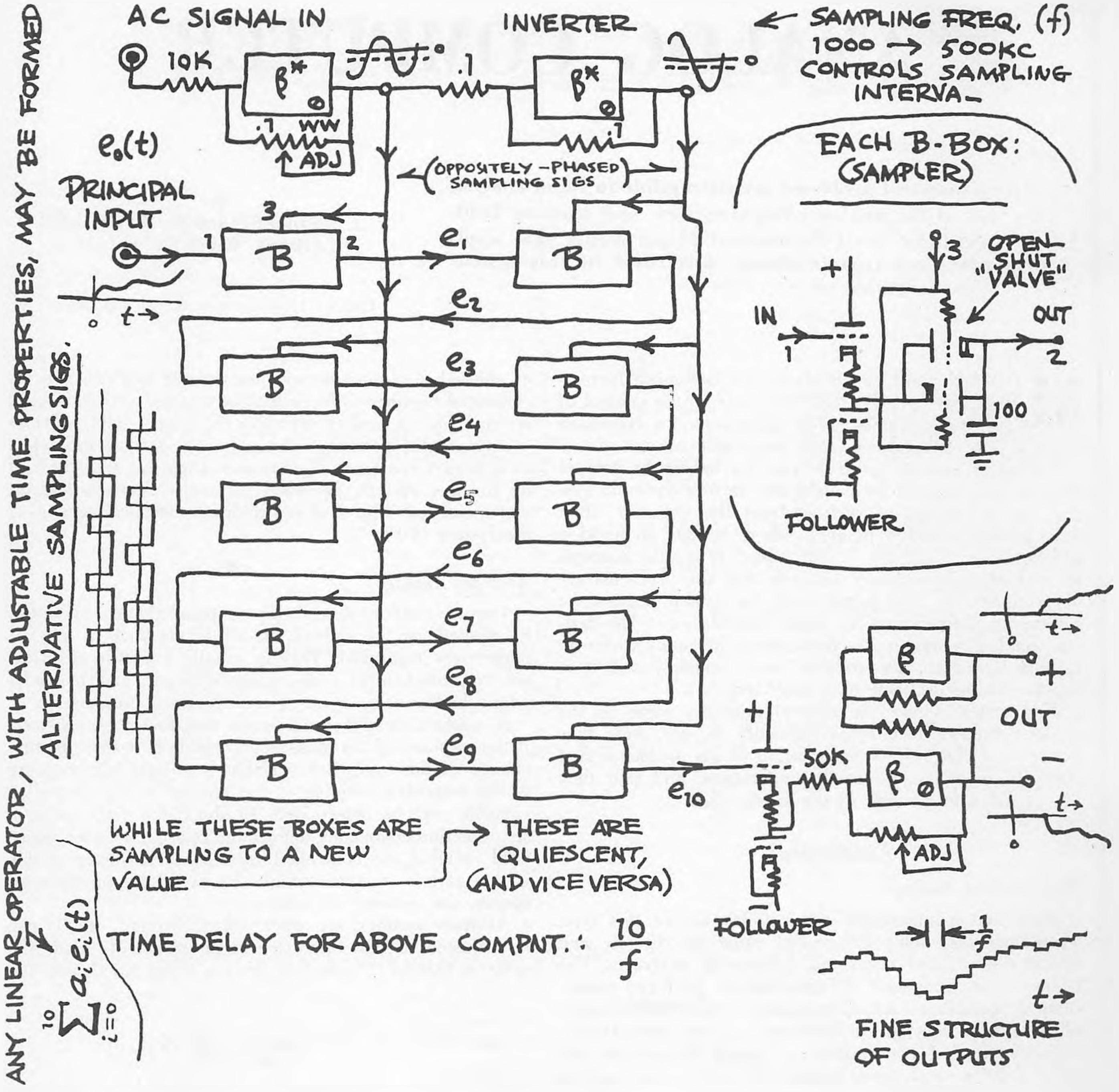
NEW CIRCUIT

AUTOMATICALLY ADJUSTABLE TIME-DELAY COMPONENT



β	STANDARD AMPL.	11	12AT7
		10	12AU7
β*	HIGH-SWING "	6	12AX7
			K4 CASE

"BUCKET BRIGADE" "TIME DELAY"

IDEA: Jan 19, 1952 — Feb. 14, 1952

FIRST MOD. 2/14/52

G A Philbrick

Solving process-control problems by

ANALOG COMPUTER

Process-control problems are susceptible to rapid analysis by use of the analog computer. New and exciting fields of operation await Instrument Departments that apply the new techniques clearly outlined in this article.

by
R. J. MEDKEFF and H. MATTHEWS
Askania Regulator Co.

October 1954—*Instruments & Automation*

MODERN plant specifications are becoming increasingly stringent in their demands on the control of process variables. Most plants are in transition through semi-automatic to fully automatic control.

The plant and its controls can no longer be treated separately, but must be considered as one dynamic system. In the design of such systems the engineer often finds himself working in areas where he has no backlog of experience to aid him. If he attempts to use the concepts of modern control theory he finds that the problems are too complex or that portions of the system cannot be sufficiently defined for "by-hand" calculations. The fast-time analog computer has been found by many to be an invaluable aid in the study of such systems, and it is rapidly taking its place as a standard tool.

This article shows the control engineer some of the analog-computer techniques currently in use, how they would be useful aids in the solution of his problems, that they are practical and easily understood, and that they are priced within range of the small office.

Definitions

Mathematical Analogs

Most analog computers in use today are of this type. These machines solve differential equations directly and are at times called electronic differential analyzers. The computer is composed of units which perform mathematical operations such as integration, summation, multiplication by a constant, differentiation (with restrictions), multiplication of two variables, lagged integration and differentiation, and many others which are combinations of the basic operations.

Each unit is composed of a high-gain d-c. amplifier and a particular feedback network which causes the amplifier to perform the desired mathematical operation on the voltage applied. All variables are voltages which are scaled in amplitude, and the independent variable is time. Fig. 1 shows a simple mechanical system, its equations, and the solution as set up by a mathematical analog computer.

Direct Electrical Analogs

Two systems can be said to be analogous if corresponding variables of one behave in the same fashion as those of the other. Inductor-capacitor-resistor circuits can be connected together to form analogs of mass-spring-dashpot systems. Corresponding variables are force—volts; velocity—current; displacement—charge; time—k times time (k is an arbitrary constant). Electrical analogs can thus be built up for any system, provided parameters can be lumped and measured. Fig. 2 shows a direct electrical analog of the system in Fig. 1.

Abstract Analogs

The two methods described presuppose that the system to be studied can be defined, its equations written, and all parameters measured. This is usually true for electrical systems, mechanical systems, aerodynamic and hydrodynamic bodies—but is rarely true for most processes.

A mathematical theorem states that for linear systems, if the response of the measured variable to a step-function change of the controlled variable is known, the response of the measured variable to any change of the controlled variable may be determined. In short, the response to a step function completely defines the relation between measured variable and controlled variable. A corollary of this is that if two systems exhibit the same step-function response, the systems are analogous.

Abstract analogs are then "black boxes," which can be adjusted so that their step response can be made the same as that of the plant or process to be simulated. The

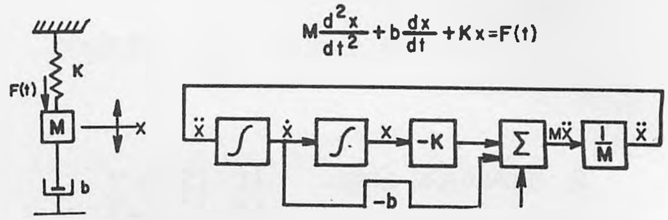


Fig. 1. Mechanical systems, equation, and mathematical analog.

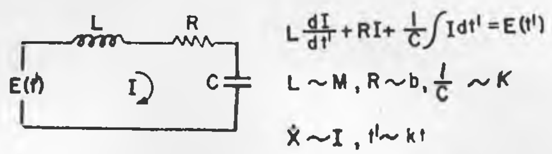


Fig. 2. Electrical analog (direct analog) of mechanical system of Fig. 1, and equation of electric circuit.

Presented at Symposium on Industrial Applications of Automatic Computing Equipment, Midwest Research Institute, 1953.

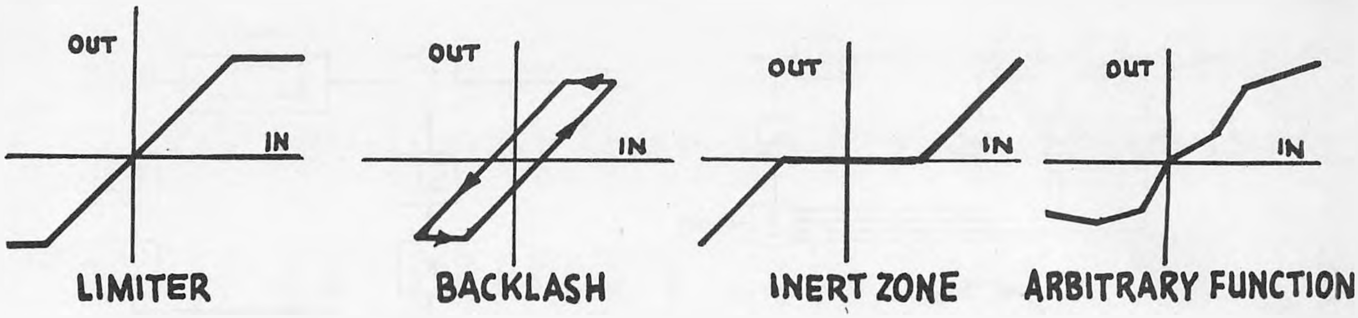


Fig. 3. Typical useful nonlinear functions.

usual procedure is to deal with small step changes so that system linearity may be assumed.

Abstract Analogs of Nonlinear Responses

All analog computers have available devices which simulate nonlinear functions such as dead zone, limits, hysteresis, backlash, and purely arbitrary functions. These are usually diode devices and simulate the desired functions by a series of straight lines. Fig. 3 shows some of the common functions. References cover this subject well.

Time Transformation

Many processes have a normal period which may run into hours. If the analog were to have the same time relationship, many of the advantages of analog study would be lost. Furthermore the requirements of the components regarding size, quality, and cost become prohibitive.

It is convenient to change the time scale so that the process period will be of the order of fractions of seconds to seconds. All of the variables behave as before with relation to each other, but have their time relationship changed by an arbitrary factor. Usually the size of analog components available will dictate a certain range of frequencies over which the analog will function. The time scale factor is chosen to relate the process to this range.

Further advantage may be gained if the frequencies are such that oscilloscopes can be used instead of more expensive recorders.

Procedures

To make the time transformation, choose the arbitrary factor (T) equal to t'/t, where t' is process seconds on analog, and t is real time in seconds.

From which: $Tt = t'$

$$dt = dt'/T$$

If this substitution is made in the equation of Fig. 2:

$$TL'dI/dt' + R'I + (I/TC')\int I dt' = E(t')$$

Analog parameters are chosen so that $TL' = L$; $R' = R$; $TC' = C$.

Note that parameter changes are only made whenever a time integration or differentiation is involved, otherwise no change is necessary.

The same discussion applies to mathematical analogs. Wherever dt appears, substitute dt'/T, and change the value of the time constant accordingly.

Process-control Studies

Most analog-computer installations are combinations of the various types (mathematical, direct, and abstract). Each type works best for specific operations. Process studies make extensive use of abstract analogs.

It will usually be necessary to vary the time base of the problem since the process time may vary from a fraction of a second to several hours.

Process Lags

Many processes have step-function responses similar to those of Fig. 4. These will be recognized as the responses to single, double, triple, and quadruple capacity plants as discussed by Oldenbourg and Sartorius. Fig. 4 also shows resistor-capacitor networks which simulate these responses.

Such networks can, by direct analogy, simulate the flow of heat through different media. Several computers developed along this principle are capable of studying three-dimensional heat-flow problems.

Process Delays

Processes exhibit a certain amount of dead time, or transportation lag (Fig. 5). The delay may be combined with any of the typical responses of plants without delay. The abstract analog in this case might be considered as an extension of the Zeigler-Nichols approach which has been found to be useful by the control engineer.

A method of simulating delay is to use an electrical delay line, made up of a cascade of inductor-capacitor sections. This can be followed by networks and mathematical operations to obtain the desired response. One of the problems with this method is that of obtaining sufficient delay. Commercial lines are of the order of microseconds, whereas even with fast-time-base computers milliseconds are needed. This means that the time scale will have to be chosen so that the actual delay on the real plant will be normalized to milliseconds on the analog.

The response of a cascade of resistor-capacitor lags approaches a delay as the number of lags is increased (Fig. 4). One group reports the use of 80 such lags, with considerable success.

The most common form of delay associated with process systems is that caused by long pneumatic lines which are used for signal purposes. Such lines should be analogous to electrical delay lines, and such lines certainly could be made to simulate a particular pneumatic line

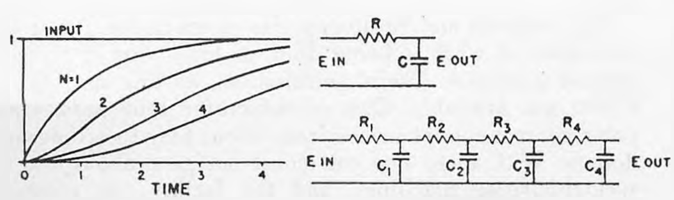
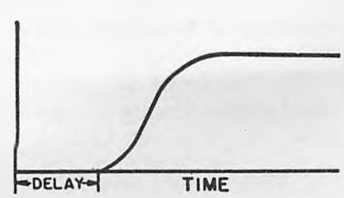


Fig. 4. Electric analog of single- and multiple-capacity plants, and response to step change in input.

Fig. 5. Plant response with delay.



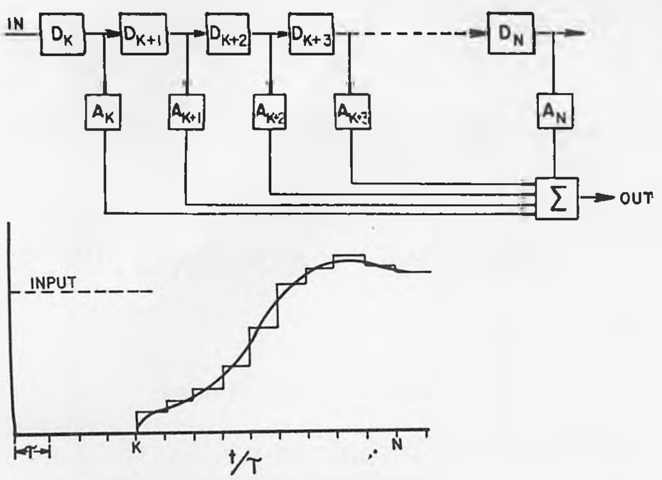


Fig. 6. Universal-analog technique for obtaining an arbitrary response.

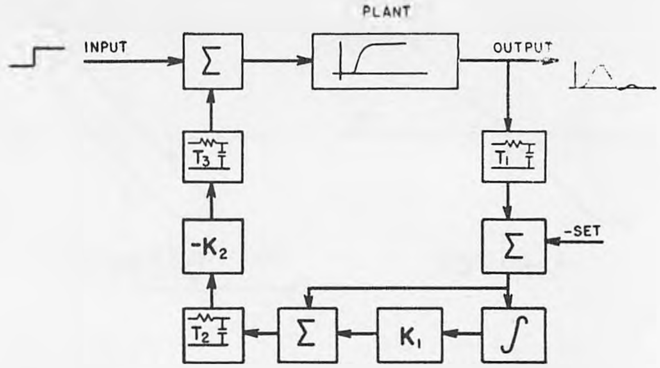


Fig. 7. Typical process-control problem, showing block diagram of simulation technique.

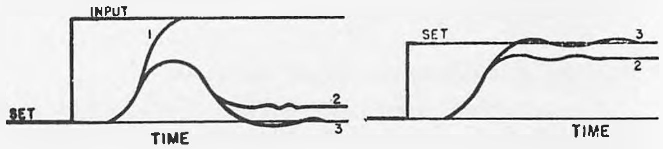


Fig. 8. A (left), response to input step disturbance; B (right), response to step change in set point.

closely. However, no sure way has been presented to predict the behavior of a line just from its dimensions.

One paper shows that three cascaded lags can simulate lines and gives experimentally determined time-constants for several sizes of tubing.* One difficulty in this problem is the separation of the line and its terminating impedance. The lines always work into bellows or other pressure-sensitive devices, and no attempt at a match ever seems to be made. The only certain method is to measure experimentally the response of the line and its load and then simulate this on the computer. The subject of pneumatic lines presents many challenging problems, much interesting work could be done with analog computers.

Universal Process Analog

An elegant approach to the problem is a universal analog based on the superposition (Duhamel's integral) principle. This machine is capable of simulating delays combined with a completely arbitrary response, and is versatile in adjustment. Fig. 6 is a block diagram showing the function of this device. The analog is built up around a delay line of forty m-derived sections. The signal may be taken off at each section, multiplied by an arbitrary constant, and all of the components are then summarized. As seen from the diagram, completely arbitrary functions of time and amplitude may be generated. The authors report considerable success with this analog.

Typical Process-Control Problem

Fig. 7 shows the block-diagram solution of a typical process-control problem. The plant is represented by a delay and a lag and simulates a process which was essentially a large tank. The measured variable was gaseous, and it was desired to control the weight flow. The flow sensor was found to be approximately a single-order lag, T_1. A hydraulic regulator performing the functions of proportional plus reset (integral) was used. Two summarizers, an integrator, a coefficient unit K_1 (calibrated in repeats per minute or inverse time constant), and a single-order lag (inherent in the regulator) simulate the regulator. K_2 sets the over-all loop gain, and the final single-order lag represents a hydraulic servo which operates the control valve.

Fig. 8A shows the response of the system to a small step-function change in upstream pressure or control variable: (1) without control, (2) with proportional control only, (3) with proportional plus reset. Fig. 8B shows the responses for a step change in set point. Time scales were chosen so that the results could be displayed on an oscilloscope, and an ordinary electronic square-wave generator was used for the step functions. The controlled responses shown were the optimum. The parameters K_1 and K_2 were determined in a few seconds once the problem was set up. Results: The computer quickly answers the following questions:

1. Will the proposed control system actually regulate the plant?
2. What are the control parameters giving optimum control?
3. What is optimum control?
4. What are the limits of parameters and time constants giving a stable system?
5. Are parameters critical, stability-wise?
6. If dead zone and/or backlash units (Fig. 3) are introduced at appropriate places in the circuit, how precise must components be?

Hydraulic circuits may be more realistically constructed to include flow limits to determine the maximum capacity required of the pumps.

Conclusion

Another useful function, unique with the fast-time analog computer, is the training of plant operating and maintenance personnel. It is difficult to explain to plant personnel the meaning of and reasons for the various control functions. With the computer the operator can be shown what should be expected of his plant, and he can see graphically on the oscilloscope the effects each control parameter has on the various process variables. The computer has been used extensively at Askania for this purpose.

The fast-time computer can also be an invaluable aid in working out starting procedures for complicated systems.

The methods and equipment discussed in this paper are indicative of what is being done in the study of process-control problems. Useful installations costing as little as $5000 are available. One manufacturer sells basic computer components at prices from about fifty to a thousand dollars. With these one can construct general-purpose or special-purpose machines, and the facility can grow as the need arises. No engineering organization is too small to be able to afford such equipment.

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* Bradner, Mead, "Pneumatic Transmission," *Instruments*, Volume 22, No. 7, July, 1949.

ELECTRONIC ANALOG METHODS IN DETAIL

Thus far the discourse has largely been concerned with the nature of the tools. Methods of application have merely been cited as examples.

The works of Rideout and his co-workers at the University of Wisconsin have been marked by a penetrating thoroughness in the detailed exploration of methods whereby analog tools may be applied to analysis to secure accurate results at high speed. Each of the cases cited is sufficiently complete as to afford an excellent perspective on both the strength and the limitation of the available tools. Particular interest therefore attaches to the problem of "making the best of" these limitations. Such questions are here discussed in fair completeness.

Reprinted from Project Cyclone Symposium on Simulation and Computing Techniques

13. PRECISION IN HIGH-SPEED ELECTRONIC DIFFERENTIAL ANALYZERS

By H. Bell, Jr. and V. C. Rideout

University of Wisconsin

The Mathieu equation, the van der Pol and other oscillator equations, and a nonlinear differential equation for fluid flow on a rotating disk have been solved on a high-speed differential analyzer. This analyzer presents a solution repetitively at a 60-cycle rate or, in the case of a self-oscillating analog, oscillates in the vicinity of 400 cps. Data is presented on a cathode ray oscilloscope. The results obtained with the electronic differential analyzer have been compared with numerical results obtained by digital computer methods for some of the problems listed above. The comparison serves to indicate the degree of precision possible in the solution of equations of these general types in a computer of this kind.

Redesign of certain computer components with a view to improving the possible precision is being made and will be described. The precision to be expected in studies now being made of such equations as those describing the frequency-modulated van der Pol oscillator will be discussed.

1. INTRODUCTION

A high-speed electronic differential analyzer consisting mainly of Philbrick equipment has been in use at the University of Wisconsin for nearly three years. In this device the solution of ordinary differential equations is presented at a rate above the flicker frequency on a cathode ray oscilloscope. Either thirty or sixty complete solutions are usually presented each second.[1,2]

A given degree of precision is somewhat more difficult to obtain with the high speed computer than with the slower machines which work on a "real time" scale. However, the high-speed machine has the advantage (where no tie-in with physical components is required) that problems with end-point conditions can be solved much more quickly.[3] Also, the oscilloscopic presentation of results is more convenient and useful for many purposes than recorder strip charts. Some efforts have been made at the University of Wisconsin to improve the precision of the high-speed computer, particularly in the solution of nonlinear differential equations and of linear differential equations with variable coefficients.

Of course, any "real time" electronic differential analyzer can be converted into a "high-speed" computer by decreasing the integrator time constants if only linear differential equations with constant coefficients are to be solved. The problem of obtaining precision then centers around the control of phase shift in the operational amplifiers, and the choice and proper use of precision linear circuit components. In the solution of other than linear differential equations the very difficult problem of obtaining fast-acting and precise multipliers and function generators must be faced. A "high-speed" computer may be slowed down to "real time" by merely changing the integrator time constant. This will give reduced phase shift errors, but amplifier zero stability becomes more of a problem.

This paper will discuss some of the problems encountered in obtaining precise results with both linear and nonlinear computer units designed for high-speed operation.

2. PRECISION IN LINEAR COMPONENTS FOR HIGH-SPEED OPERATION

The problems involved in the choice and proper use of precision components such as potentiometers, feedback capacitors, etc., will not be discussed in this paper, since they are much the same at 60-cycle rates as at "real time" rates. Of more interest here are the characteristic time constants in the operational amplifiers, or their reciprocals, the characteristic or "corner" frequencies.

Macnee[4] has pointed out that the characteristic frequencies of operational integrators and adders can account for large errors in some problems, even though these frequencies are far removed from the operating frequencies.

These errors are caused by unwanted phase shifts rather than amplitude changes. In problems in which the characteristic roots have zero or positive real parts the errors caused by small phase shifts are particularly serious.

A general solution for an operational amplifier can be made for the operational amplifier of Figure 1(a) in which the active circuit within the triangle has a transadmittance $-Y_a$. This solution is:

$$(1)\qquad e_0 = \frac{-[Y_a Z_2 Z_f - Z_2]e_1}{Y_a Z_2 Z_1 + Z_2 + Z_f + Z_1 + Z_1(Z_2 + Z_f)/Z_1}$$

This reduces to the familiar $-(Z_f/Z_l)e_l$ form if the gain $K = Y_a Z_2$ is infinite.

(a) The Operational Integrator

If $Z_l \to \infty$ and $Z_2 = R_2$, $Z_l = R_l$, and $Z_f = 1/sC_f$ in Equation (1), and if we assume that the gain $K = Y_a Z_2$ is independent of s (or free of time constants), then from Equation (1),

$$(2)\quad e_0 = \frac{-[K - R_2 C_f s]\, e_l}{1 + (R_l + R_2 + R_l K)\, C_f s} \approx \frac{-K[1 - T_2 s/K]\, e_l}{[1 + T_1 K s]}$$

where $T_2 = R_2 C_f$, and $T_1 = R_l C_f$ (the integrator time-constant).

If e_0 is solved for by Laplace transform methods for an input step function $e_l = E_l/s$, then for t in the vicinity of T_1,

$$(3)\quad \frac{e_0(t)}{E_l} \approx \frac{T_2}{T_1 K} - \frac{t}{T_1} + \frac{1}{2K}\left(\frac{t}{T_1}\right)^2.$$

This expression, which is plotted in Figure 1(b), shows that there are two main errors in the response of an integrator to an input step;[5] these are the "jump" error T_2/KT_1 and the "square-law" error $(t/T_1)^2/2K$.

It can be seen that these errors can be made quite small by the use of high-gain amplifiers. Phase shift may still give difficulty in some problems, however. If we substitute $s = j\omega$ in Equation (2) then

$$(2a)\quad \left|\frac{e_0}{e_l}\right| = K\left[\frac{1 + \omega^2 T_2^2/K^2}{1 + \omega^2 T_1^2\ K^2}\right]^{1/2}.$$

The angle of phase lag is

$$(4)\quad \phi_J = -\tan^{-1}\omega T_1 K - \tan^{-1}\omega T_2/K.$$

For frequencies in the neighborhood of $\omega_1 = 1/T_1$,

$$(5)\quad \phi_J \approx -\frac{\pi}{2} + \frac{1}{\omega T_1 K} - \frac{\omega T_2}{K}.$$

The picture is not greatly changed if Y_a contains one important time constant T_a so that,

$$(6)\quad K = Y_a Z_2 = \frac{K_0}{1 + sT_a}.$$

Equation (2) will now become

$$(7)\quad e_0 = \frac{-K_0[1 - (T_2/K_0)s - (T_a T_2/K_0)s^2]e_l}{1 + (T_a + T_1 + T_2 + K_0 T_1)s + (T_1 + T_2)T_a s^2}$$

In most operational amplifiers where $T_a \ll T_1$ the errors resulting in the response to an input step-function will not be much more serious than if T_a is not considered. The errors will similarly be small in the solution of equations whose characteristic roots have large negative real parts.

The consideration of the time constant T_a will result in an expression for phase lag ϕ_J which differs slightly from Equation (5). Using $K_0 \gg 1$, then from Equation (7), if ω is in the neighborhood of $\omega_1 = 1/T_1$,

$$(8)\quad \phi_J \approx -\frac{\pi}{2} + \frac{1}{\omega K_0 T_1} - \frac{\omega}{K_0}\left(T_a + T_2 + \frac{T_a T_2}{T_1}\right).$$

For frequencies which differ widely from $\omega_1 = 1/T_1$ more exact expressions must be used. Figure 2 shows a plot of gain and phase versus normalized frequency T_1 for some assumed integrator circuit constants. Note that the "integration band" in which the phase lag is approximately $\pi/2$ radians extends approximately from $\omega T_1 = .01$ to $\omega T_1 = 1000$. Actually, there is only one point at which $\phi_J = -\pi/2$ exactly, unless some sort of phase compensation is used. In practice the phase error is a lag for $\omega \geqq 1/T_1$ so that the last term in Equation (8) is the important one, and any phase compensation scheme must effectively introduce a lead.

(b) Adders and Scale Changers

In an adder or scale changer in which $Z_f = R_f$, then for the case of an amplifier with a single time constant we have from Equations (1) and (6), (if $Z_l \to \infty$)

$$(9)\quad e_0 = \frac{-[K_0 - (R_2/R_f)(1 + sT_a)]e_l}{[1 + (R_l + R_2)/R_f](1 + sT_a) + K_0 R_l/R_f}$$

If R_2 is relatively small and $K_0 \gg 1$,

$$(10)\quad e_0 \approx -\frac{R_f}{R_l}\cdot\frac{1 - R_2 T_a s/R_f K_0}{1 + (R_f + R_l)T_a s/R_l K_0}$$

If the gain is unity, so that $R_f = R_l$, then with the aid of other substitutions,

$$(11)\quad e_0 \approx -\frac{1 - T_2 T_a s/K_0 T_1}{1 + 2T_a s/K_0}.$$

The phase shift may be obtained from Equation (11).

(12) $\phi_A = -\tan^{-1} \omega T_a T_2/K_0 T_1 - \tan^{-1} 2\omega T_a/K_0$.

If K_0 is large, then for frequencies in the vicinity of $\omega_1 = 1/T_1$,

(13) $\phi_A \approx -\omega T_a T_2/K_0 T_1 - 2\omega T_a/K_0$.

The curves showing gain and phase versus frequency are shown in Figure 3 for an adder or scale changer (of unity gain) for a set of assumed circuit constants.

(c) Correction of Phase Distortion in Operational Amplifiers

From the above analysis and Figures 2 and 3 it can be seen that phase distortion is more serious than amplitude distortion in operational amplifiers. As pointed out by Macnee, this can cause serious errors in a computer set-up used to represent an oscillator.[4] Thus in the set-up of Figure 4, if the dotted circuit is not included and the loop is opened, a voltage $\theta = \sin \omega_1 t$ applied at the first integrator gives

(14) $-\dot{\theta} = -\sin(\omega_1 t + \pi/2 + \phi_J)$
$= -\cos(\omega_1 t + \phi_J)$

where ϕ_J is the excess phase lag in an integrator.

At the output of the second integrator

(15) $\theta = \sin(\omega_1 t + \pi + 2\phi_J) = -\sin(\omega_1 t + 2\phi_J)$

At the output of the adder, which has phase shift ϕ_A

(16) $-\theta = \sin(\omega_1 t + 2\phi_J + \phi_A) = \sin(\omega_1 t + \phi)$

The undesired phase lag $\phi = 2\phi_J + \phi_A$ will be a small angle which can be determined by use of Equations (8) and (13). Since it is small, Equation (16) can be expanded to give

(17) $-\theta = \cos\phi \sin\omega_1 t + \sin\phi \cos\omega_1 t$
$\approx \sin\omega_1 t + \phi \cos\omega_1 t$

Thus, $-\theta$ differs from $\ddot{\theta}$ by a term $\phi \cos \omega_1 t$. Since this corresponds to a negative damping term, the closed loop will give oscillations of frequency ω_1 and decrement $-\phi/2$.

One scheme for reducing the error here is to introduce positive damping by use of the potentiometer shown in Figure 4 in which $\delta = \phi$.

The phase distortion may also be reduced in the units themselves by decreasing T_a and any other time constants which may be present. One obvious method for improving the situation is to use pentodes rather than triodes so that a wider band (or smaller T_a) before feedback can be obtained.

In communication work it is common practice to add phase compensating networks to give linear phase shift over the band of frequencies to be handled. The philosophy adopted in communications is that if phase is linear and amplitude is flat for all frequency components of a message then there will be no distortion. Here the same philosophy may be adopted except that the requirements on phase shift are more stringent — zero phase shift is required within the band occupied by the "signal" in the computer.

Bode has shown that zero phase shift is possible over part of a band if the amplitude is peaked at the edge of a band. This effect results because the phase advance caused by a gain peak will compensate for the phase lag resulting from a drop-off in gain at frequencies beyond the peak. Thus, a band which has a peak at frequencies above those of interest may be used to correct the small but quite undesirable phase shifts at lower frequencies at the expense of greater phase shift further out.

An ordinary resistance-capacitance lead network will give the kind of compensation discussed above. However, it will require increased gain, and this will tend to nullify its advantages. The desired band shape may better be approximated by placing a small capacitor C_i in parallel (see Figure 5) with R_i in adders and scale changers so that Equation (10) gives, for $R_i = R_f$,

(18) $$e_0 \approx -\frac{(1 - T_2 T_a s/K_0 T_1)(1 + R_i C_i s)e_i}{1 + 2T_a s/K_0}$$

At frequencies in the vicinity of $\omega_1 = 1/T_1$, where phase shifts are small, compensation requires

(19) $R_i C_i \approx 2T_a/K_0 + T_2 T_a/K_0 T_1$.

This type of compensation was added to one channel of a Philbrick adder with measured gain and phase characteristics as shown in Figure 6 (solid curves). The phase shift was improved as shown (dotted curves) by a factor of 6 or more out to 100 kc at the expense of a slightly increased gain in the vicinity of 60 kc. Since this small gain peak is at a frequency of 1000 times the sixty cycle repetition rate, it should ordinarily cause no difficulty. The phase and amplitude in this adder may now be regarded as quite satisfactory out to about 20 kc.

This type of compensation may also be used with integrators since in general they have a phase lag error at the ordinary frequencies of operation. An oscillator set-up like the one shown in Figure 4 is a sensitive adjusting aid for these trimmer capacitors.

3. PRECISION IN NONLINEAR COMPONENTS FOR HIGH-SPEED OPERATION

(a) Limiters and Other Units Using Diode Rectifiers

Devices such as limiters, full-wave rectifiers, dead-zone units, and hysteresis units for use in high-speed computers ordinarily use pairs of diodes or diode bridge circuits.[6] In these units the associated amplifiers may give the same phase shift errors as the ordinary operational amplifiers. New problems arise in obtaining ideal characteristics from the diodes. This involves the use of large load resistances and the choice of diodes such as the 6J6 (diode-connected) which have desirable characteristics.

(b) The Photoformer as a High-Speed Function Generator

The photoformer, which uses a cathode ray tube, an opaque mask, and a multiplier phototube in a closed feedback loop has been described by a number of writers.[7,8] In the photoformer circuit developed at Wisconsin,[9,10,11] care has been taken to obtain fast response by the use of a cathode ray tube with a P-11 phosphor and by use of the smallest possible time constants in the amplifier in the feedback loop.

Because of the high speed requirements the magnetic deflection type 5WP15 flat-face tube and the techniques described by Pedersen, *et al*,[12] cannot be used. It is hoped that a flat-face tube of the electrostatic deflection type can be used in the future in place of the 5LP11A.

The masks are made by using a good grade of hard cardboard on which is pasted the graph paper with the graph to be used. The mask is cut approximately to shape with scissors and trimmed with a file.

In order to provide a kind of overall check on the precision obtainable in a problem using the photoformer, a nonlinear equation for fluid flow on a rotating disc was solved.[13] The equation, put into a form suitable for differential analyzer solution, is

$$\frac{1}{2}\frac{ds}{dx} + As^{3/2} - Bx = 0 \qquad (20)$$

where x is the radius of the disk and s the square root of the radial velocity. Figure 7 shows the computer block diagram used. The function generator here was a photoformer with a 3/2-power mask.

In Figure 8 some cathode ray traces of the solution of Equation (20) are shown for various sets of constants. These results are replotted in Figure 9 for comparison with more precise (5-figure) IBM solutions. The output velocities in the various solutions shown ranged from zero to about 4000 feet per minute, and the maximum error in the differential analyzer solution at mid-range was approximately 75 out of 3000 feet per minute, or about 2½ percent.

It may not always be possible to obtain results of a precision as satisfactory as this when the photoformer is used. In simple problems, however, this and other experiences[11] have led us to believe that an average precision of roughly one or two percent is possible even at 60 cycle repetition rates.

(c) Squarers and Multipliers

The Philbrick squaring device[14] uses the square-law current-voltage characteristic of 12AU7 triode sections operated at an experimentally selected grid potential. Both triode sections are used, one for positive and the other for negative inputs.

The device is limited in speed of operation mainly by the associated operational amplifiers. Their phase characteristics may be improved by some of the methods discussed above.

The square-law characteristic of the 12AU7 triode is also used in the Philbrick multiplier[14] which operates on the quarter-square principle (see diagram, Figure 10). This computer unit, like the squarer, is limited in speed by the associated operational amplifiers. Other errors in these devices may be expected to be caused chiefly by the slight deviations of the triodes from exact square-law characteristics.

4. TEST OF A HIGH-SPEED MULTIPLIER

In order to determine experimentally the overall precision possible with a multiplier of this type, one of the simplest differential equations, whose solution requires a multiplier, was examined:[15]

$$\frac{d^2y}{dt^2} + \epsilon (1 + k \cos t) y = 0 \quad (21)$$

This equation is Mathieu's equation and is one of considerable interest and importance.[16] A rather complete solution has been made using the ENIAC digital computer, and tabulated results are available for checking in this case.[17]

The differential analyzer set-up used for solution of the Mathieu equation is shown in Figure 11. Here the upper row of units forms the loop which solves the Mathieu equation. The quantity (1 + k cos t) was formed by a specially designed "cosine unit" which produced one cycle of this function and then gave zero output for several cycles. During this zero output period the square-wave output of the cosine unit, converted into a positive clamping potential as shown, was used to return the computing loop to zero. The square-wave output of the "cosine unit" was also used to apply initial conditions (I.C.) of either $y(0) = 0$, $\dot{y}(0) = 1$, or $y(0) = 1$, $\dot{y}(0) = 0$.

Photographs were used to record data from the cathode ray tube as in Figure 8, the results for $\epsilon = 1.0$, $k = 0.8$, and initial conditions $y(0) = 1$, $\dot{y}(0) = 0$ are plotted, together with ENIAC results, in Figure 12. The errors in the analog solution were less than two percent of the maximum value of y for all the runs made.

A stability chart obtained with this set-up is in Figure 13. This sort of data can be very quickly determined with the high-speed differential analyzer.

The Mathieu equation with an added damping term is

$$\frac{d^2y}{dt^2} + b\frac{dy}{dt} + \epsilon (1 + k \cos t) y = 0 \quad (22)$$

This equation may be converted into the ordinary undamped form,[17]

$$\frac{d^2x}{dt^2} + \epsilon' (1 + k' \cos t) x = 0 \quad (23)$$

where

$$\epsilon' = \epsilon - b^2/4, \quad k' = \epsilon k/(\epsilon - b^2/4)$$

The tabulated ENIAC results may be used to find x, and y can then be determined by use of

$$y = e^{-bt/2} x(t) \quad (24)$$

In the differential analyzer set-up the damped form of the Mathieu equation (Equation (22)) requires simply one added scale changer to supply by to the adder in the loop. Solutions for a few such cases were run off (Figure 14) and checked (rather laboriously) against ENIAC results.

Computer units with compensating capacitors (Figure 5) were not used in the solution of this problem nor in the problem discussed under 3(b), above. Some improvement in precision might be expected if these were used, and rough calculations in the case of the Mathieu equation show that the correction is in the right direction and of the right order of magnitude.

Recent studies have been made of the frequency modulated van der Pol oscillator in extension of the above work and of previous oscillator studies.[18] The equation examined on the computer was

$$(1 + b \cos \omega_L t)\, dy/dt + \epsilon (y^3/3 - y) + \int y\, dt = 0. \quad (25)$$

A sample solution for the waveform for ω_L equal to one-tenth of the oscillating frequency and $\epsilon = 0.1$, $b = 0.6$ is shown in Figure 15. Phase compensation by a method akin to that of Figure 4 was used.

The precision possible in solution of a problem such as this should be of the order of three or four percent.

5. TIME-SCALE ERRORS

It has been common practice to use a high-speed computer at a 60 cycle repetition rate, and to depend upon the 60 cycle power frequency and a linear sweep circuit to establish a time base. However, in many localities the power-line frequency, though it averages almost exactly 60 cycles, may vary from this by one percent for short periods. To avoid possible errors from this source and those from nonlinearities in sweep circuits, a circuit is now being developed which will give a precise 60-cycle or 30-cycle frequency derived from a temperature stabilized tuning fork operating at 2400 cycles. This original frequency can then be used to generate timing markers.

6. CONCLUSIONS

In the ordinary engineering problems of servo design, vibration study, and so forth the precision requirements are usually not too stringent because of the nature of the data. Here, the advantages in speed and convenience of electronic differential analyzer studies are well known. In some cases the high-speed computer offers certain additional advantages, particularly when other than initial conditions are considered. It appears that the ordinary requirements of engineering precision in servo, vibration, and similar studies can be met in the high-speed analog.

The experimental results described above indicate that with present high-speed techniques it is possible to solve certain differential equations which require the use of multipliers or function generators, with a precision of one or two percent. It might be expected that, as techniques develop, much more involved nonlinear equations may be solved with equal precision at 30- or 60-cycle repetition rates.

The rapidity with which these solutions can be obtained once a set-up is made suggests the possibility of a general graphical tabulation of the solutions of important equations. Thus, families of curves representing the solution of equations such as the Mathieu equation might be rapidly photographed and the results stored. These results could be used to search for areas of interest or to find approximate answers. The use of microfilm techniques would also be important.

ACKNOWLEDGMENT

The assistance of many staff members and graduate students at the University of Wisconsin in the work leading up to this paper is gratefully acknowledged. Also, the advice and encouragement of George A. Philbrick was most helpful.

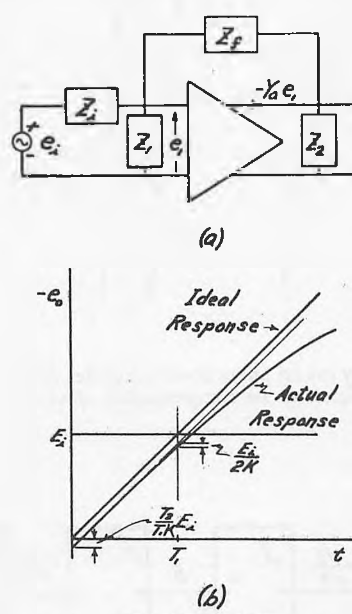

Figure 1 (a) Basic Operational Amplifier.
(b) Ideal and actual response of an operational integrator to an input step function.

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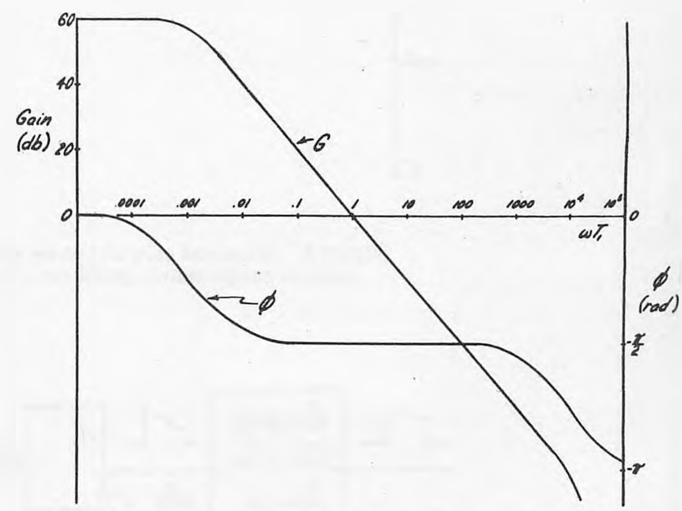

Figure 2. Approximate form of the gain and phase versus frequency for an integrator with $K_o = 1000$, $T_2 = 0$, $T_3 = T_1/10$.

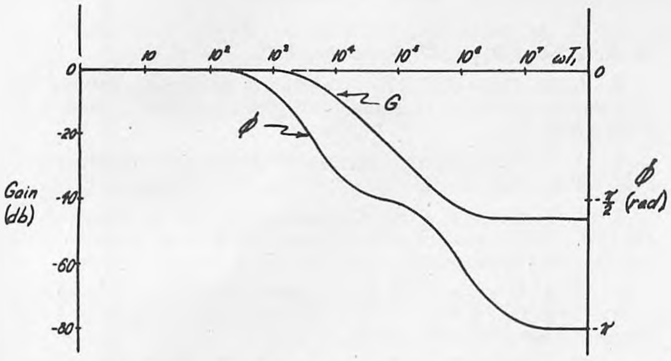

Figure 3. Approximate form of the gain and phase versus frequency for an adder or scale changer (unity gain) with $K_0 = 1000$, $T_a = T_1/10$, $T_2 = T_1/100$.

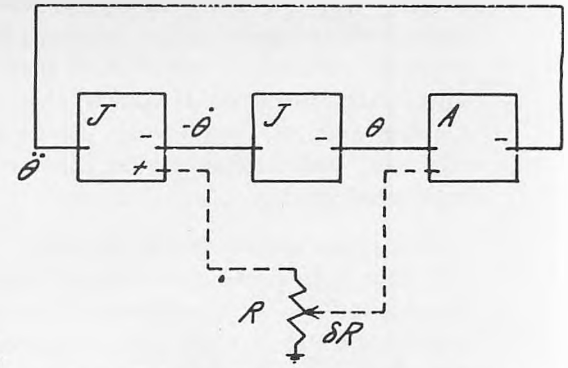

Figure 4. Computer set-up for the simple oscillator, $\ddot{\theta} + \theta = 0$, with damping added to give phase compensation. Here the J-units are integrators and the A-unit is an adder

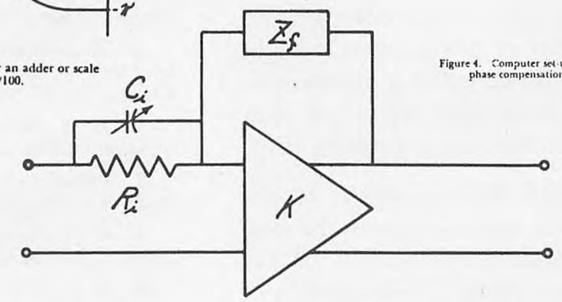

Figure 5. Elementary form of an operational amplifier with a phase compensating capacitor C_i.

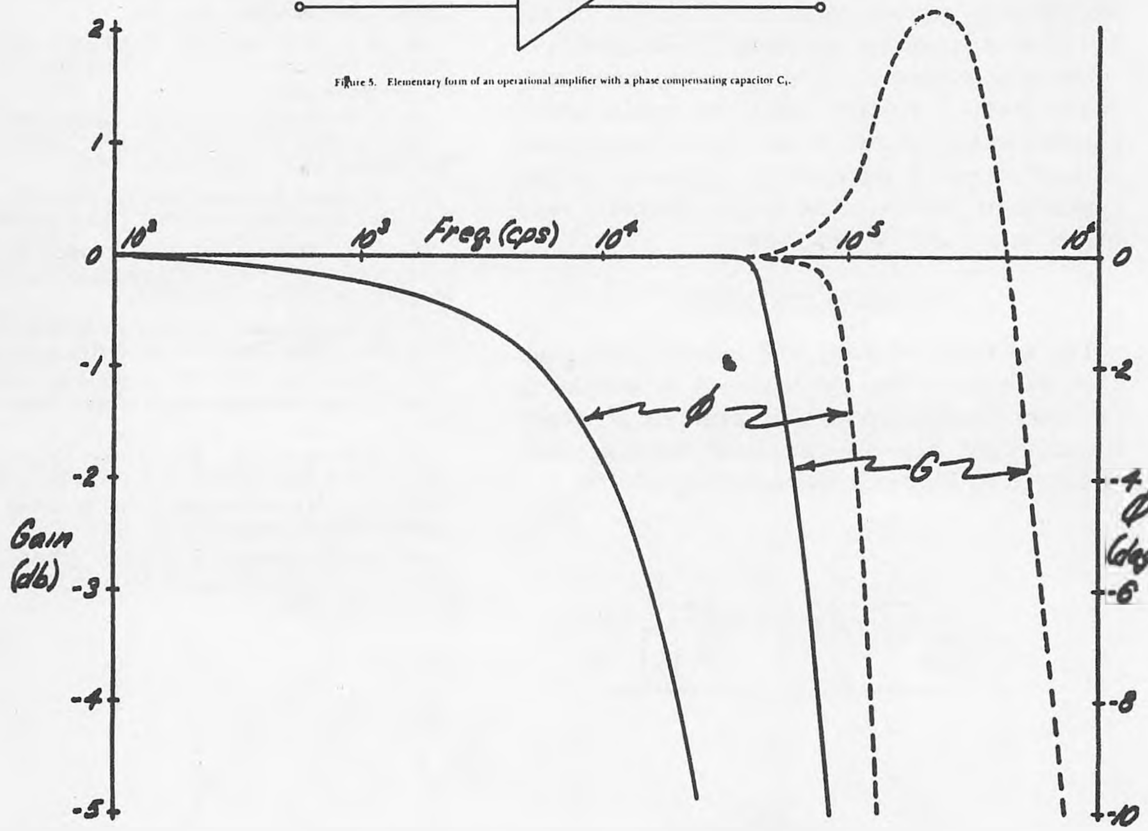

Figure 6. Measured gain and phase versus frequency for an operational amplifier of unity gain, without compensation (solid curve) and with capacitor phase compensation (dotted curve).

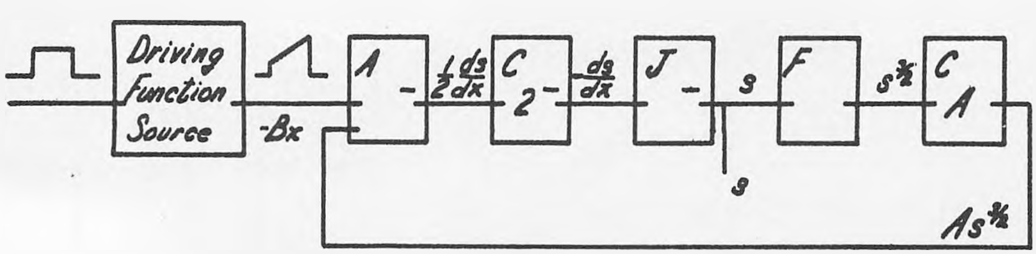

Figure 7. Computer block diagram used to solve a nonlinear equation (19). Here F is a photoformer with a 3/2 power mask, while the C-unit is a scale changer, the A-unit an adder, and the J-unit an integrator.

For Fig.8,please refer to Fig.6,Page 182

For Fig.9,please refer to Fig.7,Page 183

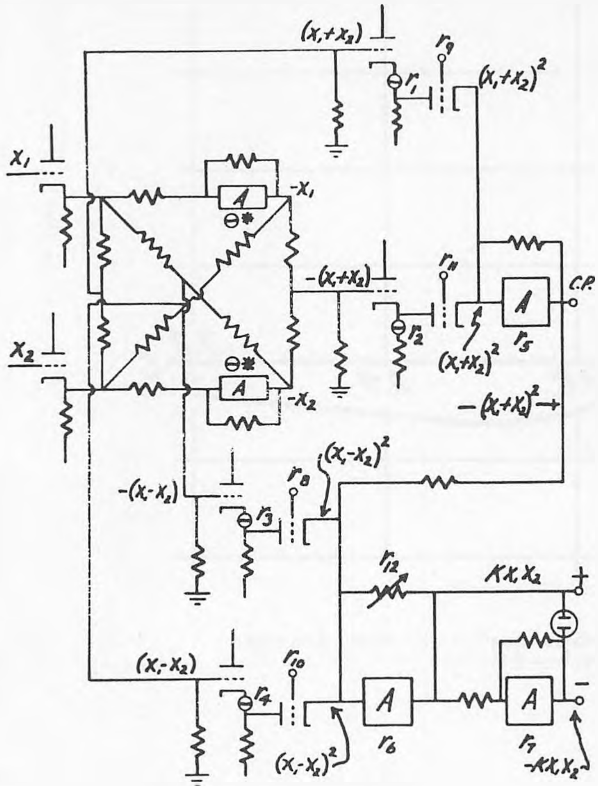

Figure 10. Circuit of a Philbrick type multiplier.

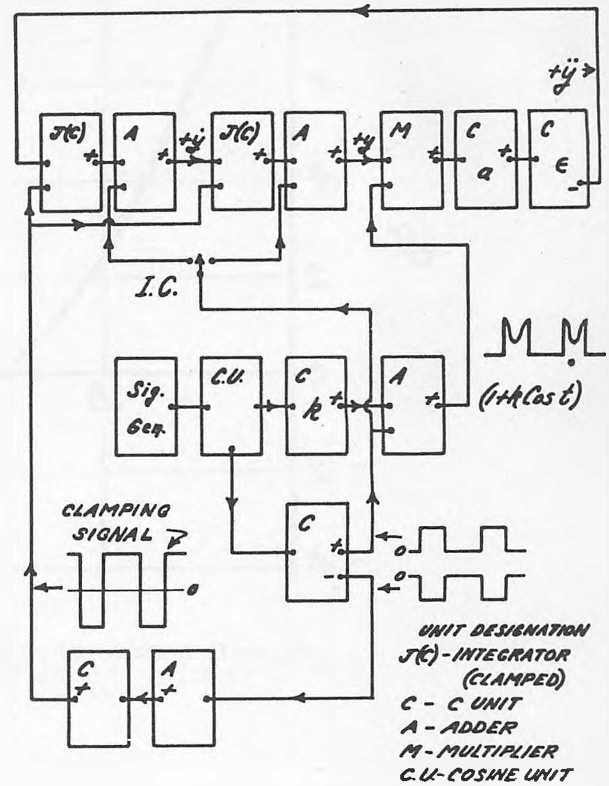

Figure 11. Block diagram of computer set-up for solution of Mathieu's equation.

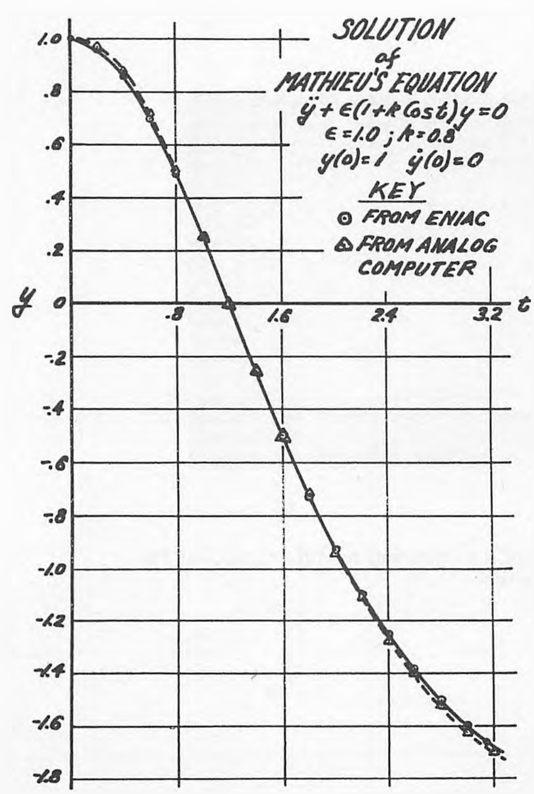

Figure 12. Some results (dotted curve) of a high-speed solution of Mathieu's equation.

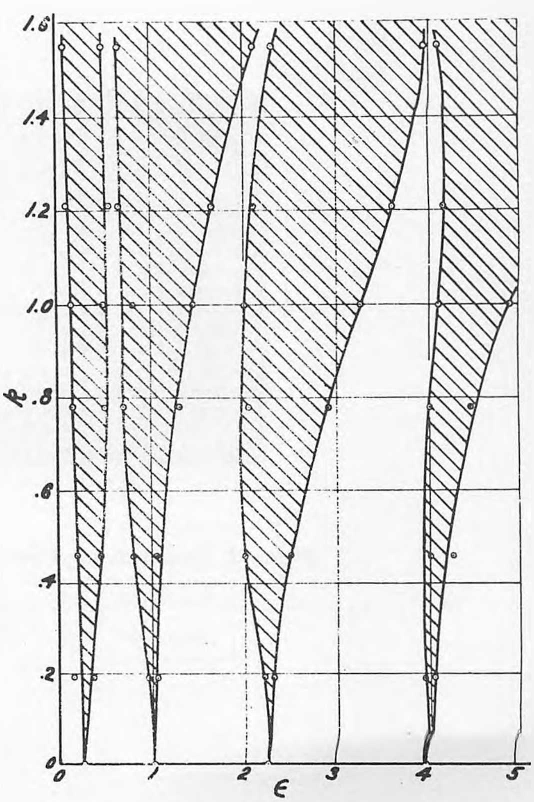

Figure 13. A stability chart for the Mathieu equation. The unstable regions are shaded.

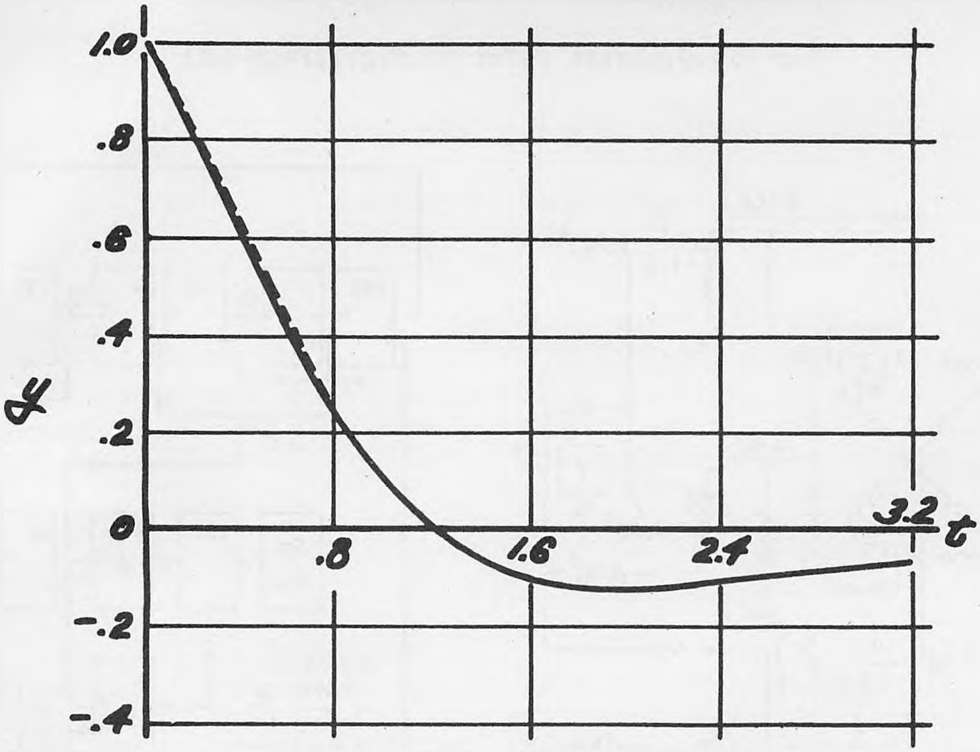

Figure 14. A sample solution of the Mathieu equation with added damping ($\epsilon = 2.0$, $k = 0.4$, $b = 2.0$).

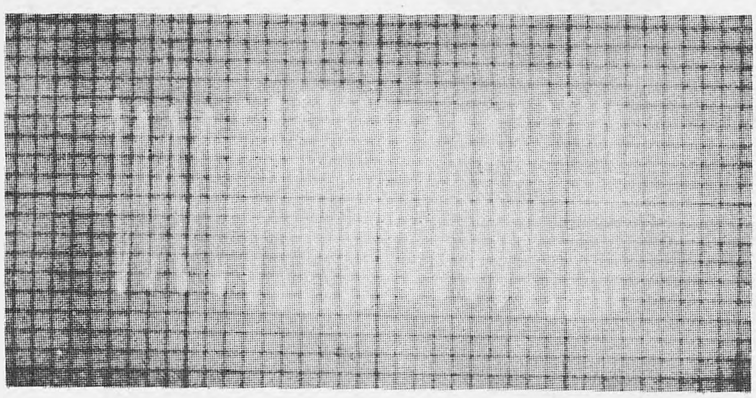

Figure 15. An oscilloscope record of a solution of the frequency-modulated van der Pol oscillator equation.

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ANALOG COMPUTER SOLUTION OF A NONLINEAR DIFFERENTIAL EQUATION

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INTRODUCTION

In a study made by the Chemical Engineering Department of the University of Wisconsin of fluid flow on a rotating disk designed to be used in spray drying, a differential equation arose which could not be solved by ordinary analytical methods[1,2]. This equation has the form

$$V \frac{dV}{dr} + AV^3 - Br = 0 \tag{1}$$

Here V is radial velocity and r is the radial distance from the center of the disk. A and B are constants which in the case of laminar flow depend upon the angular velocity of the disk, volumetric feed rate, vane height, and the density and viscosity of the liquid. If turbulent flow was assumed the same equation was obtained, with a different value for the constant A.

This equation was solved on the IBM 602A calculating punch card machine by Dr. A. M. Mark of the University of Wisconsin Computing Service. The Runge-Kutta method of approximation was used, and final results were accurate to five places. The difficulties encountered in setting up the equation and the fact that only engineering accuracy of a few percent was needed led to an examination of the possibilities of analog computer solution. It was decided to solve the equation on a high-speed Philbrick type of analog computer and to compare set-up times and precision with the results of the digital computation by punched card machine[3]. In this type of analog computer the solutions are repeated, usually at a 60 cycle rate, and may be viewed as traces on a cathode ray oscilloscope[4,5].

PREPARATION OF THE EQUATION FOR ANALOG COMPUTER SOLUTION

The constants A and B in (1) had mean values of $A = 1.66 \times 10^{-4}$ and $B = 3.6 \times 10^{7}$ for the disks studied if ft-lb-min units were used. The first step in the preparation of an equation of this sort for analog computer solution if to select new variables to give constants which lie between 0.1 and 10 so that voltages of widely different values will not be encountered in the computer during solution. In order to do this let $V = Cy$ and $r = Dx$. If these are substituted in (1) we get,

$$y \frac{dy}{dx} + ACDy^3 + \frac{BD^2}{C^2} x = 0 \tag{2}$$

Each of the new constants $A' = ACD$ and $B' = BD^2/C^2$ can be set equal to unity for the mean values given above for the original constants A and B. *If this is done* and C and D are solved for we get

$$\begin{aligned} C &= A^{-\frac{1}{2}} B^{\frac{1}{4}} = 6 \times 10^3 \\ D &= A^{-\frac{1}{2}} B^{-\frac{1}{4}} = 1 \end{aligned} \tag{3}$$

The equation now has the form

$$y \frac{dy}{dx} + A'y^3 - B'x = 0 \qquad (4)$$

The new constants A' and B' have values which are close to unity, $x = r/D$ is still the radius in feet and $y = V/C$ is now the velocity in feet per hundredth of a second.

Equation (4) is still in a somewhat undesirable form for analog computer solution because one or two multiplications are required, and multiplication is an operation which is difficult in a high-speed analog computer. This difficulty can often be overcome by a change in the functional form of one or more of the variables; in this case if $y^2 = s$ then (4) becomes

$$\frac{1}{2} \frac{ds}{dx} + A's^{3/2} - B'x = 0 \qquad (5)$$

The equation can now be solved by means of a single function generator together with an adder, an integrator and some scale changers, and no multiplier is required.

COMPUTER SET-UP

(1) Linear Components

A block diagram of the computer set-up for solution of (5) is shown in Fig. 1. The units labelled A, J and C are standard Philbrick linear computer units using operational amplifiers[6]. The adder (A), which will accept up to four inputs has unity gain. The integrator (J) has a time constant of 1/2400 sec, so that if time is measured in seconds its output is $e_{out} = 2400 \int_0^t e_{in} dt$. The constant is unity if computer time is measured in units equal to this time constant. The scale changer (C) units are simply variable gain feedback amplifiers which may be set to multiply by any desired constant. This constant should be between 0.1 and 10 if possible. All units have positive and negative outputs and will handle a $\pm$ 50 volt signal.

(2) Function Generator

The function generator[7,8] or photoformer[9] circuit is shown in block diagram form in Fig. 2. This device incorporates a cathode-ray tube, a phototube and a d-c amplifier in a feedback network which serves to hold the luminous spot on the C-R tube on the edge of an opaque mask — in this case cut in the form of a three-halves power curve. This unit was one of several built at the University of Wisconsin. The scale changers incorporating the constants k_1 and k_2 on either side of the function generator (F) in Fig. 1 were used to change scale at its input and output so that the CRT spot deflections in any given solution were between 3/4" and 1¼", for greatest accuracy. Various opaque masks of the 3/2 power relationship also had to be used.

(3) Time Scale Adjustment

In a computer of this type the independent variable in the computer must be time, in this case in units of α seconds where these units, determined by the integrator time constant are given by the equation 1 sec = 2400 αsec. There must be some linear relationship established between x, the independent variable in the problem, and the time τ in αsec in the computer. In solution of a linear problem x may equal τ

multiplied by any convenient constant which allows the solution to be completed within one computer cycle. In the solution of this nonlinear equation the independent variable must also be fed in as a linearly varying voltage, and it was convenient in this problem to measure x in volts. The relation between x (in volts) and τ (in αsec.), $x = K\tau$ included a constant K, so that the integrator equation $e_{out} = \int_0^\tau e_{in}(\tau)\, d\tau$ became

$$e_{out} = \int_0^x \frac{e_{in}(x)\, dx}{K} \qquad (6)$$

This required a multiplication by K before or after the integrator. In Fig. 1 this is accomplished by setting the first C-unit at 2K rather than 2.

(4) Generation of the Independent Variable

Since x (measured in feet in the original problem, and in volts in the computer) was to vary linearly with time, and since it had an initial value corresponding to the radius at which the liquid was fed to the rotating disk, a waveform as shown in Fig.3 had to be generated. It was desirable that x be linear with τ for one-quarter of the period of a 60 cycle wave (or 10 αsec.) and that the initial value and slope be variable. It was further necessary to provide another waveform of opposite polarity to follow the original one of Fig. 3 so that the voltages in the analog set-up could be returned to zero between successive solutions, 1/60 sec. apart. The block diagram of a device for accomplishing this is shown in Fig. 4. Very good linearity in the saw-tooth generators of Fig. 4 was achieved by use of the pentode-capacitor-charging circuit of Fig. 5.

RESULTS

The results of the analog computer solution were obtained as photographs of oscillograph traces as shown for two cases in Fig. 6. Some curves showing fluid velocity V versus disk radius x obtained from this data are shown as dotted curves in Fig. 7. The more accurate (5 figure) IBM solutions are shown as solid lines for comparison. The percent error in the analog solution is not easy to specify, but at x = 0.3 ft it has a maximum value of 5%, and is in general adequate for engineering purposes in the case of this particular problem.

When the solution was first attempted on the analog computer some weeks were spent becoming acquainted with the equipment, and in designing and building the circuits of Fig. 4, which now form a permanent part of the computer. After this was completed only five hours were needed to set up and obtain the four solutions shown in Fig. 6. Solution was equally fast by IBM digital methods, but set-up time was much greater, about 100 hours being required for wiring of the schedule board.

CONCLUSION

It appears from this study that this type of analog computation, though chiefly used for solution of linear differential equations, has possibilities for use in the solution of certain non-linear ordinary differential equations where accuracy of a few percent is adequate. The high speed feature of the computer used has the advantage that end-point conditions[5] can be required in a solution; thus for example, in this problem the initial radius can be varied by an amount which is to be determined by an allowable percentage change in final velocity without making a number of tedious runs.

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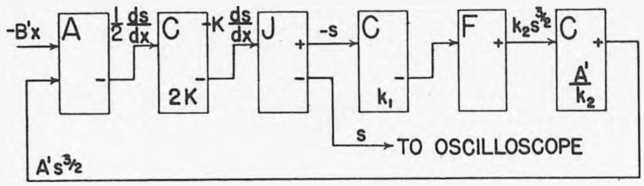

Figure 1. Computer block diagram for solution of the equation $\frac{1}{2}\frac{ds}{dx} + A's^{3/2} - B'x = 0$

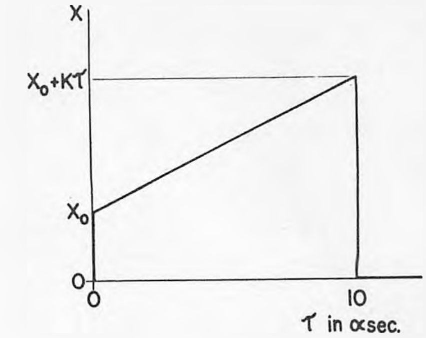

Figure 3. Waveform required to supply the independent variable in the form of a voltage.

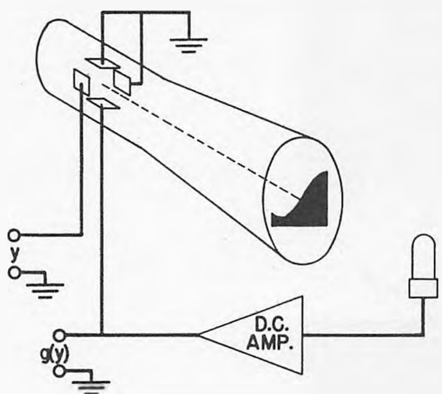

Figure 2. Block diagram of the function generator.

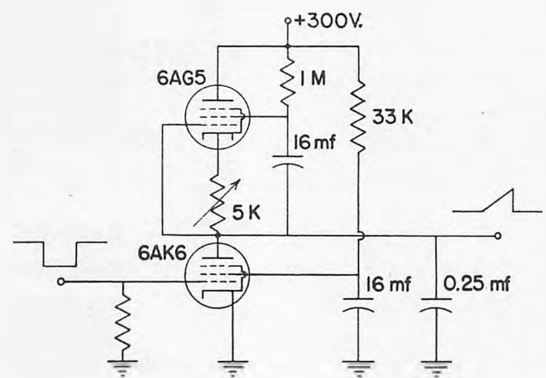

Figure 5. Saw-tooth generator using a pentode charging circuit.

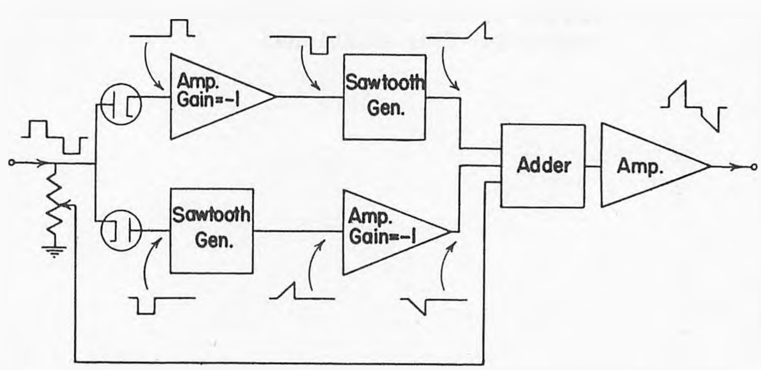

Figure 4. Block diagram of the device used to supply the waveform of Figure 3.

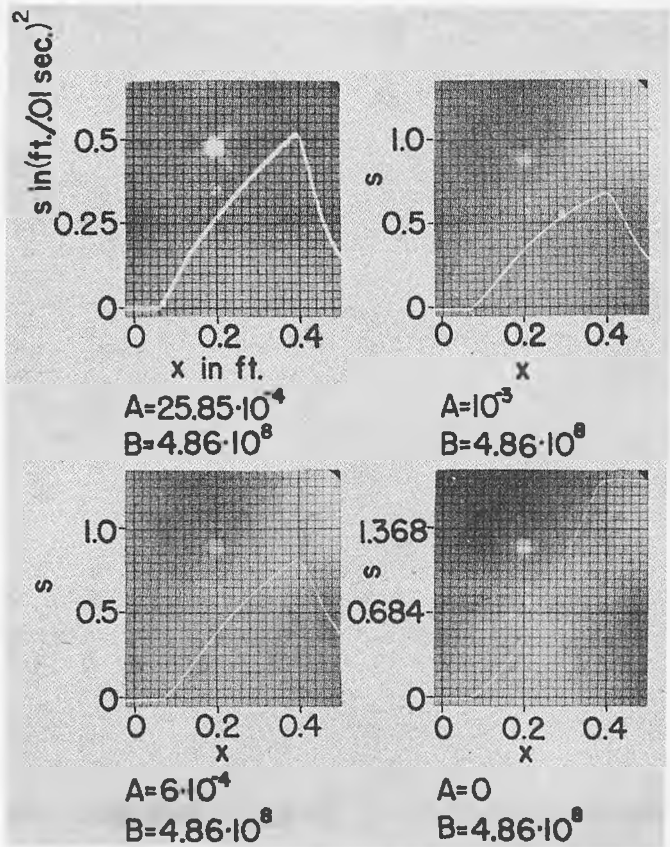

Figure 6. Photographs of the oscilloscope traces for four solutions.

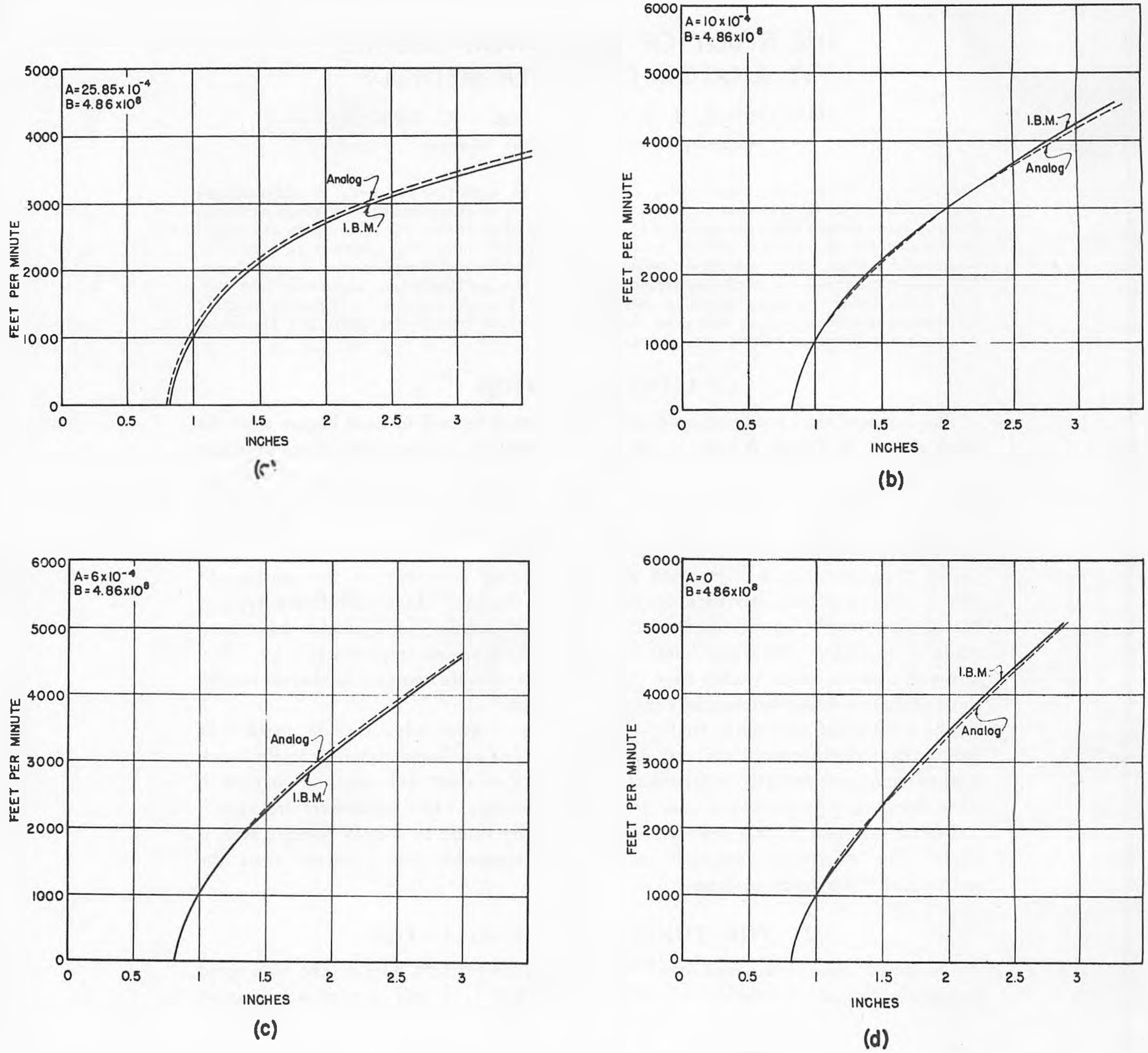

Figure 7. Results obtained from analog solution compared with IBM digital computer results.

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THE STUDY OF OSCILLATOR CIRCUITS BY ANALOG COMPUTER METHODS

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Abstract—A high speed electronic analog computer can be applied to the study of self-oscillating circuits. The use of the computer is considered to be superior to experimenting with actual oscillators in many cases, for the following reasons: (1) It is possible to conveniently produce nearly any desired g_m characteristic by using an arbitrary function generator. This means that optimum g_m curves for a particular oscillator circuit can be determined without resorting to the construction of special tubes. (2) Theoretical analysis of oscillators frequently requires that an idealized g_m characteristic be used. Since either idealized or actual g_m curves may be synthesized on the computer, it is possible to check the idealized theoretical analysis with either the idealized or actual experimental results. (3) The circuit parameters can be varied quickly and conveniently.

I. INTRODUCTION

The mathematical study of oscillator circuits may be said to have begun with the work of van der Pol.[1,2] A number of analytical methods giving approximate solutions for such circuits have been advanced; they are, in general, quite tedious, even though power series approximations are used. To these was added the more powerful method involving the use of the mechanical differential analyzer by Brainerd and Weygandt,[3] and Herr,[4] for the solution of van der Pol's equation.

The application of a high-speed electronic analog computer to the solution of van der Pol's equation has been demonstrated by Macnee.[5] In the Philbrick type of analog computer[6,7] in use at the University of Wisconsin, some studies have been made of oscillators using non-linear functions which are not representable by a few terms of a power series. Studies have also been made of more complex oscillators having third or fourth order differential equations.

The high-speed electronic analog computer has a great advantage in speed over the mechanical differential analyzer. In the solution of oscillator circuits, the Philbrick type of computer actually oscillates at a frequency of about 400 cps, and changes in wave form may be seen on an oscilloscope immediately when parameters are varied. Values of parameters for a desired solution are easily found by merely turning several knobs. The electronic computer is, however, somewhat less accurate than the mechanical differential analyzer.

II. THE TUNED-PLATE OSCILLATOR

The simple tuned-plate oscillator[1,2,3,4] was selected for first tests of the high-speed computer method of solution. In the circuit of Fig. 1, if grid current is neglected,

$$(1) \qquad (p_1C + \frac{1}{R} + \frac{1}{p_1L})e_p = - i_p$$

In (1) and all equations to follow, $p_1 = d/dt$, e_b, e_c and i_b are instantaneous plate voltage, grid voltage and plate current, respectively, and e_p, e_g and i_p are the variational components of e_b, e_c and i_b.

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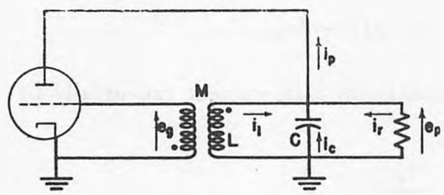


Fig. 1—The simple tuned-plate oscillator.

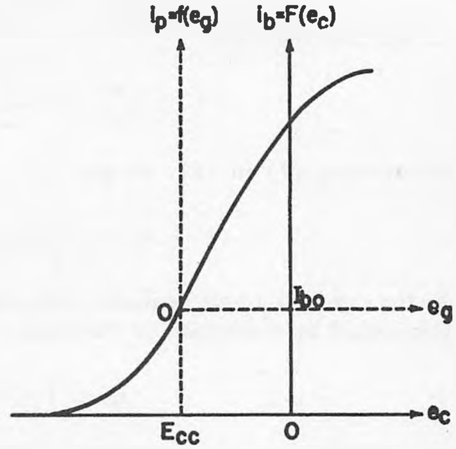


Fig. 2—The transfer characteristic for a pentode.

If the oscillator tube is a pentode, i_b may be considered to be independent of e_b, and we may write

(2) $$i_b = F(e_c)$$

If, in Fig. 2, the origin is taken at the point of operation 0′ on the curve of this function, variational coordinates e_g and i_p may be introduced, and (2) becomes

(3) $$i_p = f(e_g)$$

For the circuit of Fig. 1,

(4) $$e_g = - Mp_1 i_L = - ke_p,$$

where $k = M/L$. Combining (1), (3), and (4), and multiplying through by $\sqrt{L/C}$, we get

(5) $$(p_1\sqrt{LC} + \frac{1}{R}\sqrt{\frac{L}{C}} + \frac{1}{p_1\sqrt{LC}})e_p = -\sqrt{\frac{L}{C}}\, f(-ke_p$$

Define

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$\delta_0 = \frac{1}{R}\sqrt{\frac{L}{C}} = \frac{\omega_0 L}{R}$$

$$\rho = \sqrt{\frac{L}{C}}$$

Then

(6) $$(\frac{p_1}{\omega_0} + \delta_0 + \frac{\omega_0}{p_1})e_p = -\rho f(-ke_p)$$

Equation (6) may be normalized by changing to a different time scale in which $\tau = \omega_0 t$ and therefore,

$$p = \frac{d}{d\tau} = \frac{1}{\omega_0}\frac{d}{dt} = \frac{p_1}{\omega_0}. \tag{7}$$

Substituting (7) in (6), we get

$$(p + \delta_0 + \frac{1}{p})e_p = -\rho f(-ke_p) \tag{8}$$

In the case of a triode oscillator tube, the variational plate current can usually be approximated by a function of the form

$$i_p = f(e_g + \frac{e_p}{\mu}) \tag{9a}$$

$$= f(-\frac{M}{L}e_p + \frac{e_p}{\mu}), \tag{9b}$$

where μ is a constant. If we define $M/L - 1/\mu = k$, the triode oscillator equation reduces to (8). In fact, since an equation of the form of (8) also describes the tuned-grid oscillator, the dynatron oscillator[2], and certain mechanical vibrating systems[8], we can let $e_p = x$ and arrive at a general second-order oscillator equation,

$$(p + \delta_0 + \frac{1}{p})x = -\rho f(-kx) \tag{10}$$

In the solution of (10) by means of a high-speed analog computer,[5,6,7] integrators, scale changers, and an adder are required, as well as a function generator to form the experimentally determined function (3) or (9). The Philbrick integrators used in this study have a time constant of 1/2400 second, so that their response to an input $e_1(t)$ is

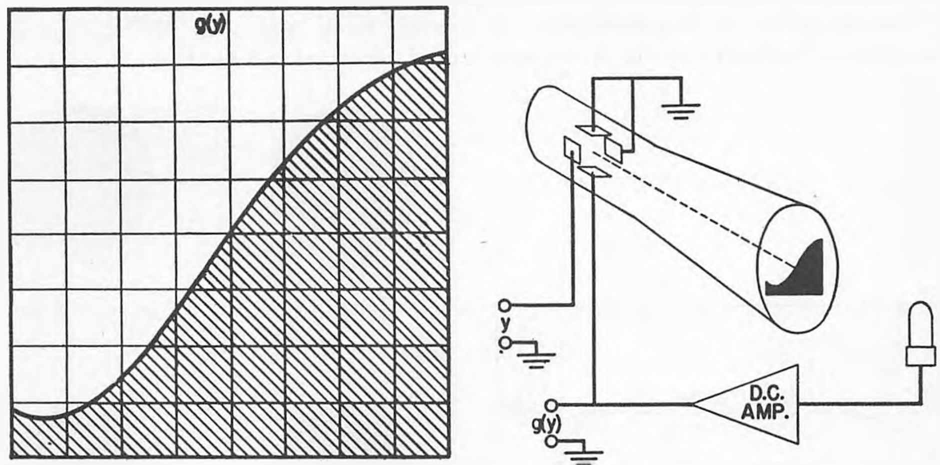
$$e_2(t) = 2400\int_0^t e_1(t)dt. \tag{11}$$

If the units of τ in the analog system are chosen so that one unit of τ equals 1/2400 sec., the integrator equation may be expressed as

$$e_2(\tau) = \int_0^\tau e_1(\tau)d\tau. \tag{12}$$

This choice for τ also means that when the normalized oscillator frequency ω/ω_0 is one radian per second, the actual computer frequency is one radian per unit of τ, or 2400 radians (383 cycles) per second. The computer frequency must therefore be multiplied by $\omega_0/2400$ to give the corresponding actual oscillator frequency.

The non-linear tube characteristic may be approximated by a diode limiter circuit, biased to represent the tube cut-off characteristic. An additional diode limiter may be used to approximate plate current saturation. However, a much more accurate and flexible method of forming the non-linear function is by means of the photo-electric



(a) (b)

Fig. 3—(a) Mask representing g(y). (b) Photoelectric arbitrary function generator giving an output g(y) in response to an input y.

type of function generator[9,10,11,12]. In this device, a mask is used as shown in Fig. 3, and an input voltage y is used to form an output voltage $g(y)$, where the function $g(y)$ is determined by the curve of the mask. In the function generator constructed at the University of Wisconsin, the output voltage equals the input voltage if a 45 degree straight-line mask $[g(y) = y]$ is used. However, in order to achieve optimum accuracy with an arbitrary function, it may be necessary to expand or compress the ordinate and abscissa. Thus to get a function $f(x)$ of the variable x, an input $y = k_1x$ may be used with a mask expressing some function $g(y)$, such that an output

$$k_2 f(x) = g(y) = g(k_1 x)$$

is obtained. If, as in (10), it is desired to form $-\rho f(-kx)$ where only x is available as an input, it is necessary to precede the function generator by a scale changer set at $-kk_1$, and to follow it by another scale changer set at $-\rho/k_2$. Thus (10) becomes

$$(p + \delta_0 + \frac{1}{p})x = -\rho' g(-k'x), \tag{13a}$$

where $k' = kk_1$ and $\rho' = \rho/k_2$.

Equation (13a) may be rewritten in the following manner:

$$px = -\delta_0 x - \frac{x}{p} - \rho' g(-k'x). \tag{13b}$$

For practical oscillator circuits, the constants δ_0, k', and ρ' will usually lie between 0.1 and 10.

The computer solution of (13) is formed, as shown in Fig. 4, by generating the terms on the right side of (13b) from px, and then adding them to form px. The closed loop constrains the computer to follow (13).

Voltages corresponding to x, px, and x/p are available for display on an oscilloscope. These quantities are proportional to the currents i_R, i_C, and i_L in Fig. 1. Typical waveforms are shown in Fig. 5 for four sets of constants and mask functions.

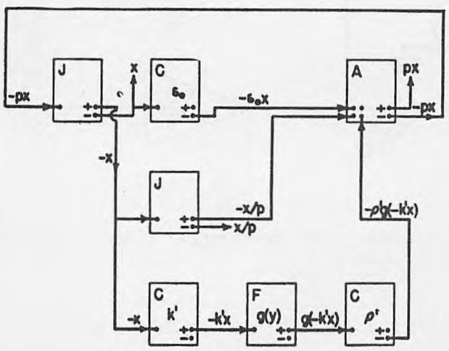


Fig. 4—Analog computer set-up for solution of the simple oscillator equation. J represents an integrator. A an adder, F an arbitrary function generator, and C represents a scale changer, which multiplies by a constant.

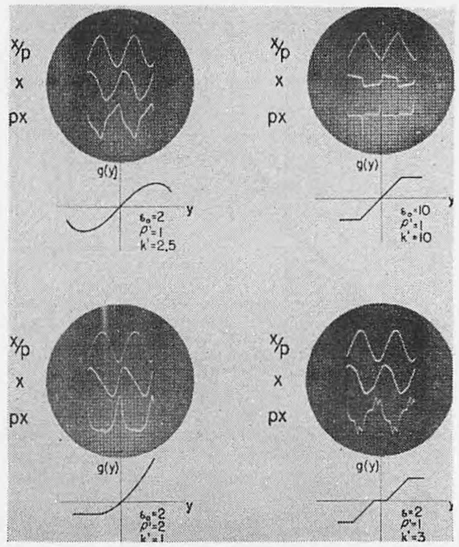
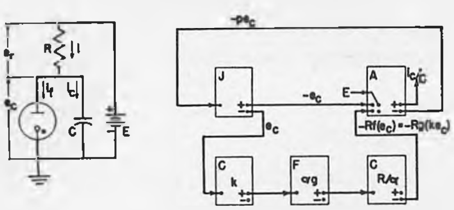


Fig. 5—Typical waveforms for four different non-linear functions in the simple oscillator.



(a) (b)

Fig. 6—(a) Glow-tube relaxation oscillator. (b) Computer set-up for the glow-tube oscillator equation.

III. A GLOW-TUBE OSCILLATOR

The simple glow-tube oscillator of Fig. 6(a) provides an interesting example for computer solution. In particular, it illustrates the versatility of the type of arbitrary function generator used, because of the double-valued nature of the glow-tube characteristic,

$$i_f = f(e_c) \tag{14}$$

The operation of this circuit is determined by the differential equation,

$$RCp_1\, e_c + e_c + Ri_f = E \tag{15}$$

By a change of time scale, (15) may be normalized to

$$pe_c + e_c + Ri_f = E \tag{16}$$

Equation (16) may be solved by using the computer set-up of Fig. 6(b). The arbitrary function generator is required to form the double-valued function (14), which has the form shown in Fig. 7(a)[8]. This can be done by use of a mask of the form shown in Fig. 7(b)[13]. As in the preceding example, it may be necessary to make the mask such that $ag(ky)$ is delivered rather than $f(y)$, where a and k are scale expansion factors. The function generator must be preceded and followed by scale changers to introduce the circuit parameter R and to compensate for the expansion or compression of the mask scales.

A cathode-ray tube display of the voltage corresponding to e_c results in the familiar exponential saw-tooth waveform.

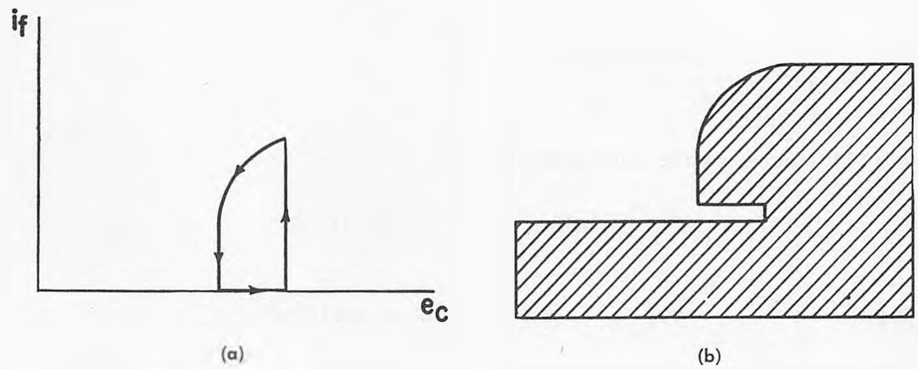
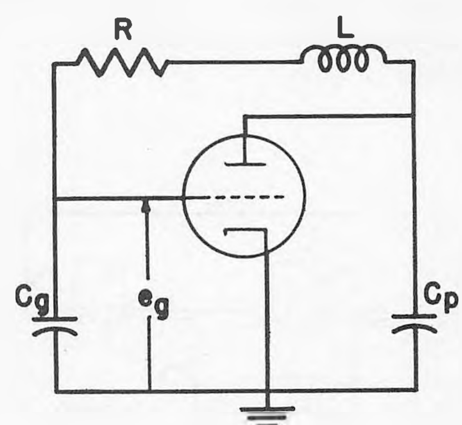


Fig. 7—(a) Typical glow-tube characteristic under oscillatory conditions. (b) Mask for generating the double-valued glow-tube characteristic.

Fig. 8—The Colpitts oscillator.



IV. THE COLPITTS OSCILLATOR

The Colpitts oscillator, Fig. 8, is one of the simplest circuits which gives rise to a third-order non-linear differential equation. Assuming a pentode tube and neglecting grid current, the equation for this circuit is

$$(17)\quad \frac{LC_gC_p}{C_p + C_g}\, p_1^3 e_g + \frac{RC_gC_p}{C_g + C_p}\, p_1^2 e_g + p_1 e_g = -\frac{i_p}{C_p + C_g}$$

where $p_1 = d/dt$, the current i_p is the tube plate current defined by (3), and the parameters are as shown in Fig. 8.

If we let

$$
\begin{aligned}
\omega_0{}^2 &= \frac{C_p + C_g}{LC_pC_g} \\
\delta_0 &= \frac{R}{\omega_0 L} \\
\rho &= \frac{1}{\omega_0 (C_p + C_g)} \\
\mathrm{p} &= \frac{d}{d\tau} = \frac{p_1}{\omega_0},
\end{aligned}
\tag{18}
$$

then (17) reduces to the normalized form

$$p^3e_g + \delta_0 p^2 e_g + pe_g + \rho f(e_g) = 0 \tag{19a}$$

or

$$p^3 e_g = -\delta_0 p^2{}_g - pe_g - \rho' g(k'e_g) \tag{19b}$$

The computer set-up for the solution of (19b) is shown in Fig. 9. Although an analytical solution of the Colpitts circuit equation is much more difficult than that of the simple tuned-plate oscillator, the analog computer set-ups differ only slightly.

Figure 10 shows a set of waveforms and phase-plane diagrams for a particular non-linearity and typical constants.

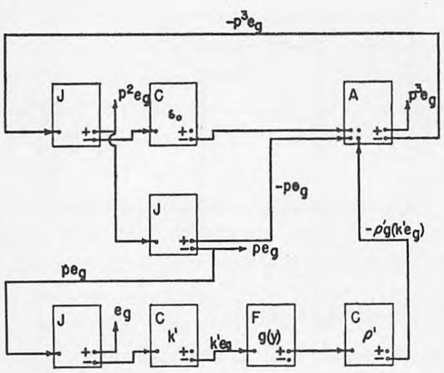


Fig. 9—The Colpitts analog set-up.

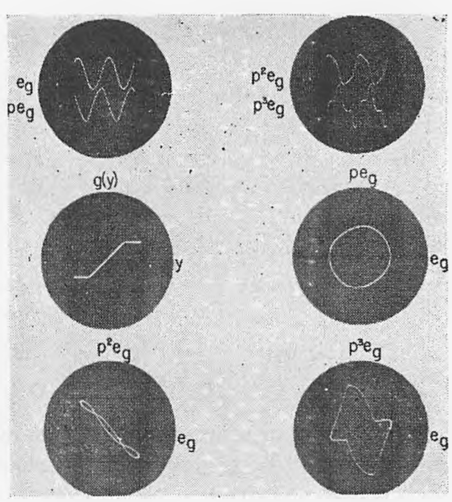


Fig. 10—Typical waveforms, non-linear characteristic $g(y)$, and phase-plane diagrams for the Colpitts oscillator, with $\delta_0 = 2$, $\rho' = 8$, and $k' = 1$.

V. OSCILLATORS WITH FOURTH-ORDER DIFFERENTIAL EQUATIONS

The high-speed analog computer has also been applied to the study of reactance-tube oscillators[14]. In general, the technique is the same as in the preceding examples. It is possible to describe these circuits either by two simultaneous second-order differential equations, one of which is non-linear, or by a single fourth-order non-linear differential equation. The computer set-ups for these two possibilities, although mathematically identical, are quite different physically.

The computer has been used to show that periodic solutions exist for certain reasonable values of circuit parameters, and further studies of the fourth-order equation are in progress.

VI. CONCLUSION

The examples given illustrate the general technique for high-speed analog computer solution of oscillator problems. In each case, the arbitrary function generator is used for the important non-linear function. If necessary, more than one such generator could be used; for example, a second function generator could be used to represent the grid current characteristic of a Class C oscillator.

If the oscillator tube plate current cannot be approximated by a function of a linear combination of e_g and e_p such as (9), an electronic function generator capable of forming a general function $f(e_g, e_p)$ is required. Such a function generator is not available at present. However, in some cases the function of two variables may be expressible in a form such as $ke_pf(e_g)$, and may therefore be obtained by the use of the present function generator and a multiplier.

In the examples given, only the steady-state solutions of the oscillator equations have been considered. We have also examined the build-up of oscillations by inserting an electronic switch in the computer set-up in such a way as to render the computer solution alternately oscillatory and non-oscillatory.

The analog computer method of solution is well adapted to the study of lock-in or synchronization phenomena in oscillators. It has been observed that, for a non-linear function with odd symmetry, synchronization occurs for synchronizing voltages which are odd multiples of the oscillator frequency. If the non-linear function does not have odd symmetry we have found that synchronization is also possible for even multiples, as suggested by Herr[4].

An analog computer of the type used in these investigations might more properly be called an electronic differential analyzer, since, like the mechanical differential analyzer[15], it permits the direct solution of the differential equations of the system being studied. We may say that a "differential equation analog" has been used rather than an electric circuit analog. In some cases, such as the phase-shift oscillator, in which the differential equations are complex and no inductors are required, the electric circuit analog is preferable. In most oscillator circuits, the use of the differential equation analog gives the advantage that no large, high-Q, calibrated inductors are required. Also, all variables are directly available as voltages for convenient display on an oscilloscope.

ACKNOWLEDGMENTS

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A Differential-Analyzer Study of Certain Nonlinearly Damped Servomechanisms

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A SERVOMECHANISM must hold its error or the difference between input and output, to as small a value as possible. If a large error appears as a result of a rapid change in the value of an input position signal, then high acceleration and high output speed are desirable, and these in turn are favored by a low damping ratio. When the error is near zero, the speed should be reduced rapidly so that overshoot does not occur, and this requires a high damping ratio. Thus the use of nonlinear damping which has some inverse relationship to the error has been proposed by several investigators.[1-5]

Another approach to the design of a nonlinear servomechanism involves a consideration of the practical limit of the available torque of the servo motor. If some maximum torque T_m is available, then the best servo performance in response to an input step will be obtained if T_m is first applied in such a direction as to accelerate the load and reduce the error, and is then reversed so that $-T_m$ is applied to reduce output speed to zero. Such a servo is called a relay servo and for optimum response it requires a sensing and switching device.[3,6,7]

The nonlinear damping approach has certain advantages, such as the fact that the ordinary stability criteria may be applied for small errors, but it cannot ordinarily be expected to give as rapid a response to an input step as the relay or maximum-torque servo. However, the relay servo has difficulty with other than step-function inputs, and is not too satisfactory in its operation when errors are small.

In the course of investigating a number of nonlinearly damped servos with the aid of a high-speed differential analyzer[8] it was discovered that one of these nonlinearly damped servos tended to operate at very nearly maximum torque except when errors were small. Thus this servo combines the best feature of the relay servo and of the nolinearly damped servos. It was first built in simple form and tested for response to an input step by J. B. Lewis.[4] The differential analyzer study made it possible to compare the Lewis servo with others, and to extend the study to other than step inputs, and to the case where the servo contained more than one lag. As a result of this work some major modifications of this servo are suggested which give significant improvement in its operation.

Response Criteria

A number of criteria for comparison of servo responses have been suggested.[9,10,11] In this work the integral

$$I=\int|e|dt \tag{1}$$

was used, where $|e|$ is the absolute value of the error. For a step-function input applied to the simple linear second-order servo this integral is minimized by a damping ratio of $\zeta=0.65$, giving an error overshoot of 7 per cent.

The choice of this criterion was based on a compromise between computer adaptability and suitability of the measure obtained. Thus use of the absolute value of error rather than error squared reduces the unwanted emphasis on the large initial error, which is unavoidable with step-function inputs. The per cent of overshoot obtained *with this* criterion is within the range generally accepted as suitable for a large class of servo applications. The use of time weighting, although desirable from some points of view, gives rise to difficulties in computer setups which are not in proportion to the advantages gained.

Unfortunately the absolute-value-error integral is not easily handled analytically. However, an analytic solution can be obtained for the second-order linear system. For a step-function input of amplitude P and the underdamped case, the absolute value integral is

$$I=\frac{p}{\omega_0}\left[\frac{2\epsilon^{\sigma\theta}}{1-\epsilon^{-\sigma\pi}}+\sigma\sin\theta+\cos\theta-2\epsilon^{\sigma\theta}\right] \tag{2}$$

where

$\sigma=\zeta(1-\zeta^2)^{-1/2}$
$\tan\theta=(1-\zeta^2)^{1/2}\zeta^{-1}$

The computer setup used to determine this integral is shown in Figure 1. This setup was used with various servos tested with different inputs to give a numerical basis for comparison.

Computer Solution of Linear and Nonlinear Servomechanisms

Figure 2 shows the basic servo used in this study.[3] If the nonlinear device is inactivated and the amplifier has no lags, we have a simple second-order linear servo in which

$$\left[p^2+\left(\frac{B+N}{I}\right)p+\frac{K}{I}\right]c=\frac{K}{I}r \tag{3}$$

where

I = the inertia of motor plus load
B = the viscous friction damping coefficient
$N=k_tk_1k_2$ is the tachometer damping coefficient
$K=k_1k_2$ is the loop stiffness or gain

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The material in this paper is based in part on a thesis presented by R. R. Caldwell in partial fulfillment of the requirements for the Doctor of Philosophy degree at the University of Wisconsin.

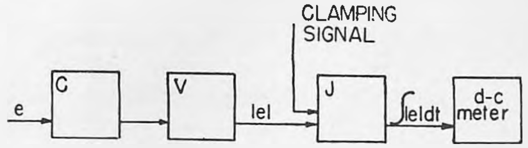


Figure 1 (left). Analogue computer setup for determination of absolute-error-integral criterion. Here the A-unit is an adder, the J-unit is an integrator, and the V-unit a full-wave rectifier, or absolute-value unit

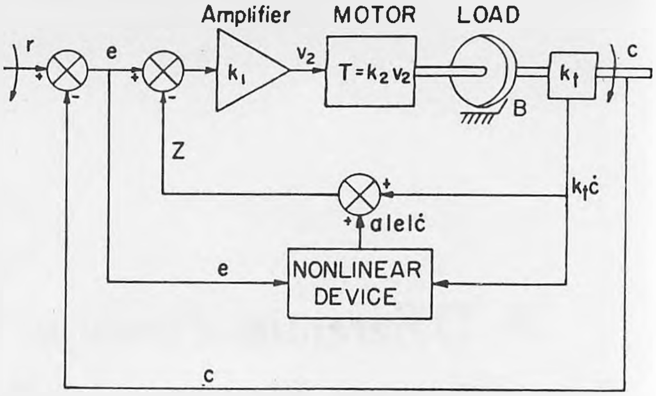


Figure 2 (right). Block diagram of basic servo studied in the paper. The linear form results if the nonlinear device is inoperative

We are chiefly concerned with the case where $B<<N$. The reference damping ratio is therefore chosen as

$$\zeta = \frac{N}{\sqrt{2KI}} \tag{4}$$

The characteristic frequency is

$$\omega_0 = \sqrt{\frac{K}{I}} \tag{5}$$

By choosing a new time scale we can normalize with respect to ω_0 so that equation 3 becomes

$$[p^2+2\zeta(1+\alpha)p+1]c=r \tag{6}$$

where $\alpha=B/N$.

An analogue computer setup for solution of this second-order linear case is shown in Figure 3. A limiter has been added to limit the maximum motor torque. One or more lags, such as the one shown dotted in Figure 3, may be added if a higher order servo is to be investigated.

The type of nonlinear damping found to be most effective in this study was the kind in which a term proportional to the product of absolute error and output velocity $|e|\dot{c}$, was subtracted from the tachometer damping term.[4] This results in a total damping term (if $B/N<<1$) of $2\zeta\dot{c}-a|e|\dot{c}$. The block diagram of Figure 2 indicates how this damping may be applied in a servo. In practice, a rectifier might be used to obtain $|e|$, which might then be applied to the field of the tachometer to produce the $a|e|\dot{c}$ term. Thus normalized equation 6 becomes, for the Lewis nonlinearly damped case

$$\ddot{c}+2\zeta\dot{c}-a|e|\dot{c}+c=r \tag{7}$$

The analogue computer setup for this case is shown in Figure 4. The error integrals were computed as shown in the figure and used to aid in the comparison of transient responses.

Results

Step Input Signals

The peculiar advantages of the nonlinear servo of Figure 2 result from the fact that the nonlinear term not only reduces the damping when error is large but may actually make it negative if the proper magnitude relationship exists in equation 7. The constants were so adjusted that the damping did become negative for all but small initial errors. Thus, not only is the response to a step function better than the linear case, as shown in Figure 5(A), but the initial reduction in error is even faster than for the linear case with zero damping, as shown in Figure 5(B). The absolute-error-integral I is 4.0 units for the linear case and 2.8 units for the nonlinear case. The results of Figure 5 are for a servo of second order, with viscous damping B equal to zero, and with no torque limiting.

This servo, as shown in 5(C), gives much the same form of response for a wide range of input amplitudes, and in no case does overshoot occur. Reduction of overshoot is an advantage found in many nonlinear servos, but the smaller time of recovery obtained for smaller input steps is a unique advantage of this servo. The phase plane plot of Figure 5(D) also indicates improvement in response of the nonlinear servo in this simple case, for, as shown by MacDonald,[2] the area under this curve is inversely proportional to recovery time.

Figure 5 also shows the effect of torque limiting in this servo. Torque limitation was set at a value equal to the maximum torque attained by the linear system. This limitation had little effect on the time of response of the nonlinear system. The output torque is shown for the linear case in Figure 5(E) and for the corresponding nonlinear case in Figure 5(F). Note that in the latter case the torque tends to switch between values of $+T_m$ and $-T_m$, until the error is nearly reduced to zero. Thus, for this torque-limited case the operation of the nonlinear system is essentially the same as that of the optimum relay servo.

Figure 6 shows a comparison of the phase-plane trajectories for the optimum relay servo and the continuously damped torque-limited Lewis servo. Thus the torque-limited condition of this servo makes the best possible use of the maximum available torque.

Double-Step Input Signals

If, in the nonlinear system, the error is allowed to overshoot zero, the system may become unstable. Such a case is shown in the phase-plane plot of Figure 7(A), where stability is marginal, and the error is reduced to zero in only one out of three cases. This instability is the result of a change in the sign of the restoring torque which occurs when the nonlinear damping term $a|e|\dot{c}$ becomes too large. At the critical point the torque sign reversal causes the error to increase rather than decrease.

For the simple step input the parameters may easily be adjusted to eliminate the possibility of overshoot. However,

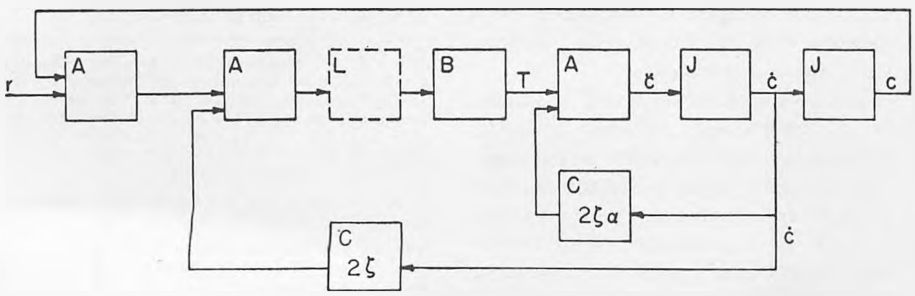


Figure 3. Computer setup used in the solution of the linear form of the servo of Figure 2. Here the B-unit is a limiter, and the L-unit is a lag (1/1+pT) which can be added as shown, or in other positions

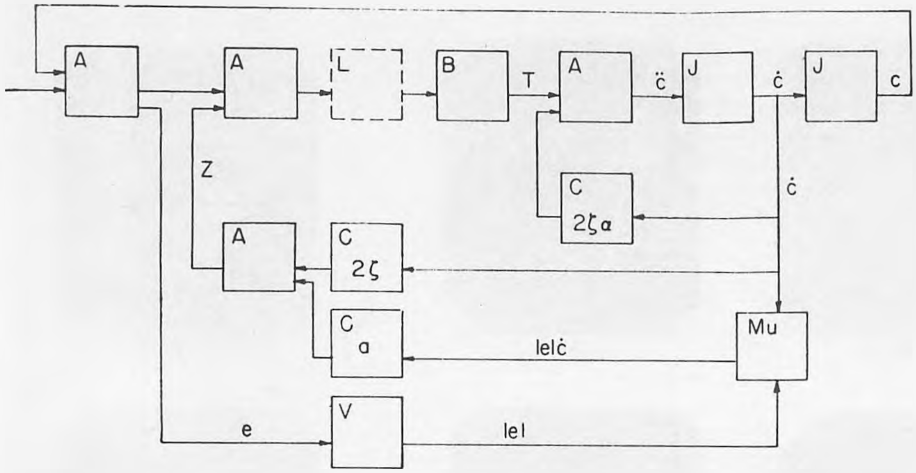


Figure 4. Computer setup used in solution of the nonlinear form of the servo in Figure 2; the Mu-unit is a multiplier

This tendency towards instability can be reduced by limiting the magnitude of the nonlinear term $a|e|\dot{c}$ to some experimentally determined maximum value. Using this type of limiting, it was found that a considerable improvement in stability could be obtained without substantially increasing the transient response time. This limiting treatment was also particularly effective in improving the servo response to sinusoidal input functions. Another very promising scheme, which has not yet been investigated, uses the $a|e|\dot{c}$ term only if the product $e\dot{c}$ is positive. This would always give full linear damping in the first and third phase-plane quadrants where $\dot{c}$ and e have opposite signs.

an input pulse, or a double-step input consisting of a positive step followed by an unequal negative step, can easily cause unstable action of the nonlinear system. This occurs when the input is suddenly reduced before the error caused by the original step is fully corrected. This may result in a change to a negative error while the velocity $\dot{c}$ is in the direction to correct a positive error. If the new negative error reduces the damping to zero, instability may result. The error will tend to increase without limit in either the first or third quadrant of the phase plane, much as in the case shown in Figure 7(A).

Sinusoidal and Saw-tooth Inputs

For a linear system, if either the transient or the frequency response is known then the other is completely determined. For the nonlinear system this is not the case. Thus it is necessary to extend the investigation of the nonlinear system beyond the study of transient response to

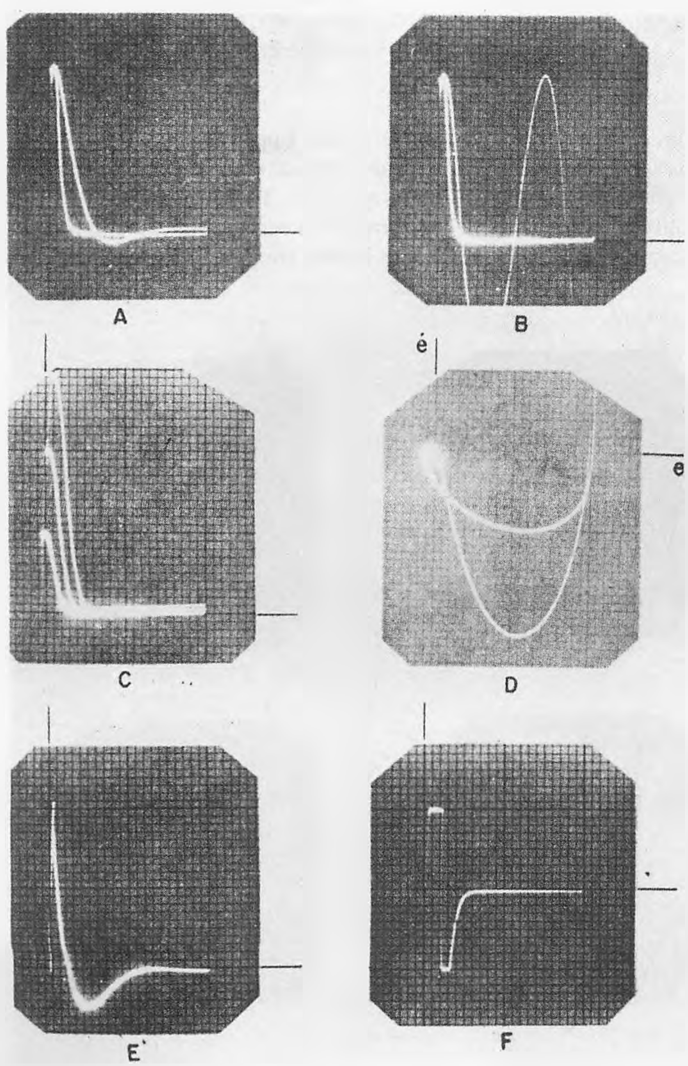


Figure 5 (left). Comparison of a nonlinearly damped servo with a linear second-order servo

A. Upper curve is the error for a step input to a linear servo ($\zeta = 0.65$) and the lower curve is the error for a step input to the Lewis servo ($\zeta = 2.2$, $a = 0.7$)

B. Nonlinear servo error for an input step compared to that for the linear case with zero damping

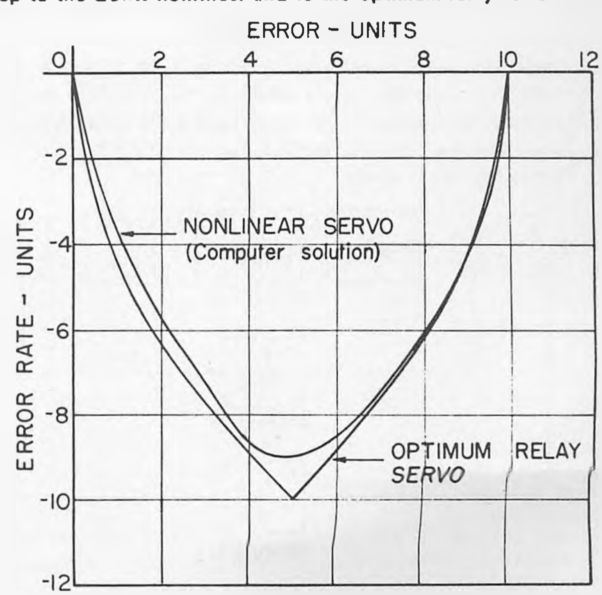
C. Nonlinear servo error for three values of input step

D. Phase-plane plots for equal input steps; upper curve for the linear case, lower curve for the nonlinear case

E. Torque in the linear case (scale = 5 volts per large division)

F. Torque in the nonlinear case, with torque limited to the peak value found in the corresponding linear case, shown in D (scale = 10 volts per large division)

Figure 6 (below). Comparison of the phase-plane trajectories for an input step to the Lewis nonlinear and to the optimum relay servo



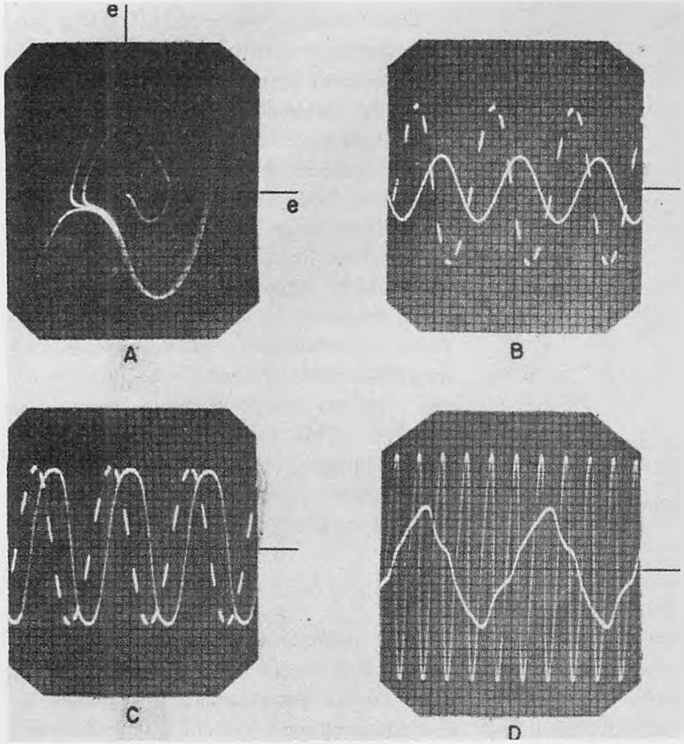


Figure 7

A. Instability in a nonlinearly damped servo
B. Response of a linear second-order servo ($\zeta=0.65$) to a sine wave, ($f=1.57$ cycles per second, input shown dotted)
C. The torque-limited Lewis nonlinear servo response to the sine wave of B
D. Frequency dividing (by five times) in a nonlinear servo ($f=4.92$ cycles per second)

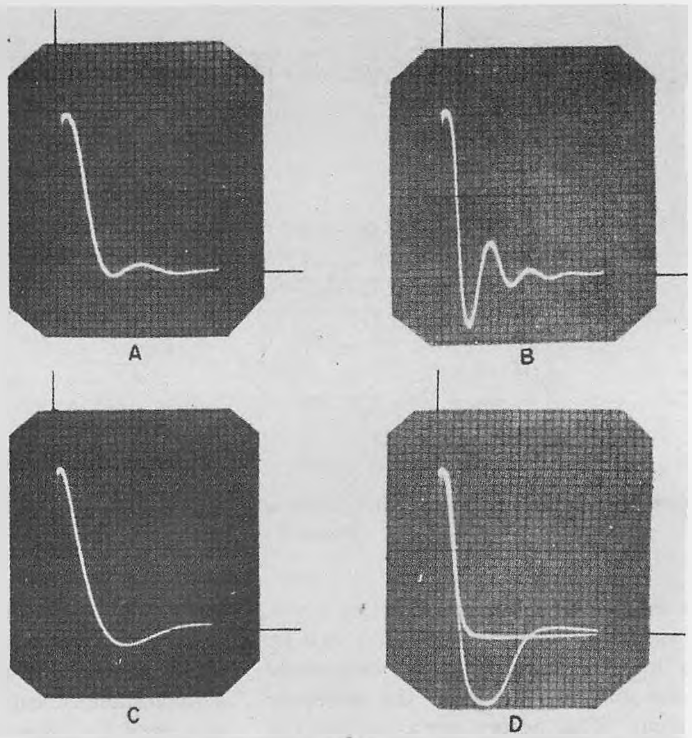


Figure 9

A. Optimized linear case, servo A of Figure 8, $T_A=0.6$, $\zeta=0.87$, $I=4.3$
B. Nonlinear case, servo A, $T_A=0.6$, $\zeta=2.2$, $a=0.7$, $I=3.9$
C. Optimized linear case, servo B of Figure 8, $T_B=0.6$, $\zeta=0.9$, $I=4.5$
D. Nonlinear case, servo B, $T_B=0.6$, $\zeta=3.7$, $I=3.3$ (upper trace), $\zeta=2.2$, $I=5.6$ (lower trace)

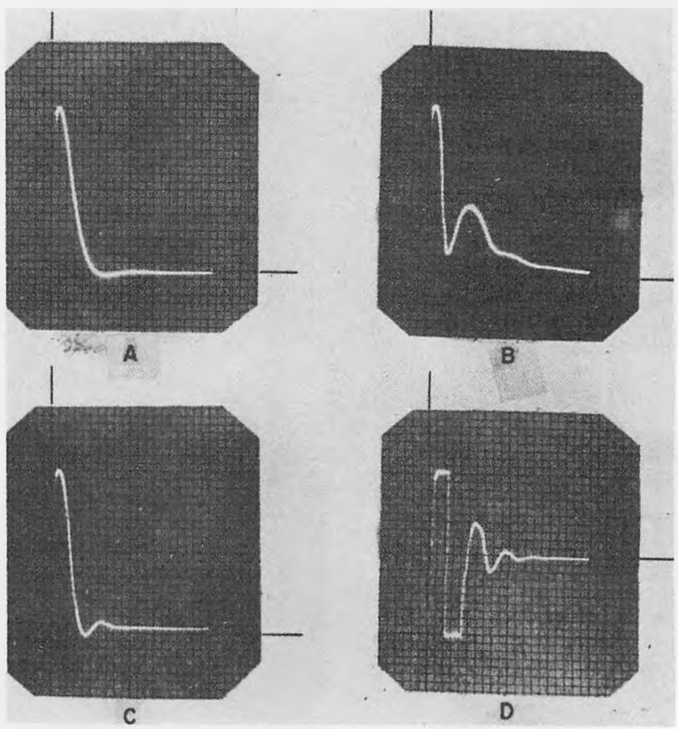
various step-function inputs. The computer setup used for the transient study is quite satisfactory for sinusoidal inputs. An audio oscillator was used as a sine-wave source since for this high-speed computer 382 cycles per second is equivalent to 1 cycle per second in real time.

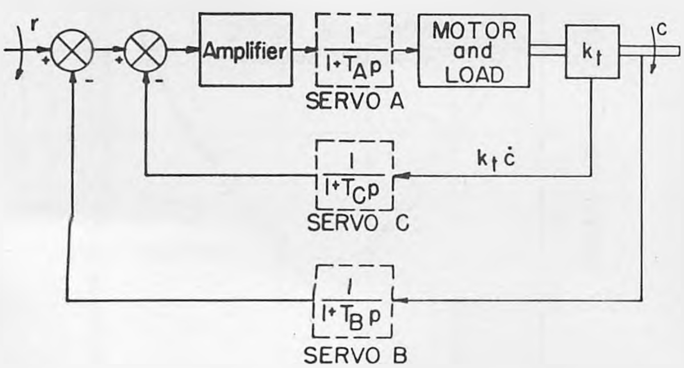
The first noticeable effect of the nonlinear system is distortion of the output wave. This of course is to be expected, since the behavior of a nonlinear system is a function of amplitude. For the higher frequencies the nonlinear system does give an improvement in response as shown by a comparison of Figures 7(B) and (C). This improvement is in the form of increased output amplitude. At the higher frequencies larger torques

Figure 8 (below). Servo block diagram showing locations of added lags

Figure 10 (right)

A. Optimized linear case, servo C of Figure 8, $T_C=0.3$, $\zeta=0.65$, $I=3.4$
B. Nonlinear case, servo C, $T_C=0.3$, $T_m=\pm 20$, $I=4.15$
C. Nonlinear case, servo C, $T_C=0.3$, $T_m=\pm 10$, $I=3.0$
D. Torque for case C above





are required for the output to follow the input. These increased torques can be supplied by the nonlinear system. The improvement of the phase of the output relative to the input is negligible. A lead network is more effective for the reduction of the phase lag.

Figure 7(D) shows a pathological effect caused by the nonlinear damping. This frequency division occurs for certain combinations of input amplitude and frequency. For magnitudes above or below the critical values the system will return to normal operation. This frequency division results when the $a|e|\dot{c}$ term is too large, so that a torque condition results which is similar to that discussed for the double-step case. Limiting the magnitude of $a|e|\dot{c}$ will return the system to normal operation. Varying degrees of limiting were found to be effective depending on the input amplitude and frequency. However, even with substantial limiting sufficient to remove all frequency-dividing effects, the high-frequency response of the nonlinear system is better than that of the linear.

Triangular input waves were also used, but gave no more information than the sine-wave inputs.

Step Inputs for Higher Order Servos

Although an unlimited number of higher order systems is possible, this study was limited to three cases corresponding to the addition of one or two simple lags in each of three positions in the system. The three locations of the inserted lags are shown by the broken-line blocks in Figure 8. The response of the nonlinear system is quite different in each case. Relatively large time-constant lags were used in an effort to emphasize the effect of lag. For smaller time constants, responses will differ from the second-order case by correspondingly smaller amounts.

For servo A, the nonlinear system transient response had a smaller error integral than the linear system, but had a rather large overshoot. The linear and nonlinear responses are shown in Figures 9(A) and (B). Thus for this servo the linear system may be preferred to the nonlinear if small overshoot is desirable.

With additional lags inserted, as in servos B and C, the transient responses of the linear and nonlinear systems are each similar in shape to their second-order counterparts. Figures 9(C) and (D) show the linear and nonlinear responses resulting for servo B. The two traces of Figure 9(D) result from use of the optimum damping ratio $\zeta = 3.7$, and the damping ratio $\zeta = 2.2$ used in the second-order system. This new optimum adjustment of ζ is of course necessary to fit the new system resulting because of the insertion of the lags.

For servo C, the linear and nonlinear responses are shown in Figures 10(A) and (B). The response of the nonlinear system is far from desirable. The unusual increase of the error in the positive direction is the result of excessive reverse torque brought about by the lagging $\dot{c}$ term. This condition may be corrected by another nonlinearity, such as the torque limiting which is present in any servo motor. This results in the nonlinear transient response shown in Figure 10(C), with the corresponding torque as in Figure 10(D).

Noise

The operation of the nonlinear servo was studied for the case where random noise was added to various input signals. Noise peaks, if large enough, were found to give rise to the same instability problems found with double-step inputs. Smaller noise voltages gave effects which were much the same as those observed for the linear servo.

Conclusions

The nonlinearly damped servo described here has been shown by computer studies to have significant advantages over the corresponding linear servo, as well as over a number of other nonlinear servos. These studies have also shown that the responses to the various types of input signals must be carefully examined. Thus in the Lewis servo the nonlinear term $a|e|\dot{c}$, which is used to modify the damping, must in turn be modified to prevent undesirable responses or actual instability from occurring in the case of other than simple step input signals. In the computer studies made at the University of Wisconsin, a limiting of the $a|e|\dot{c}$ term was found to be helpful. It appears that servo performance may also be improved by reducing $a|e|\dot{c}$ to zero when the product $e\dot{c}$ is negative.

The complexities of even this relatively simple nonlinear servo are such that when the time constants and other constants are known for a practical case, it should probably be studied on a computer for the various input signals which might be encountered before the design is crystallized. It is hoped, however, that the study reported here will provide some kind of a guide in such design work, and act as an aid to future investigation.

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Discussion

Thomas M. Stout (University of Washington, Seattle, Wash.): I would like to offer a simplification of equation 2 of the paper and describe a related experimental procedure for determining the integrated absolute or squared error. This equation may be simplified by observing that

$$\sin\theta = (1-\zeta^2)^{1/2} \tag{1}$$

and

$$\cos\theta = \zeta$$

With these substitutions, the second and third terms reduce to

$$\sigma\sin\theta + \cos\theta = 2\zeta \tag{2}$$

It is also convenient to write the equation in terms of an angle θ_1, such that

$$\tan\theta_1 = -\tan\theta \tag{3}$$

and

$$\theta_1 = \pi - \theta \tag{4}$$

where $\pi/2 < \theta_1 < \pi$. This angle is a direct measure of the time required by the system to first reach zero error, which is

$$t_1 = \frac{\theta_1}{\omega_0\sqrt{1-\zeta^2}} \tag{5}$$

Making this change, the equation for the

Table I. Integrated Absolute Error for a Second-Order Servomechanism Subject to a Step Input of Magnitude P

ζ	$\frac{I\omega_0}{P}$
0.1	6.384
0.2	3.341
0.3	2.364
0.4	1.924
0.5	1.712
0.6	1.618
0.65	1.606
0.7	1.608
0.8	1.631
0.9	1.804
1.0	2.000

integrated absolute error of a second-order system subjected to a step input becomes

$$I=\frac{2P}{\omega_0}\left[\frac{\epsilon^{-\sigma\theta_1}}{1-\epsilon^{-\sigma\pi}}+\zeta\right] \qquad (6)$$

This form of the equation is particularly useful when ζ approaches unity, where the other equation leads to an indeterminate form. For $\zeta=1$, equation 6 of the discussion gives directly

$$I=\frac{2P}{\omega_0} \qquad (7)$$

which can also be obtained by direct integration of the error for the critically damped case. For $\zeta>1$, the integrated absolute error is the same as the integrated error and can be found[1] by adding the time constants of the closed-loop transfer function.

For purposes of comparison and to save future workers some labor, values computed from equation 6 are given in Table I of the discussion.

In some experimental studies similar to those described in the paper, we have been using a slightly different method for determining the integral of the error. In our case, a voltage proportional to the error was applied to an integrating amplifier in which the usual input resistor had been replaced by a diode rectifier. With this arrangement, only errors of one sign are integrated; the diode characteristics determine whether the absolute or squared error, or some intermediate power of the error, is used. Part of the integral is found by application of a positive signal to the input of the simulated servomechanism, and the remainder is found by applying an equal negative input, which simply means removing the previous signal.

This technique, although less elegant than the procedure described in the paper, is useful when the time constants in the computer simulation are fairly large, and it eliminates the need for an absolute value unit.

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1. See reference 9 of the paper.

R. R. Caldwell and V. C. Rideout: We wish to thank Mr. Stout for pointing out the simplifications possible in equation 2 of the paper.

APPLICATIONS OF ANALOG COMPUTORS

The remaining articles in this volume are largely devoted to applications rather than methods or equipment.

This sequence commences with an exemplary contribution of Roedel concerned with mechanical suspension and coupling problems which demonstrates superbly the benefits of high speed exploration of parameter changes.

APPLICATION OF AN ANALOG COMPUTOR TO DESIGN PROBLEMS FOR TRANSPORTATION EQUIPMENT

by Jerry Roedel

Presented at:
International Instrument Conference, Sept. 22, 1954

EMPTY BOX CAR IMPACT

To illustrate the techniques of setting up and solving a problem on an analog computor, it is best to choose a situation which results in the most elementary equation of motion. Such a problem would exist if we could consider the impact of two railway cars as lumped constant system. For purposes of illustration, consider that the two empty box cars shown in Fig. 10 may be replaced by two equivalent masses $2M_{eq}$ as shown in Fig. 11. Further assume that the spring rate of each of the cars is an equivalent lumped spring constant k and that the damping of the cars may be represented by an equivalent viscous damping constant D. To simplify the problem still more, consider that the springs are blocked out so the two car are free to move only parallel to the track, and that the motion of interest is the relative motion between the cars. This system is a single degree of freedom system. By applying Newton's laws of motion [8] or Lagrangian Analysis, [9] the following second order differential equation may be derived to completely describe the motion of this dynamic system.

$$M_{eq}\ddot{x} + D\dot{x} + kx = 0 \qquad (1)$$

This is an equation of force equilibrium for the system. To adapt the problem for computation we rearrange the terms to set the acceleration term equal to the negative sum of the velocity and displacement terms shown in equation 2.

$$M_{eq}\ddot{x} = -(D\dot{x} + kx) \qquad (2)$$

To set up the computor to solve this equation, assume that the term $M_{eq}\ddot{x}$ is available at the input of a coefficient box as shown in the lefthand side of Figure 12a. If the scale of the coefficient box is set at the value $\frac{1}{M_{eq}}$ the output will be the term $\ddot{x}$. This output is now connected to the input of an integrator as shown in Figure 12b where it is integrated with respect to time to result in the velocity term $\dot{x}$. The prime indicates that this is velocity minus the initial condition. The initial condition of velocity v_o may be added to give true initial velocity in an addition unit as shown in Fig. 11c. The output is now true velocity $\dot{x}$. This velocity upon integration results in the displacement term x, as shown in Fig. 12d. The two terms x and $\dot{x}$ must now be modified by coefficient units set to values k and D respectively as shown in Fig. 12e. All the terms which appear on the righthand side of equation 2 are now available. By connecting these two inputs into an adder, the term $M_{eq}\ddot{x}$ will exist at the output as shown on the righthand side of Fig. 12f. It is now seen that if the assumed value of $M_{eq}\ddot{x}$ is available at the input on the lefthand side of Fig. 12f, the same term will appear on the righthand side. Therefore by connecting these two points together as shown in Fig. 12g the computor loop is forced to follow the equation. If a voltage proportional to an initial velocity v_o is now applied at the initial condition adder; velocity vs. time, displacement vs. time, or acceleration vs. time, plots may be observed by connecting an oscilloscope at the points marked (1), (2) or (3) respectively in Fig. 12g. Since scaling of the problem is relatively complex but straight forward,[10] an explanation of the actual scaling will be omitted from this text. If a unit initial impact velocity is applied to the adder, the curves shown in Fig. 13a will result for the displacement and velocity between the cars. To illustrate how car weight affects the response of these cars in impact, we will double and half the weight of the cars. For cars which are twice as heavy we turn the dial in the first scale changer to one-half its former value and the curves appear as in Fig. 13b. If the weight of the cars is halved the curves appear as in Fig. 13c. Similar changes are made in the values of D, k, and v_o to produce the curves shown in Figs. 13d to 14d.

Nonlinear effects may be simulated by inserting a nonlinear operation at the appropriate point in the computing loop. If, for example, the equivalent spring k was to have a nonlinearity of the type exhibited when a spring hits a solid stop, a unit of the B type in Fig. 9 would be inserted in the loop between the displacement term x and coefficient unit.

With this elementary problem we have tried to demonstrate some of the basic principles in the use of an operational analog computor. An attempt has also been made to demonstrate the usefulness of this new tool to scan the effects of changing problem parameters. With this basic background it is now possible to consider two more complex problems which have arisen in the design of transportation equipment.

TRAILER-ON-FLAT-CAR

One of the more current problems which has arisen is the prediction of the dynamic loads which might be encountered in the widely publicized hauling of highway trailers on flat cars. A photograph of a model of a system proposed by the Pullman-Standard Car Manufacturing Company is shown in Fig. 15. Consider the condition of a flat car carrying one loaded highway trailer being impacted by a loaded hopper car as shown in Fig. 16. This system may be drawn schematically as shown in Fig. 17 for purposes of mathematical analysis. Utilizing Lagrangian analysis the three equations of motion shown by equations 3, 4, and 5 are derived for this system.

$$\ddot{y}_1 = -[2\zeta_1 \omega_1 \dot{y}_1 + \omega_1^2 y_1] + \frac{m_3}{m_2 + m_3}[2\zeta_2 \omega_2 \dot{y}_2 + \omega_2^2 y_2] \qquad (3)$$

$$\ddot{y}_2 = + \frac{m_1}{m_1 + m_2}[2\zeta_1 \omega_1 \dot{y}_1 + \omega_1^2 y_1] - [2\zeta_2 \omega_2 \dot{y}_2 + \omega_2^2 y_2] + \frac{m_4}{m_3 + m_4}[2\zeta_3 \omega_3 \dot{y}_3 + \omega_3^2 y_3] \qquad (4)$$

$$\ddot{y}_3 = \frac{m_2}{m_2 + m_3} \left[2\zeta_2\omega_2\dot{y}_2 + \omega_2^2 y_2 \right] - \left[2\zeta_3\omega_3\dot{y}_3 + \omega_3^2 y_3 \right] \qquad (5)$$

These equations are set up in a manner similar to that used for the empty box car impact. Fig. 18 is a diagram showing the setup of the computor for the solution of this equation. By applying an initial impact velocity v_0 between the cars, solutions are obtained as shown in Figs. 19 and 20. It is seen that the dynamic response of the system is dependent upon the stiffness of the trailer hold-down equipment. Other system parameters could also easily be varied to show the effects of such variation on the force distribution in the system.

AUTOMOBILE RIDE

As a final illustration, the problem of automobile ride is considered. The problem of an automobile riding up a 15° ramp is considered for an idealized case. Consider the automobile shown in Fig. 21 with the car free only, in pitch about the center of gravity, vertical motion of the center of gravity and horizontal motion along the road. In this case the two front springs and the two rear springs must act as a unit because all roll of the car is neglected. A schematic diagram for an automobile suspension system to cover this case is shown in Fig. 22. In this diagram one additional symplification is made by neglecting the spring of the tires. This simplification is necessary to keep the complexity of the problem within limits for easy manipulation in a demonstration. The equations of motion of this system shown by equations 6 and 7 have been derived in an earlier paper by the Goodyear Aircraft Company.[11]

$$m\ddot{x} + 2D_1(\dot{x} - L_1\dot{\theta} - \dot{x}_1) + 2D_2(\dot{x} + L_3\dot{\theta} - \dot{x}_2) + 2(k_1 + k_2)x$$
$$-2(k_1L_1 - k_2L_2)\theta = 2k_1x_1 + 2k_2x_2 \qquad (6)$$

$$mi^2\ddot{\theta} - 2D_1L_1(\dot{x} - L_1\dot{\theta} - \dot{x}_1) + 2D_2L_3(\dot{x} + L_3\dot{\theta} - \dot{x}_2) + 2(k_1L_1^2 + k_2L_2^2)\theta$$
$$-2(k_1L_1 - k_2L_2)x = -2k_1L_1x_1 + 2k_2L_2x_2 \qquad (7)$$

The computor setup for this problem is shown in Fig. 23. The results for various speeds of the car are shown in Figs. 24 and 25.

CLOSURE

An attempt has been made to demonstrate the simplicity and usefulness of analog computors in design studies. The main emphasis has been on the solution of the problem because little or no electrical knowledge is necessary for one to use these devices as engineering tools. All of these problems can be solved with commercially available equipment varying in price from $5,000 up.

FOOTNOTES

1,2,3,4,5,6 Roedel, J., "An Introduction to Analog Computors," ISA Journal, Vol. 1, No. 8, Pages 9-15, August, 1954.

7 Roedel, J., "The Electronic Differential Analyzer", paper presented at Case-Industry Conference, Case Institute of Technology, Cleveland, Ohio, April 13, 1954.

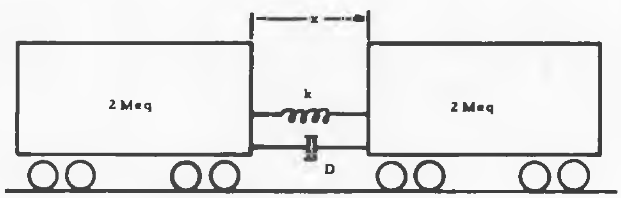
8 Johnson, W. C., "Mathematical and Physical Principles of Engineering Analysis", McGraw-Hill Book Co., Inc., New York, Chapter II, Pages 14-15, 1944.

9 Slater & Frank, "Mechanics", McGraw-Hill Book Co., Inc., New York, Chapter IV, Pages 69-88, 1947.

10 Reswick, J., "Scale Factors for Analog Computers", Product Engineering, Vol. XXV, No. 3, Pages 197-201, March, 1954.

11 "Geda Analysis of a Standard Automobile Suspension System", Goodyear Aircraft Corporation, Akron, Ohio, Report No. GER-5262, March 12, 1953.

BOX CARS ON A RAILROAD HUMP TRACK
FIGURE 10



$$M_{eq}\ddot{x} + D\dot{x} + kx = 0$$

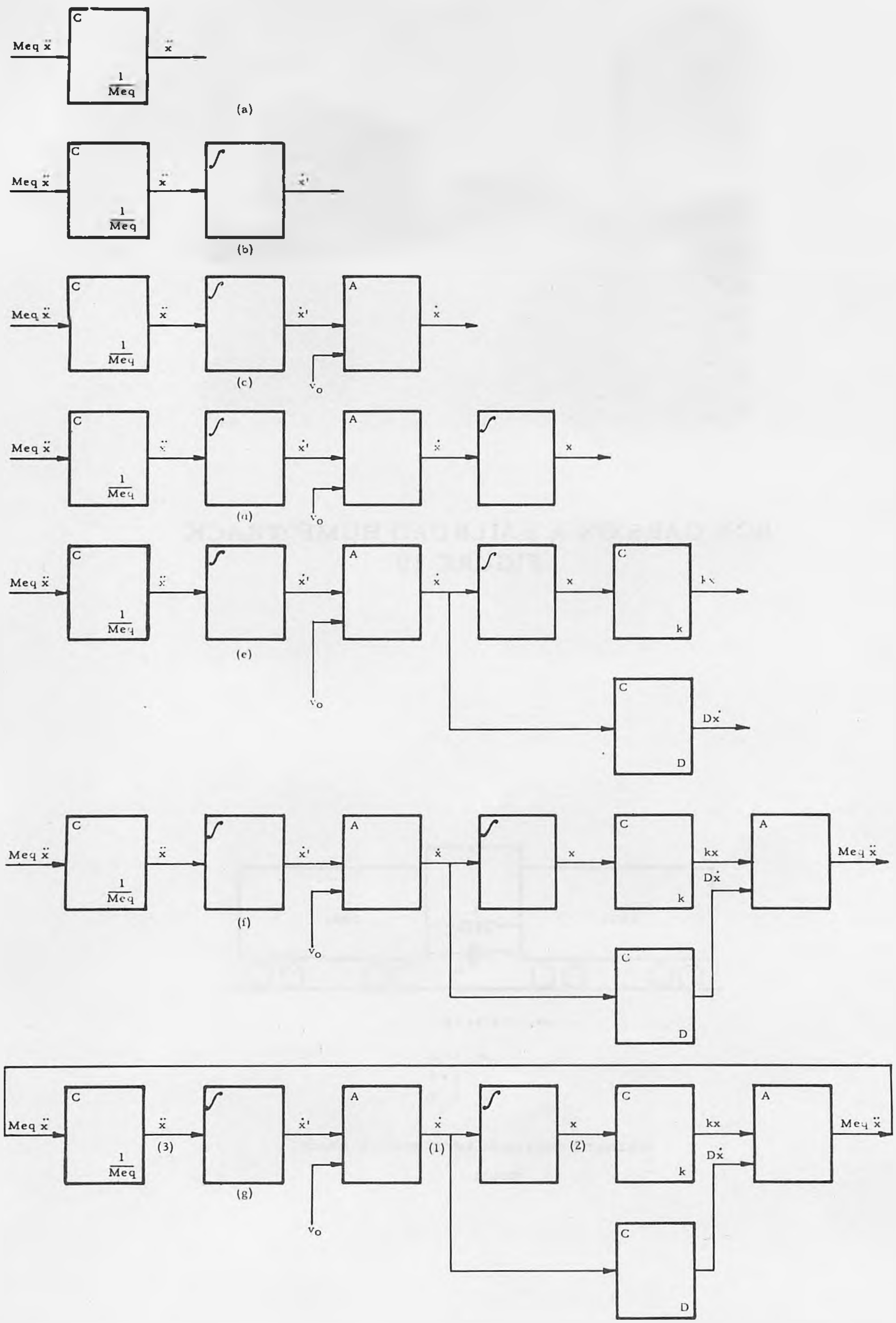
At $t = 0$,

$x = 0$

$\dot{x} = v_0$

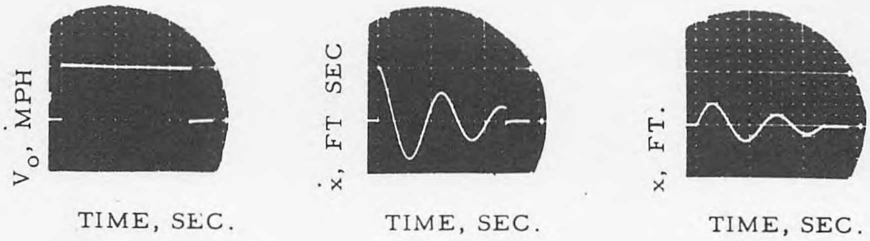
SCHEMATIC DIAGRAM OF EMPTY BOX CAR IMPACT

FIGURE 11



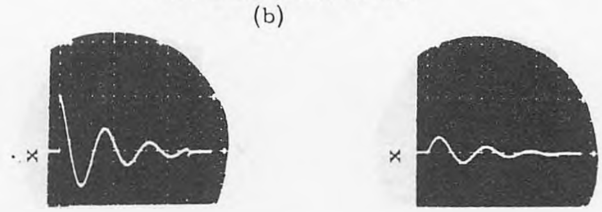
STEPS IN COMPUTOR SETUP FOR EMPTY BOX CAR IMPACT

FIGURE 12



(a)

DOUBLE THE MASS
(b)

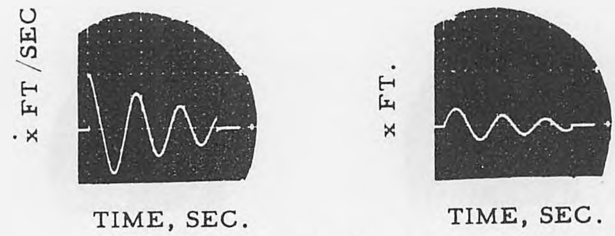


ONE-HALF MASS
(c)

DOUBLE THE DAMPING
(d)

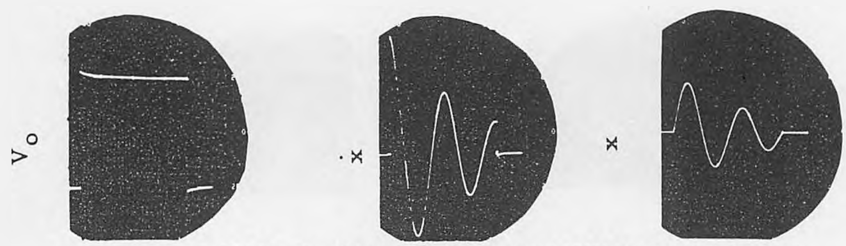
ONE-HALF DAMPING
(e)

SOLUTIONS OF EMPTY BOX CAR IMPACT
FIGURE 13

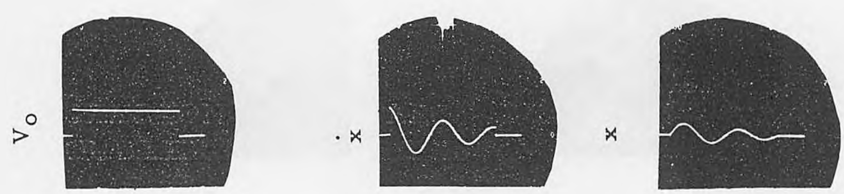


DOUBLE THE SPRING RATE
(a)

ONE-HALF SPRING RATE
(b)

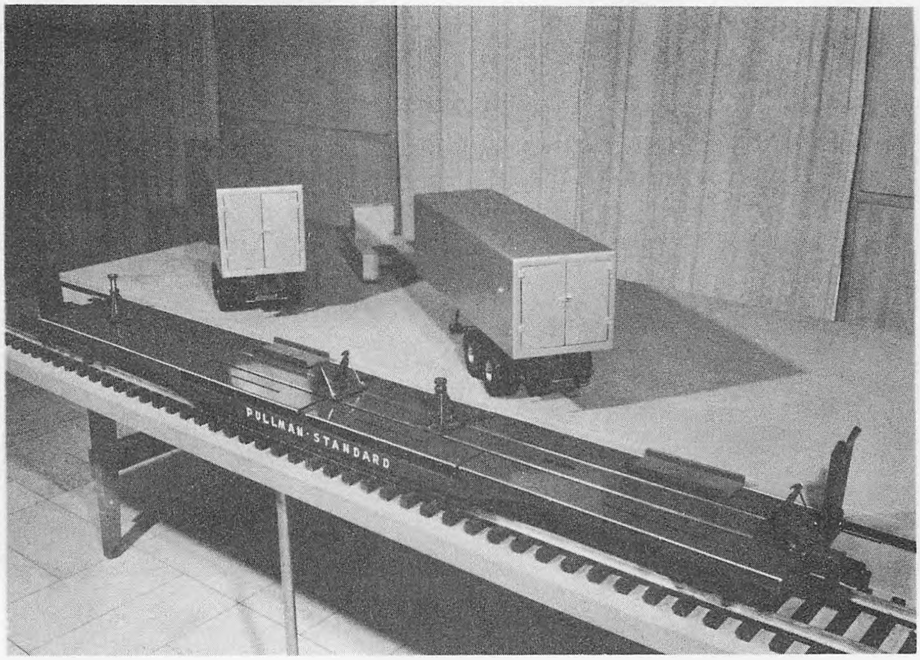


DOUBLE THE INITIAL STRIKING VELOCITY
(c)

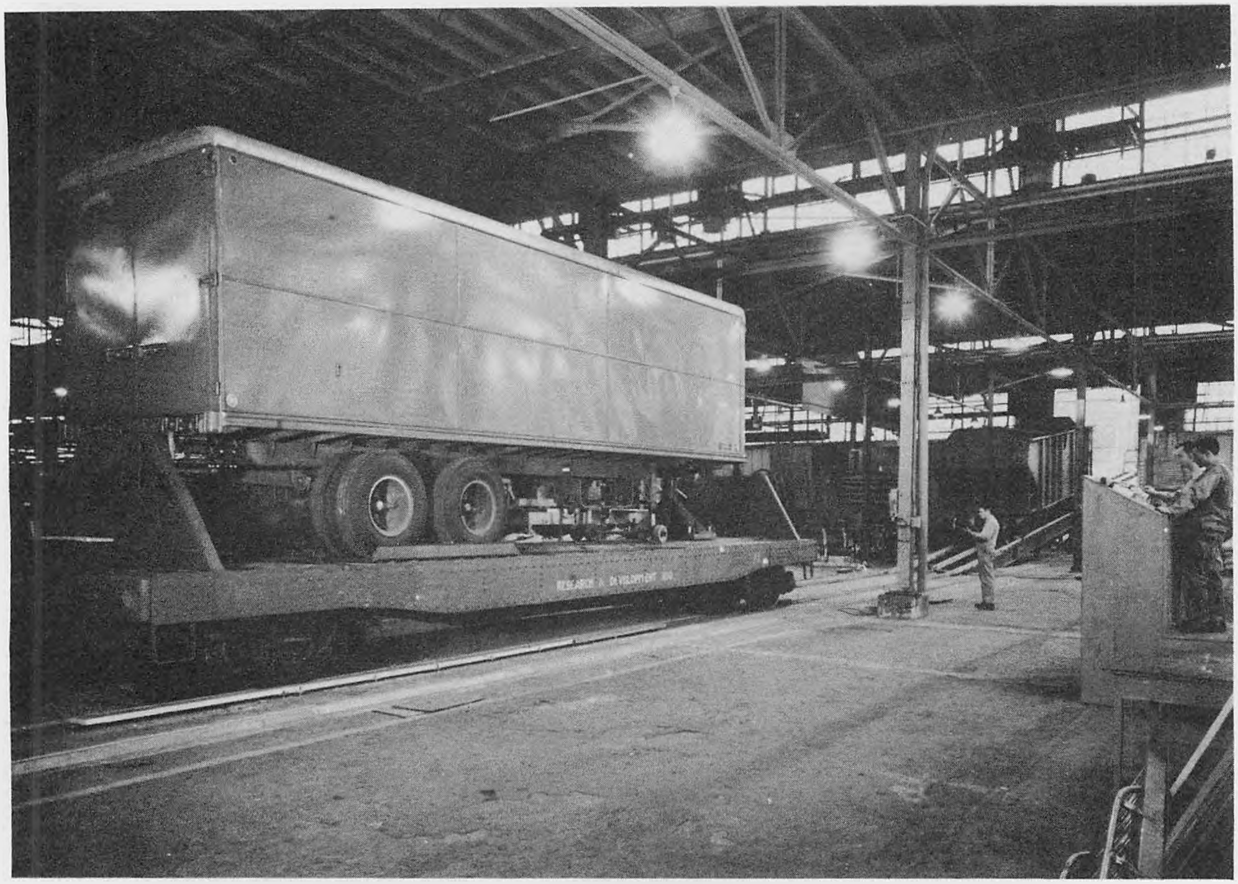


ONE-HALF INITIAL STRIKING VELOCITY
(d)

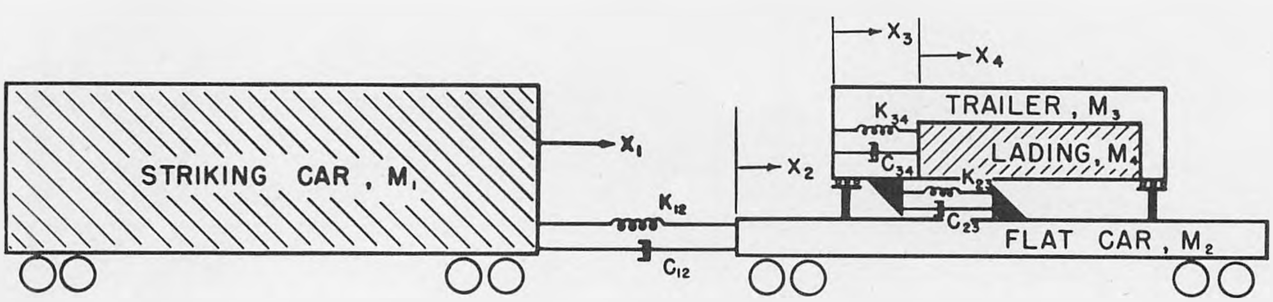
SOLUTIONS OF EMPTY BOX CAR IMPACT
FIGURE 14



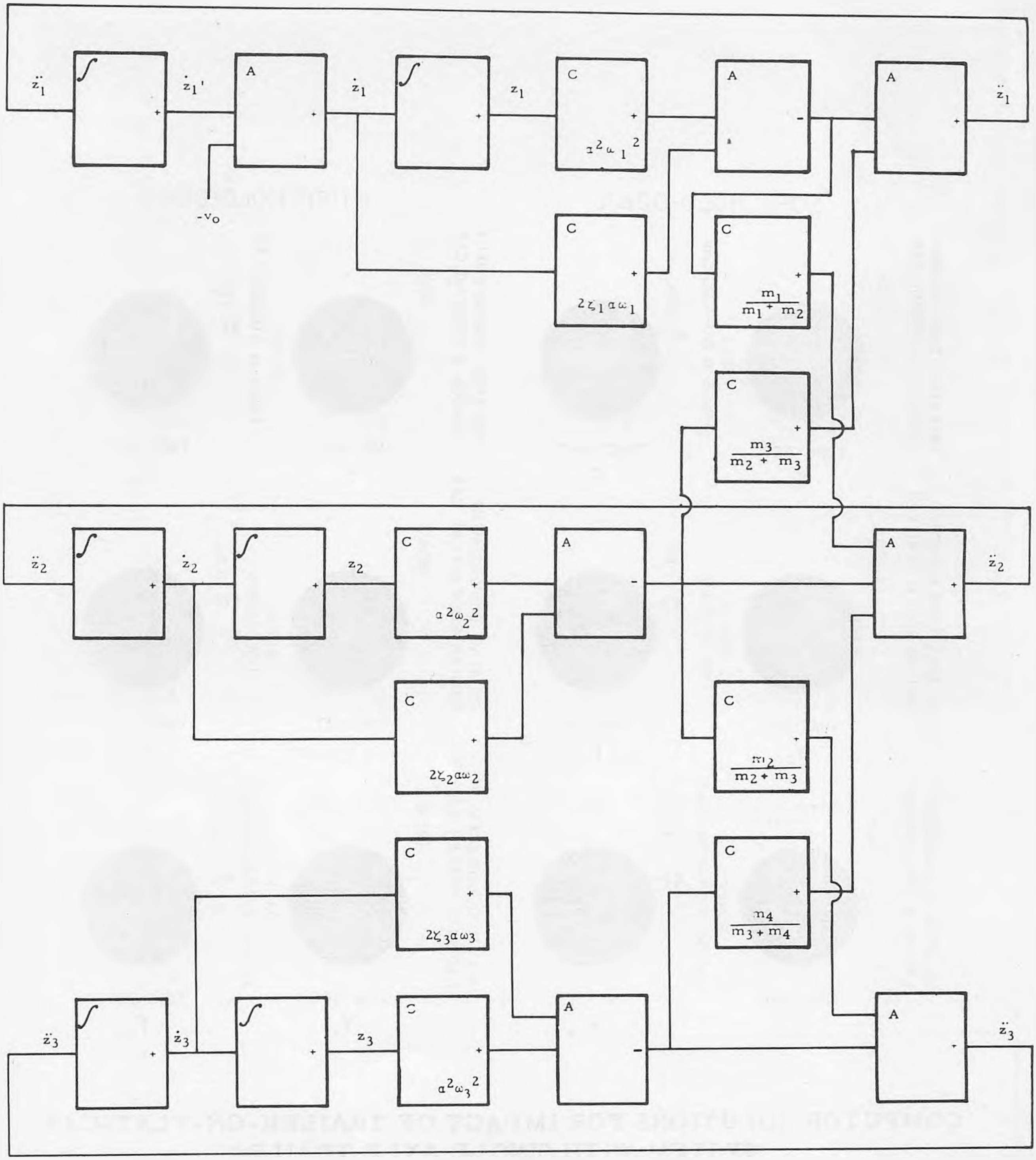
PULLMAN-STANDARD FLAT CAR FOR TRAILERS
FIGURE 15



PULLMAN-STANDARD EXPERIMENTAL TRAILER-ON-FLAT CAR SYSTEM ON TEST TRACK
FIGURE 16

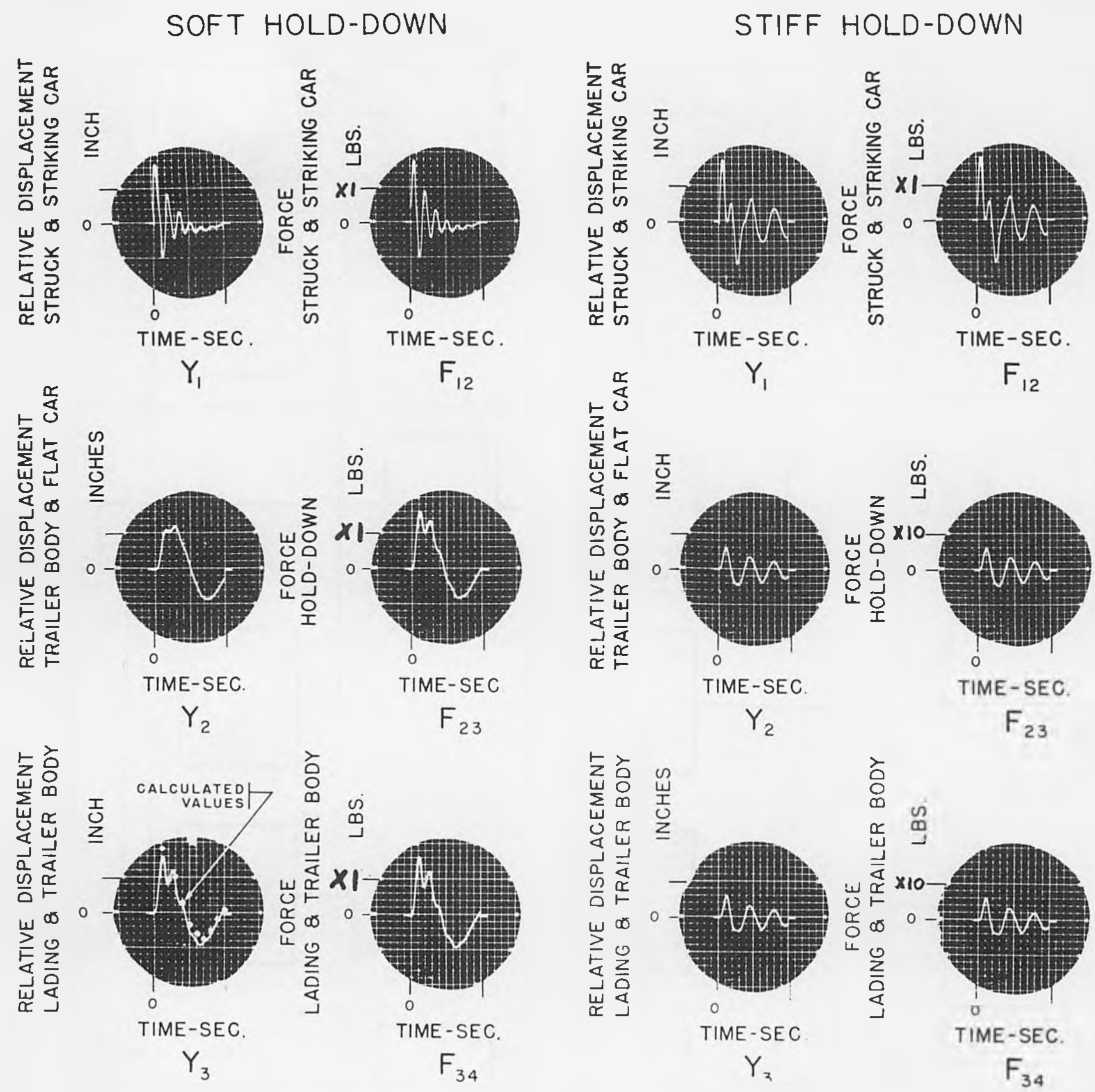


SCHEMATIC DIAGRAM OF TRAILER-ON-FLAT CAR SYSTEM
FIGURE 17

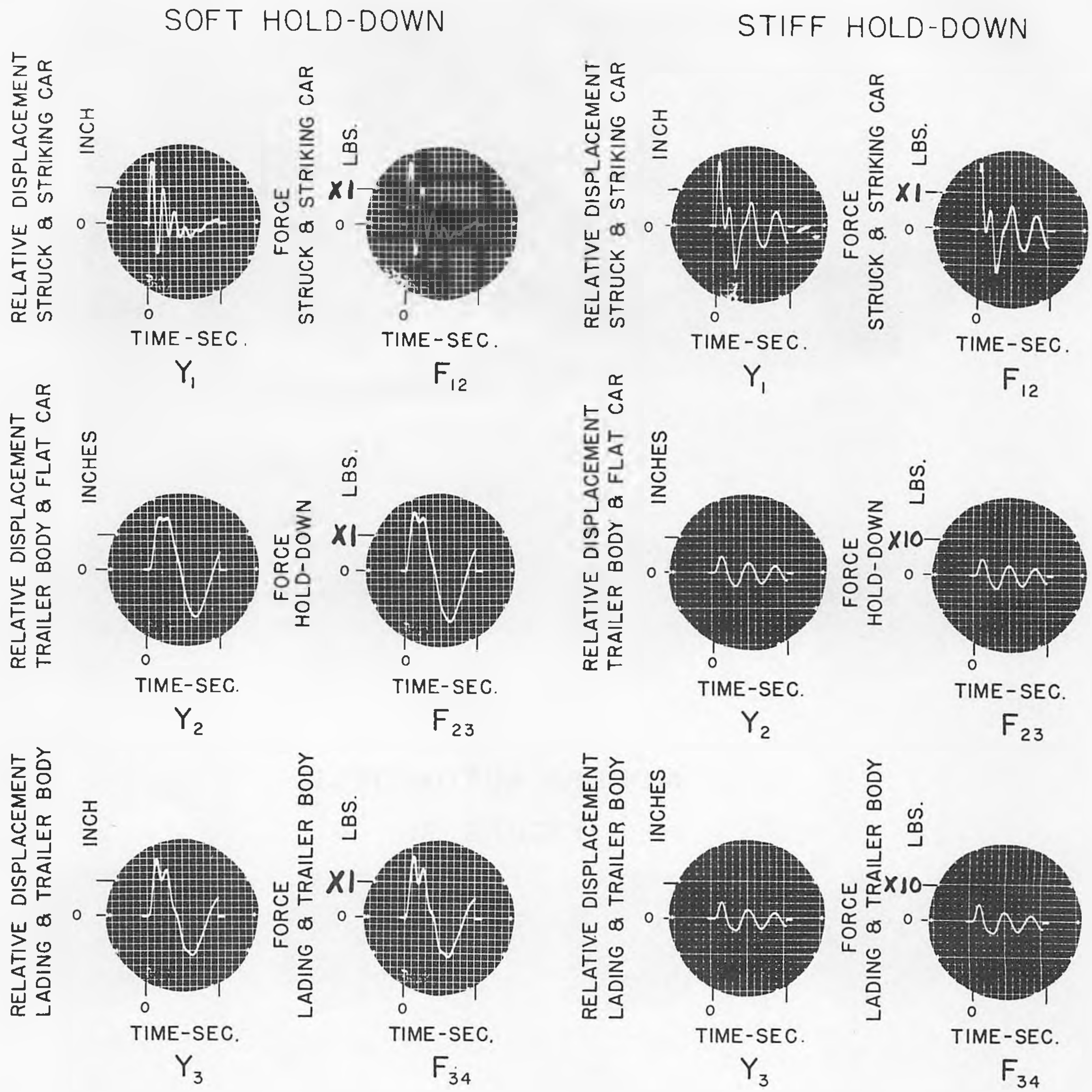


BLOCK DIAGRAM OF COMPUTOR SETUP

FIGURE 18



COMPUTOR SOLUTIONS FOR IMPACT OF TRAILER-ON-FLAT-CAR SYSTEM WITH SINGLE AXLE TRAILER
FIGURE 19

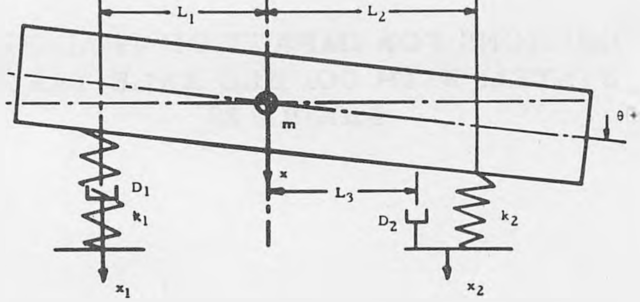


COMPUTOR SOLUTIONS FOR IMPACT OF TRAILER-ON-FLAT-CAR SYSTEM WITH DOUBLE AXLE TRAILER

FIGURE 20

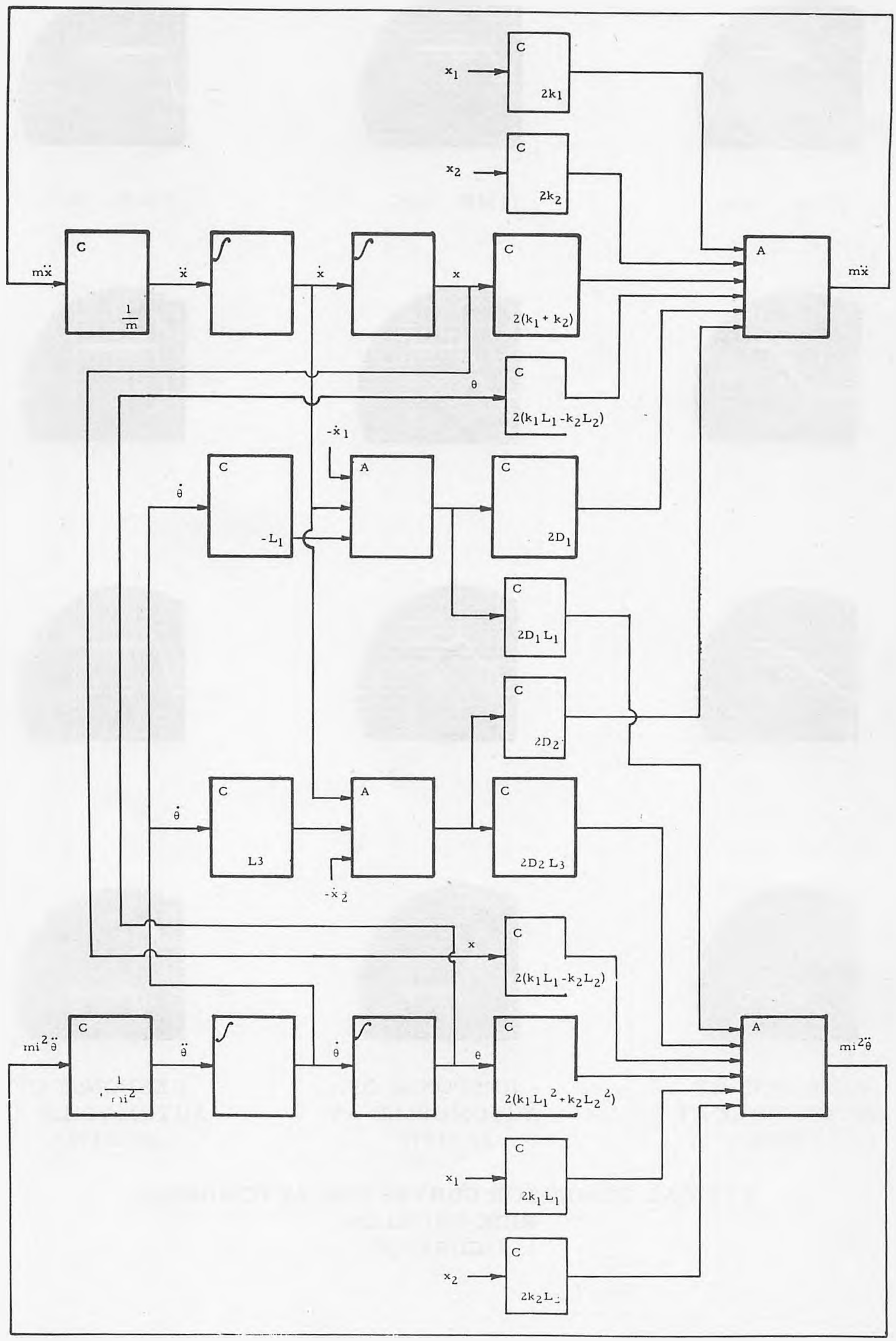
MODERN AUTOMOBILE

FIGURE 21



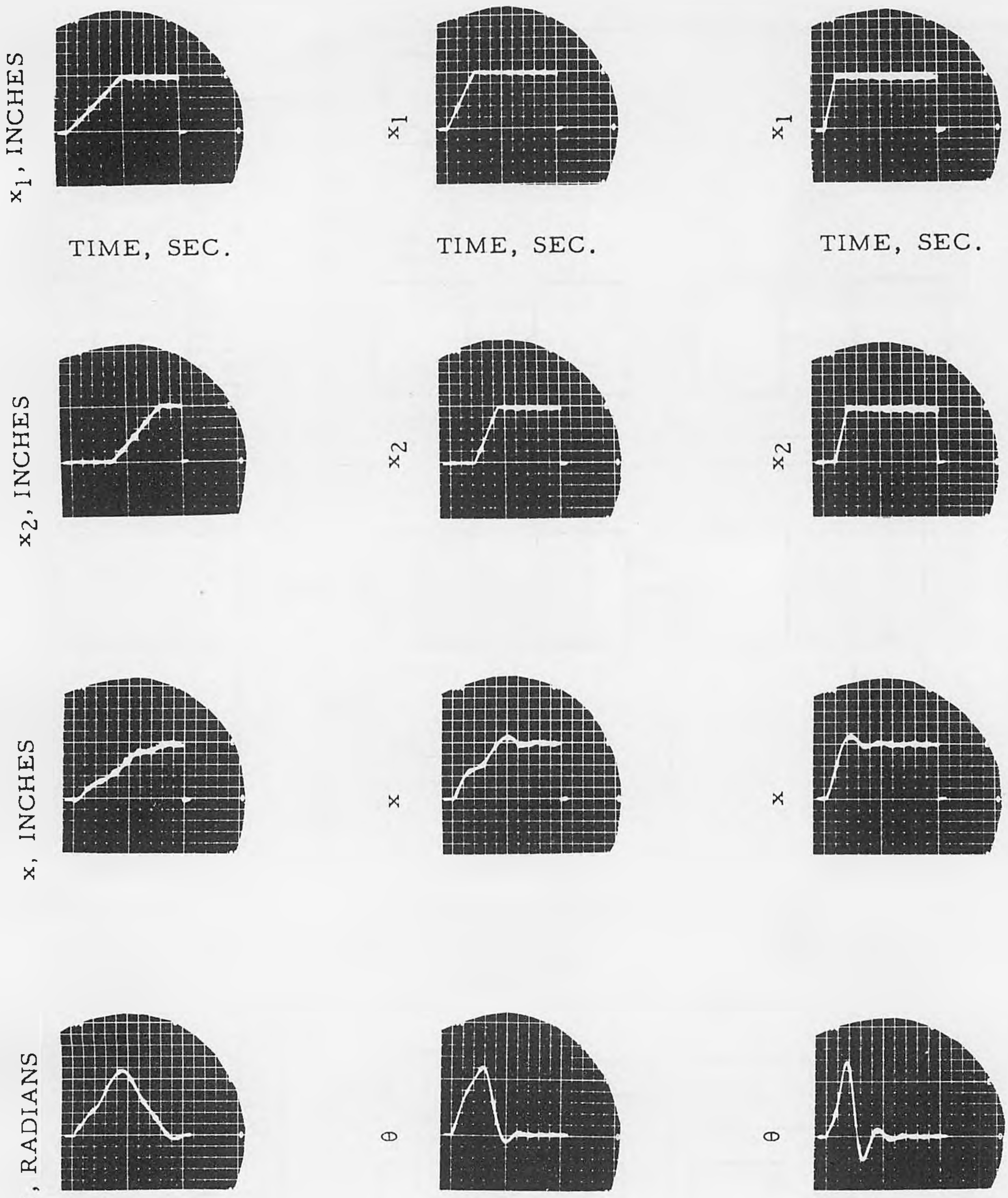
SCHEMATIC DIAGRAM OF AUTOMOBILE SUSPENSION SYSTEM

FIGURE 22



COMPUTOR BLOCK DIAGRAM FOR AUTOMOBILE RIDE PROBLEM

FIGURE 23

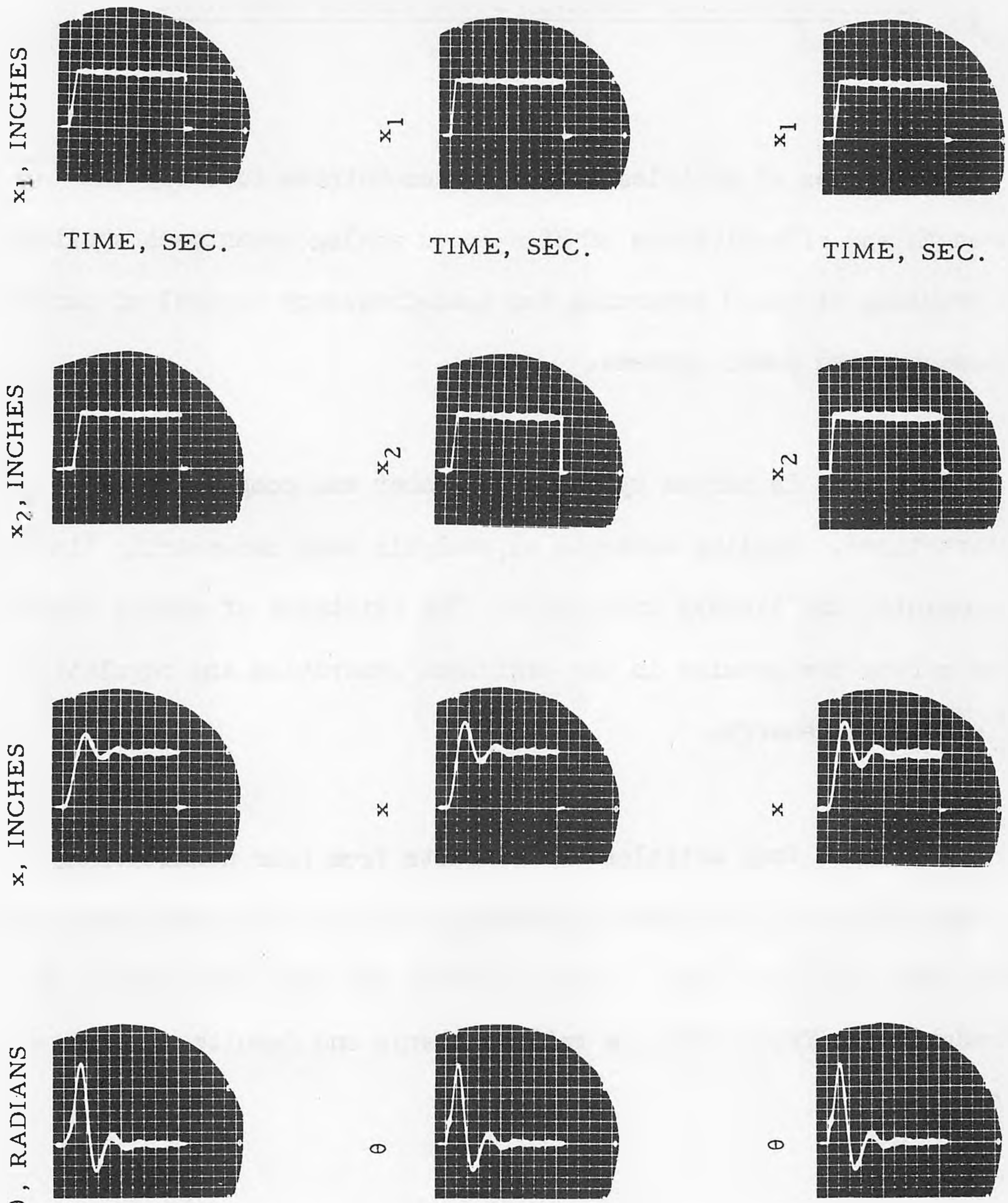


RESPONSE OF AUTOMOBILE AT 5 MPH

RESPONSE OF AUTOMOBILE AT 10 MPH

RESPONSE OF AUTOMOBILE AT 20 MPH

TYPICAL COMPUTOR CURVES FOR AUTOMOBILE RIDE PROBLEM
FIGURE 24



RESPONSE OF AUTOMOBILE AT 30 MPH

RESPONSE OF AUTOMOBILE AT 50 MPH

RESPONSE OF AUTOMOBILE AT 60 MPH

TYPICAL COMPUTOR CURVES FOR AUTOMOBILE RIDE PROBLEM
FIGURE 25

ANALOG STUDIES OF ELECTRIC POWER SYSTEM TRANSIENTS

The series of articles to follow demonstrate forcibly the strength and effectiveness of high-speed analog techniques applied to problems of speed governing and load-frequency control of large interconnected power systems.

This area is marked by the vast number and complexity of interactions. Earlier attempts at analysis were necessarily limited, incomplete, and largely inaccurate. The existence of analog tools thus offers new promise in the efficient generation and regulation of electrical energy.

The first four articles are extracts from four publications by Paynter, which, for ease of reading have been slightly rearranged from their original form. Also presented are important papers of Obradovic (in Yugoslavia) as well as Koenig and Schultz of Allis-Chalmers.

In particular, the "universal amplifiers" of Obradovic may have especial interest. These permit the change in form of the operation simply by switching input and feedback circuits.

ASCE TRANSACTIONS
Paper No. 2569
Vol. 118, 1953

SURGE AND WATER HAMMER PROBLEMS

By Henry M. Paynter,[1] J. M. ASCE

Synopsis

In the course of extensive investigations in the fields of speed and pressure regulation at the Massachusetts Institute of Technology (M.I.T.), at Cambridge, use has been made of certain electrical-fluid analogies. Also, as an aid to obtaining direct solutions while retaining the basically nonlinear features in the components of the systems being studied, an electronic computer has been found useful.

This paper describes the foundation for these techniques, the types of analogies, methods developed, and equipment employed, together with a few representative results and conclusions drawn from these studies. The particular cases of pressure transients in a uniform pipe and surges in a simple tank have been selected for discussion. However, no such restrictions are inherent in the methods developed.

Emphasis has been placed on the utilization of analog techniques to extend, refine, and clarify analytical procedures to secure a more thorough understanding of the basic hydraulic phenomena and to furnish results that are generally useful and accessible. Brief mention is made of some improved analytical procedures developed in connection with these studies.

Introduction

The interconnection of hydraulic and steam power plants within modern electric power networks gives rise to numerous complex problems concerning both the influence of load fluctuations and frequency control equipment on the stable operation of the generating units, and, conversely, the effects of the transient behavior of the hydraulic and mechanical components of the units on the performance of the electrical network. A representative hydroelectric system is shown schematically in Fig. 1, in which B is the gate opening, H is the head, P is the power, N is the speed, and Q is the discharge. As subscripts, c denotes "conduit," l denotes "transmission line," r denotes "reservoir," and t denotes "surge tank." In the course of this paper, use will be made of dimensionless incremental variables—the "per-unit" variables familiar to electrical engineers. These will generally be denoted by the lower case letters corresponding to the physical variables; thus, $b = \frac{\Delta B}{B_0}$, $h = \frac{\Delta H}{H_0}$, $p = \frac{\Delta P}{P_0}$, $n = \frac{\Delta N}{N_0}$, $q = \frac{\Delta Q}{Q_0}$, in which the numerators in each case represent changes from a reference condition and the denominators correspond to this reference or index condition, denoted by the subscript zero.

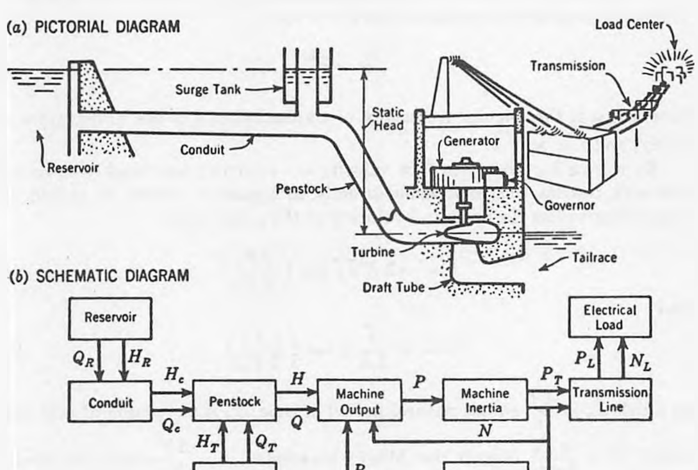


Fig. 1.—Typical Hydroelectric Installation

Most earlier investigations in the regulation field were not extended much beyond the point at which immediate and practical questions were answered. Thus, the classical researches into water hammer and surge phenomena that formed the subject of many pioneering investigations[2,3,4,5,6,7] remained adequate as long as the various generating stations were loosely tied and operated in relative isolation. However, with the growth of interconnection to the level at which nearly all the plants in a particular system and even entire systems are coupled together, these earlier methods of analysis and special solutions have become inadequate.

Straightforward extension and generalization of the pioneer studies are hampered by the presence of inescapable nonlinearities in the various physical components of the problem. These nonlinearities occur in the hydraulic elements of a hydroelectric plant, for example, in the friction loss in the conduit and penstock, the flow through the wicket gates and tank orifices, and the turbine power and efficiency characteristics. Basically, all physical systems are nonlinear and all analyses arising through linearization procedures are but approximate, agreeing with fact only for those cases in which the deviations resulting from nonlinearity are insignificant; in the present context such is usually the case for equilibrium stability determinations and for small changes in operating conditions.[8,9,10,11,12,13,14,15]

However, there are many instances in which linear treatments are not sufficiently accurate, including, for the hydraulic system, surges and pressure swings for large changes in demand flow. Problems such as these require graphical or numerical solutions, in which techniques are used similar to those employed for water hammer studies[4,6,16,17,18] or surge tank studies.[5,19,20,21] However, to these basic methods must be annexed expressions that reflect the actual plant and load characteristics as well. The tediousness and consumption of time inherent in these methods emphasize the need for more effective techniques for solution, without disregarding essential nonlinearities.

As a first step toward developing new techniques and refined solutions in this field, both for the purpose of extending basic knowledge and to answer immediate questions arising in practice, the M.I.T. Hydrodynamics Laboratory began, in 1947, an investigation into the basic dynamic phenomena involved in the transient performance of power system prime movers, which necessarily required research into the field of hydraulic transients. Certain of these studies are outlined here as well as in a related paper by the author.[22]

Analogs and Computers

One fruitful approach toward understanding these problems has been found in the use of electrical analogs and computers, by means of which solutions may be obtained rapidly and in immediately useful form. These devices fall naturally into two classes,[23,24,25] analogs and computers.

Analogs.—In analogs the prototype system, in effect, is duplicated by a model system whose equivalent components behave according to laws analogous to those governing the prototype.

Computers.—In computers the basic algebraic and differential equations are solved either by (*a*) Digital computation (numerical calculation by discrete steps) or (*b*) analog computation (involving calculation by continuous variation of analogous variables).

Comparisons.—Generally, then, the establishment of a formal analogy between the basic equations describing the behavior of two different physical systems permits the use of previously determined solutions for new cases. Indeed, this is one of the major benefits derived from mathematical analysis itself.

As problems become more complex, with more variables and system parameters, the utility of analog techniques becomes more marked. In particular, most mechanical (or dynamical) problems can be represented by suitable electrical analogs. Since electrical terminology and concepts have become highly developed in the fields of unsteady and periodic motion, these analogies will usually prove fruitful. For a detailed introduction to the basic structure of analog techniques the reader is referred to the literature.[26,27,28,29,30]

With respect to computers, some form of high-speed computer is useful if the system under study involves a large number of variables or many different solutions for varying parameters and conditions. Even linear analyses became awkward for these cases. Computers are also helpful if the system possesses one or more significant nonlinear features that render analytic solutions impossible.

Many problems in the field of speed and pressure regulation possess both of these attributes. In this paper the latter aspect has been emphasized, especially since a number of novel results have been obtained.

Some Physical Concepts

All continuous devices to transport energy over distances possess many features in common. Such mechanisms may be mechanical shafts, electrical transmission lines, pressure pipe lines, or open channels, all of which can be conceived of as a series of inertial elements linked together by elastic or flexible couplings capable of storing potential energy. In other words, these systems may be visualized in terms of a simple dynamical model consisting of mass cars

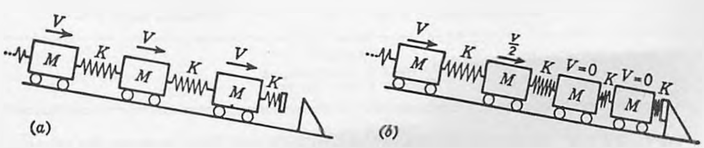


Fig. 2.—Mass-Spring Model

M and coupling springs K, as shown in Fig. 2. The energy terms associated with these elements are

$$\text{KE} = \tfrac{1}{2} M\, V^2 \qquad (1a)$$

and

$$\text{PE} = \tfrac{1}{2} K\, X^2 \qquad (1b)$$

in which KE is kinetic energy and PE is potential energy when the cars are moving with a velocity V and the springs are compressed a distance X.

If the leading car is suddenly stopped, the springs must each in turn absorb the kinetic energy of the adjacent car, and convert it into potential energy. Furthermore, a definite time delay occurs at each unit, in order to decelerate the mass and build up the compression. These two features may be obtained from the conservation of energy and momentum principles as follows:

Energy—

$$\text{KE} = \text{PE} \qquad (2a)$$

$$\tfrac{1}{2} M\, V^2_{\max} = \tfrac{1}{2} K\, X^2_{\max} \qquad (2b)$$

or

$$X_{\max} = \sqrt{\frac{M}{K}}\, V_{\max} = Z_0\, V_{\max} \qquad (3)$$

in which Z_0 = surge impedance $= \sqrt{\frac{M}{K}}$.

Momentum—

$$M\, \Delta V = F\, \Delta T \text{ gives } M\, V_{\max} \sim K\, X_{\max}\, \Delta T \qquad (4)$$

From which

$$\text{delay } \Delta T \sim \frac{M\, V_{\max}}{K\, X_{\max}} = \frac{M}{Z_0\, K} = \sqrt{\frac{M}{K}} \qquad (5)$$

and

$$c \sim \frac{1}{\Delta T} \sim \sqrt{\frac{K}{M}} \qquad (6)$$

in which c is the propagation velocity. In the case of elastic pressure waves, the corresponding values for the surge impedance Z_0 and propagation velocity c become

$$Z_0 = \frac{c}{g} = \frac{\Delta H}{\Delta V} \qquad (7)$$

in which g is the acceleration due to gravity, and for unbounded fluid,

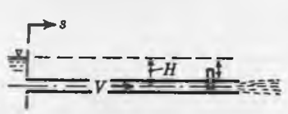
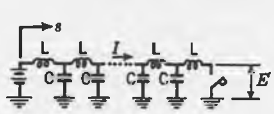
$$c = \sqrt{\frac{E_b}{\rho}} \qquad (8)$$

in which ρ is the density, with Z_0 relating the change in head to the change in velocity. These concepts prove generally useful for all types of transmission problems.

Electrical Analogy

It is demonstrated in Table 1 that a uniform, frictionless pipe line is formally analogous to a uniform, dissipationless electrical transmission line. Thus the

TABLE 1.—Electrical-Hydraulic Analogy

Line	(a) Hydraulic System: Uniform frictionless pipe line	(b) Electrical System: Uniform lossless transmission line
		
1	Inertia equation: $-\frac{\partial H}{\partial s} = \frac{1}{g}\frac{\partial V}{\partial t}$	Voltage drop: $-\frac{\partial E}{\partial s} = L\frac{\partial I}{\partial t}$
2	Continuity equation: $-\frac{\partial V}{\partial s} = \frac{w}{E_b}\left(1 + \frac{E_b}{E_y}\frac{D}{e}\right)\frac{\partial H}{\partial t}$	Line charging: $-\frac{\partial I}{\partial s} = C\frac{\partial E}{\partial t}$
3	Wave equations: $\frac{\partial^2 H}{\partial t^2} = c^2\frac{\partial^2 H}{\partial s^2}$	Wave equations: $\frac{\partial^2 E}{\partial t^2} = c^2\frac{\partial^2 E}{\partial s^2}$
4	$\frac{\partial^2 V}{\partial t^2} = c^2\frac{\partial^2 V}{\partial s^2}$	$\frac{\partial^2 I}{\partial t^2} = c^2\frac{\partial^2 I}{\partial s^2}$
5	Propagation velocity: $c = \sqrt{\frac{E_b/\rho}{1 + \frac{E_b}{E_y}\frac{D}{e}}}$	Propagation velocity: $c = \sqrt{\frac{1}{LC}}$
6	Surge impedance: $Z_0 = \frac{c}{g}$	Surge impedance: $Z_0 = \sqrt{\frac{L}{C}}$
	Reflections:	Reflections:
7	Open end: Pressure node: $\Delta H = 0$	Grounded end: Voltage node: $\Delta E = 0$
8	Reflection factor: $r = -1$	Reflection factor: $r = -1$
9	*Closed end*: Velocity: $\Delta V = 0$	Open end: Current node: $\Delta I = 0$
10	Reflection factor: $r = +1$	Reflection factor: $r = +1$

ANALOGY
Head H ⟷ *Voltage E*
Velocity V ⟷ Current I

water hammer waves and surges of the hydraulic engineer became the traveling waves, electrical surges, and switching transients of the electrical engineer. The fruits of this analogy are (1) the ability to make use of the many useful tools and concepts developed by the electrical engineer since 1900 and (2) the realization that, when viewed in this light, hydraulic and electrical engineers have similar problems and a common language.

In short, velocity V is analogous to current I, and head H is analogous to voltage E.

As long as dissipation phenomena are negligible in both systems these analogies are strictly valid. Nevertheless, in the practical case resistance and losses must be taken into account. For the complex forms of pipe networks, even in steady flow, computers are useful. With steady flow one successful method of making friction proportional to an exponential power of the flow is embodied in the pipe flow analyzer.[31] For unsteady flow problems, using electronic computers, it is possible to account for this frictional effect satisfactorily through special components described subsequently.

However, even in the loss-free form, many concepts of electrical engineering have direct application to problems of unsteady flow. For example, the ordinary alternating-current vector diagrams furnish valuable clues to the behavior of pipe lines subjected to alternations of flow and pressure. Moreover, the use of surge impedance and the other generalized circuit constants can be extremely profitable.

Resonance Phenomena

An excellent example of the use of electrical concepts for unsteady flow problems can be shown in connection with the resonance phenomena associated with the rhythmic motion of a gate or valve at the end of a pipe line. In such cases, the resultant pressure fluctuations along the pipe may often be substantially in excess of the maximum at the valve for the full gate stroke, for which the governor or gate mechanism timing is usually specified and adjusted. It will also be of interest to compare the results obtained by these techniques with those obtained by following the classical methods.

The solutions of the electrical wave equations for long alternating-current transmission lines, as derived in any standard work on power transmission,[32,33,34] may be put in the form:

$$\vec{E}_x = \vec{E}_s \cosh(\overrightarrow{\alpha x}) - \vec{I}_s \vec{Z}_0 \sinh(\overrightarrow{\alpha x}) \qquad (9)$$

and

$$\vec{I}_x = \vec{I}_s \cosh(\overrightarrow{\alpha x}) - \left(\frac{\vec{E}_s}{\vec{Z}_0}\right) \sinh(\overrightarrow{\alpha x}) \qquad (10)$$

in which x is the distance along the transmission line from the sending end; $\vec{E}_x$ is the voltage vector at point x; $\vec{E}_s$ is the voltage vector at the sending end; $\vec{I}_x$ is the current vector at point x; $\vec{I}_s$ is the current vector at the sending end; $\vec{\alpha}$ is the vector propagation constant; and $\vec{Z}_0$ is the vector surge impedance.

If the line is short-circuited (analogous to open reservoir) at $x = L$, then $E_L = 0$, and the sending end voltage and current are related by the expression,

$$\vec{E}_s = \vec{I}_s \vec{Z}_0 \tanh(\overrightarrow{\alpha L}) \qquad (11)$$

Furthermore, if the line resistance is negligible (analogous to the frictionless pipe), the propagation constant $\vec{\alpha}$ is given by

$$\vec{\alpha} = \vec{j}\,\frac{\omega}{c} \qquad (12)$$

in which ω is the angular frequency of transmission; c is the propagation velocity; and $j = \sqrt{-1}$.

By the analogy between flow velocity ⟷ current, and head ⟷ voltage, one may rewrite this expression directly in hydraulic terms, to obtain corresponding values for head and velocity at the gate, thus:

$$\vec{h} = -2\, K\, \vec{v}\, \vec{j} \tan\left(\frac{\pi}{2}\frac{T_n}{T_0}\right) \qquad (13)$$

and

$$\vec{v} = +\frac{\vec{h}}{2\,K}\, \vec{j} \cot\left(\frac{\pi}{2}\frac{T_n}{T_0}\right) \qquad (14)$$

in which $T_n = \frac{4\,L}{c}$ equals natural period of pipe; T_0 is the period of gate oscillation; $K = \frac{c\, V_0}{2\, g\, H_0}$ equals the Allievi parameter; $h = \frac{\Delta H}{H_0}$ equals the relative head change; and $v = \frac{\Delta V}{V_0}$ $\left(\text{numerically equal to } \frac{\Delta Q}{Q_0}\right)$ equals the relative velocity (or flow) change.

Eqs. 13 and 14 demonstrate that the head variation will always lag behind the flow variation by an angle of 90° and that the amplitude ratio of head relative to flow is given by the expression:

$$\left[\frac{h}{v}\right] = 2\, K \tan\left(\frac{\pi}{2}\frac{T_n}{T_0}\right) \qquad (15)$$

For small variations, however, the normalized flow v may also be found from the expression:

$$(1 + v) = (1 + b)\sqrt{1 + h} \qquad (16)$$

in which $b = \Delta B/B_0 =$ relative gate opening, or, approximately

$$v = b + (\tfrac{1}{2})\, h \qquad (17)$$

Combining Eqs. 15 and 17,

$$\left[\frac{h}{b}\right] = -2\cos\phi_h \text{ (with } h \text{ lagging } b \text{ by angle } \phi_h) \qquad (18)$$

and

$$\tan\phi_h = -\frac{1}{K}\cot\left[\frac{\pi}{2}\frac{T_n}{T_0}\right] \qquad (19)$$

A graphical representation of these expressions is shown in Figs. 3 and 4. These results were first obtained by D. Gaden[8] using interval equations derived by L. Allievi.

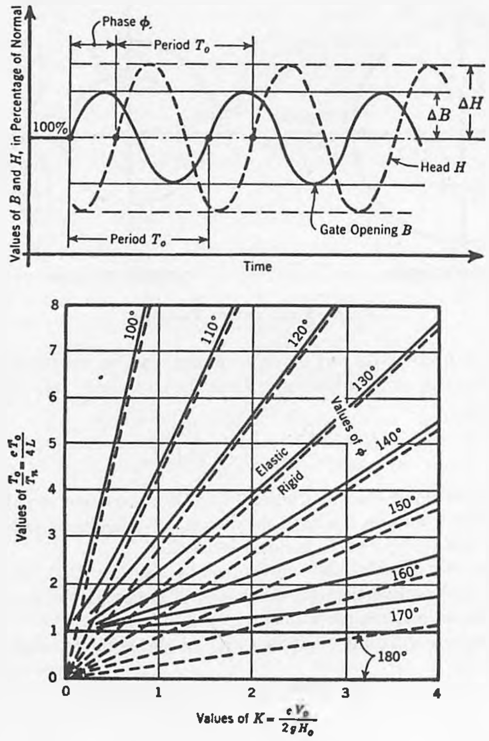


Fig. 3.—Water Hammer Resonance

For example, in Fig. 4 assume $K = 5$ and $\frac{T_0}{T_n} = 2.5$ to locate the point A; then extend a line OB through point A to point B on the semicircle. The values $\frac{h}{b} = 1.92$ and $\phi_h = 164°$ are read directly. The significance of these curves arises from a consideration of the variations in velocity (or flow) and input torque (or power) caused by oscillations of gate opening. The velocity and input torque in the normalized (or per-unit) dimensionless form may be expressed by the equations:

Flow—

$$(1 + v) = (1 + b)\sqrt{1 + h} \qquad (20)$$

and power—

$$(1 + p) = (1 + h)(1 + v) = (1 + b)\sqrt{(1 + h)^3} \qquad (21)$$

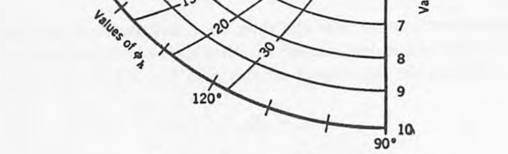


Fig. 4.—Resonance Diagram

For small increments Eqs. 20 and 21 become

$$v = b + (\tfrac{1}{2})\,h \qquad (22)$$

and

$$p = b + (\tfrac{3}{2})\,h \qquad (23)$$

These sinusoidally varying increments can be conceived as rotating vectors (as illustrated in Fig. 5) and the resulting amplitude ratios, making use of the previously determined expressions, become

$$\left[\frac{v}{b}\right] = \sin\phi \qquad (24)$$

and

$$\left[\frac{p}{b}\right] = \sqrt{\tfrac{1}{2}(5 + 3\cos 2\phi)} = \sqrt{1 + 3\cos^2\phi} \qquad (25)$$

with $\phi = \phi_h$, the lagging angle between the gate vector b and the head vector h, as before, increasing from 90° (for $T_0/T_n \rightarrow \infty$) up to 180° (for $T_0/T_n = 1$). Thus, both the velocity v and the power p always lag behind the gate b, as demonstrated in Fig. 5, in which it should be noted that the tips of the v-vector and p-vector, as well as the multiples of h, all execute circular loci as the frequency increases.

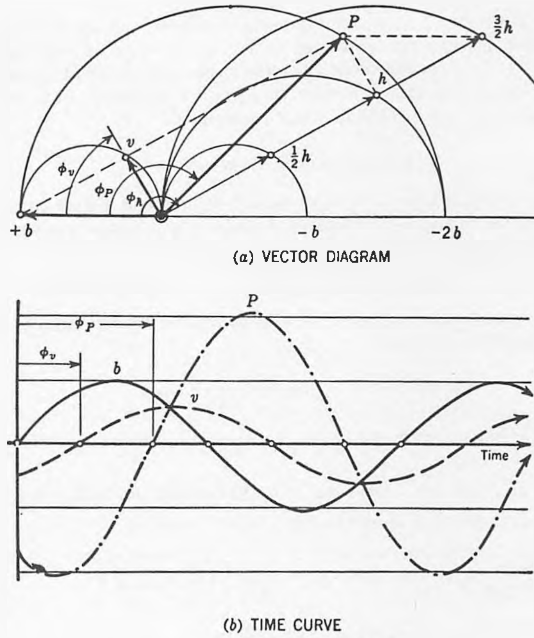


Fig. 5.—Response Curves

It may be concluded from these analyses that the input power to the prime mover inherently lags behind the gate opening, no matter how slowly the gate is moved, with increasing phase shift ϕ_p for higher frequencies; this circumstance is critical in determining the transient performance of hydroelectric units and clearly indicates one of the fundamental limitations of such units for participation in system load and frequency control programs.

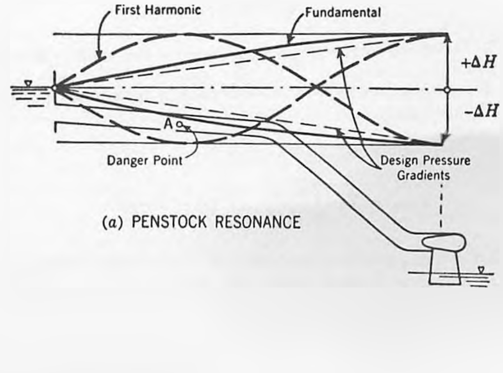
In the Appendix, a numerical example is given showing the use of Figs. 3, 4, and 5 and comparing the results with those obtained by conventional methods.

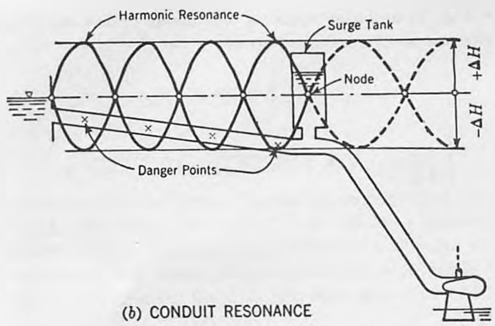
Application of Results

From the foregoing it is evident that oscillations of the gate or valve produce standing waves in the pipe or systems of pipes with the familiar loops and nodes of all vibratory continuous systems. At the gate end there will usually be a pressure loop, and at the upstream end there is a node at the free surface.

From purely intuitive considerations it is clear that only resonances of the fundamental and odd harmonics can occur under such conditions; but for all of these cases, the full pressure variation will occur at each loop, making the resonance of the harmonics, in general, more dangerous than that of the fundamental.

Moreover, it can be shown that it is possible, under certain conditions, to produce pressure swings along the pipe (through oscillations of the gate that are consistent with the governor timing and stroke) nearly as large as the design values established by the conventional methods. These extreme fluctuations can exist both for the fundamental and the lower harmonics. For example, the pipe line illustrated in Fig. 6(a) would be in serious danger of bursting or collapse near point A under the conditions shown.





(b) CONDUIT RESONANCE

Fig. 6.—Cases of Resonance

Another dangerous case of simultaneous resonance can arise between the gate, penstock, and conduit as shown in Fig. 6(b). In this case, the surge tank will not serve to trap the elastic waves since a node exists at its base, and, therefore, renders the tank inoperative. Failure resulting from excessive positive or negative pressures may occur at the loops.

Useful Surge Concepts

Considerable insight into the analysis and behavior of simple surge tanks can be gained through consideration of the energy principles involved in the elementary problem of sudden, full-load rejection with a cylindrical tank. The basic dimensions and physical constants of such a tank are indicated in Fig. 7.

For the limiting case of a frictionless conduit, the conservation of energy may be expressed in the form:

$$\text{KE} + \text{PE} = \text{constant} = (\text{KE})_0 \qquad (26)$$

or, specifically,

$$\tfrac{1}{2} M_c V^2 + M_t g h = \tfrac{1}{2} M_c V_0^2 \qquad (27)$$

with M_c and M_t being the mass of water in the conduit and tank, respectively.

With the data of Fig. 7 there results

$$\frac{1}{2}\left(\frac{w}{g} A_c L_c\right) V^2 + (w A_t Y)\frac{Y}{2} = \frac{1}{2}\left(\frac{w}{g} A_c L_c\right) V_0^2 \qquad (28)$$

or

$$\alpha V^2 + \beta Y^2 = \alpha V_0^2 \qquad (29)$$

in which α and β are constants.

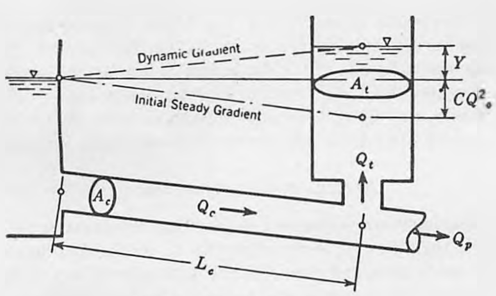


Fig. 7.—Surge Tank Elements

This expression is therefore the equation of an ellipse which can be always transformed into a circle by the process of normalization, that is, by making horizontal and vertical scales homogeneous. Thus, dividing through by the term (αV_0^2),

$$\frac{\alpha V^2}{\alpha V_0^2} + \frac{\beta Y^2}{\alpha V_0^2} = 1 \qquad (30)$$

$$\left(\frac{V}{V_0}\right)^2 + \left(\frac{Y}{Y_0}\right)^2 = 1 \qquad (31)$$

in which

$$Y_0 = V_0 \sqrt{\frac{A_c L_c}{A_t g}} = V_0 Z_0 \qquad (32)$$

with $Z_0 = \sqrt{\frac{A_c L_c}{A_t g}}$ defined as the surge impedance of the tank. This situation is sketched in Fig. 8(a), in which it is seen that a maximum positive surge occurs at point 2, for which

$$Y_{max} = Y_0 \qquad (33)$$

or

$$Y_{max} = V_0 \sqrt{\frac{A_c L_c}{A_t g}} = \text{“free” surge} \qquad (34)$$

From Fig. 8(a) it is also clear that the tank will oscillate indefinitely as a result of continuous interchange of energy without dissipation.

In the practical case, however, conduit friction will modify the basic energy expression into the form:

$$\text{PE} = \text{KE} + (\text{work done against friction}) \qquad (35)$$

Thus the total change of water level in the tank will be increased by the presence of friction, although the surges will dampen out successively as a result of dissipation, as shown in Fig. 8(b).

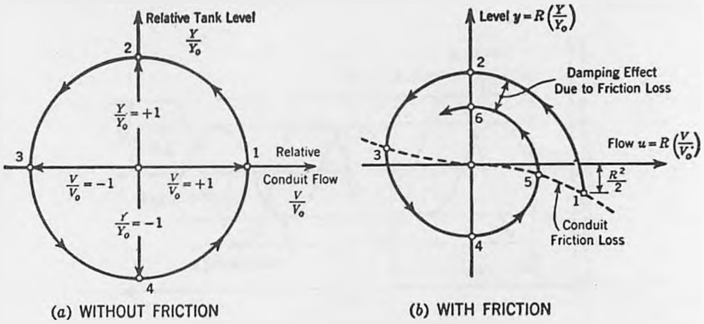


Fig. 8.—Surge Energy Diagrams

The effect of friction, for any given transient, can be measured if the ratio of friction loss to free surge is known. Thus the parameter may be defined as

$$R = \frac{2 H_f}{Y_0} = \frac{2 H_f}{V_0}\sqrt{\frac{A_t g}{A_c L_c}} \qquad (36)$$

in which H_f is the rated conduit friction loss. For a particular value of R the surge oscillations would be modified as shown. It should be noted that the initial friction loss relative to Y_0 is given by $R/2$.

It has been found useful and convenient to multiply both the horizontal (conduit flow = u) and vertical (tank level = y) scales by the constant factor R, to obtain the normalized plots shown in Fig. 9.

Further analysis will show that in terms of the notation used in the diagram—

$$\text{PE} = \text{KE} + (\text{work done}) \qquad (37)$$

becomes—

$$\frac{z^2}{2} = \frac{u^2}{2} + \int f\,dz \qquad (38)$$

Eq. 38 resolves the problem of computing surges into the evaluation of the work integral on the right-hand side, corresponding to the areas A and B in Fig. 9. This can be done by several different procedures and has been successfully achieved using the electronic computer.

The value of the particular formulation of the problem as implied by Eqs. 37 and 38 lies in the considerable precision that may be derived using even approximate methods of integration. Employing such techniques, tables of surges that are accurate to four significant figures have been prepared for the normal range of tank design. Similar procedures have been developed for restricted orifice and differential tanks.

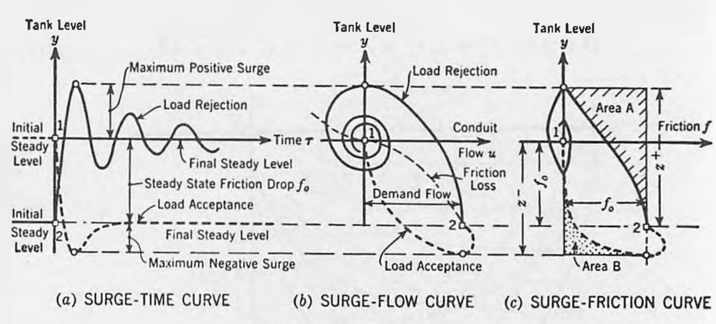


Fig. 9.—Surge Tank Transients

It is also of interest to note that the tank constant R serves as a dimensionless hydraulic similitude index similar to the Reynolds, Froude, and Mach numbers. In other words, two simple surge tanks A and B will be dynamically similar if

$$R_a = R_b \qquad (39)$$

or

$$\frac{2 H_{fa}}{V_a \sqrt{\frac{A_{ca} L_{ca}}{A_{ta} g}}} = \frac{2 H_{fb}}{V_b \sqrt{\frac{A_{cb} L_{cb}}{A_{tb} g}}} \qquad (40)$$

This conclusion agrees with the tank similitude analyses of A. H. Gibson[35,36] and W. F. Durand.[37]

Surge Tank Problems

The excellent early studies of R. D. Johnson[5] were essentially limited only to establishing design constants and dimensions for fixed positive and negative surge heights. Mr. Johnson did not attempt the determination of the various relationships between diverse values of the many system constants, but rather established working design values for a particular choice of these constants; these restrictions appreciably simplified the analysis of performance, leading directly to the type of results that Mr. Johnson sought.

However, many tanks now installed are often required to operate under conditions quite different from those selected for design, and the behavior under these conditions has sometimes been unfavorable and unexpected. The problem of determining the response of tanks under all reasonable disturbances that might be encountered has been neglected in the past, largely because of the altogether forbidding length of time required by even the most elementary investigations using conventional methods.

With these needs in view, the research efforts reported in this paper have been devoted, in the surge tank field, to the following problems concerning present and proposed tank installations:

1. The behavior of tank systems for operating conditions other than those used for design; and
2. The true stability margins for surge tanks subjected to specified disturbances of appreciable magnitude.

It has long been realized that the actual transient behavior of surge tanks is considerably influenced by the action of the turbine governor and other plant and load characteristics. However, the magnitude of this effect depends on the comparative values of the governor timing and the natural period of the tank. Investigations by the writer have established the limitations and range of validity of three simplifying assumptions as to plant effects, as illustrated in Fig. 10—constant gate opening, constant discharge, and constant input power.

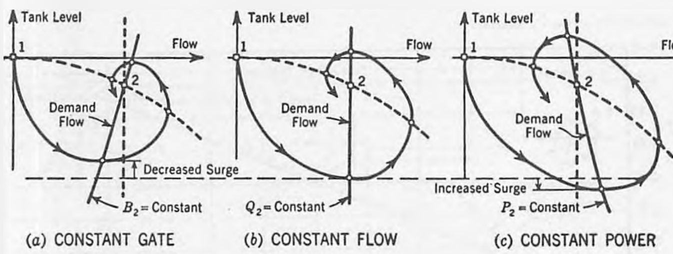


Fig. 10.—Demand Flow Assumptions

1. *Constant Gate Opening (B_2 = Constant) at the Final Steady-State Value.*—This is the situation most representative of a unit under manual control by the operator, and produces the least surge, under otherwise comparable conditions, as a result of increased positive damping action.

2. *Constant Discharge (Q_2 = Constant) at the Final Steady-State Value.*—This was the basis of the methods of Mr. Johnson and most other investigators and produces a greater surge than assumption 1.

3. *Constant Input Power (P_2 = Constant) to the Runner at the Final Steady-State Value.*—This assumes that efficiency changes are small and the turbine regulates over a time that is very short compared to the tank period. This assumption gives materially greater surges, and can even lead to instability, as explained subsequently in Fig. 14.

This method, when suitably linearized, was the basis of the classical tank stability analyses[7] that have been modified by later work[38,39,13] of J. Calame and D. Gaden, G. Evangelisti, and the writer.[39]

In addition to the conventional problem of a sudden or step-change in demand conditions, the electronic computer has been applied successfully in the examination of two other types of problems—synchronous or resonant load changes and true static and dynamic stability margins.

4. *Synchronous or Resonant Load Changes, of the Pulsing and Oscillatory Type.*—Such a loading program may result in abnormally high surges and pressure swings as the period of the disturbance approaches the natural period of the tank. Previously, this problem had been investigated only for a handful of cases.[40]

5. *True Static and Dynamic Stability Margins for Tanks Operating Both Under Steady-Load Conditions and Under Appreciable Changes in Demand.*—The simplified techniques of the linear stability theory, for example, cannot account for the known augmentation to stability through the differential principle, giving rather the same value for the critical tank area for all common types of tank. Moreover, the engineering concept of stability implies far more than the various classical definitions of mathematics and mechanics. If a surge tank is to be stable in a real sense, it must be able to reach and maintain an effective steady operating condition in a sufficiently short time after any reasonable load disturbance. Simply having the oscillations die out in a time just short of eternity is not adequately meeting these requirements, yet quantitative, generally applicable knowledge has been lacking in these matters.

With the conclusion of a first stage in the current research program at M.I.T., it is possible to give a few preliminary examples of the many results obtained.

Simple Tank Equations

For computational purposes, it is usually convenient to return to the basic differential equations in the physical form; for the simple tank, following the nomenclature of Fig. 6(*b*),

Continuity—

$$-A_t \frac{dY}{dt} = Q_p - Q_c \qquad (41)$$

Acceleration—

$$-\frac{L_c}{A_c g}\frac{dQ_c}{dt} = Y + C\,Q_c^2 \qquad (42)$$

Assuming a uniform conduit and tank, with square law friction, these may be normalized to the form:

Continuity—

$$-\frac{dy}{d\tau} = v - u \qquad (43)$$

Acceleration—

$$-\frac{du}{d\tau} = y + \tfrac{1}{2}u^2 \qquad (44)$$

in which $y = R\,(Y/Y_0)$; $u = R\,(Q_c/Q_0)$; $\tau = 2\pi t/T_{ct}$; $v = R\,(Q_p/Q_0)$; $R = 2\,H_f/Y_0$; and $T_{ct} = 2\pi\sqrt{\frac{A_t L_c}{A_c g}}$ = free period. As a particular case one

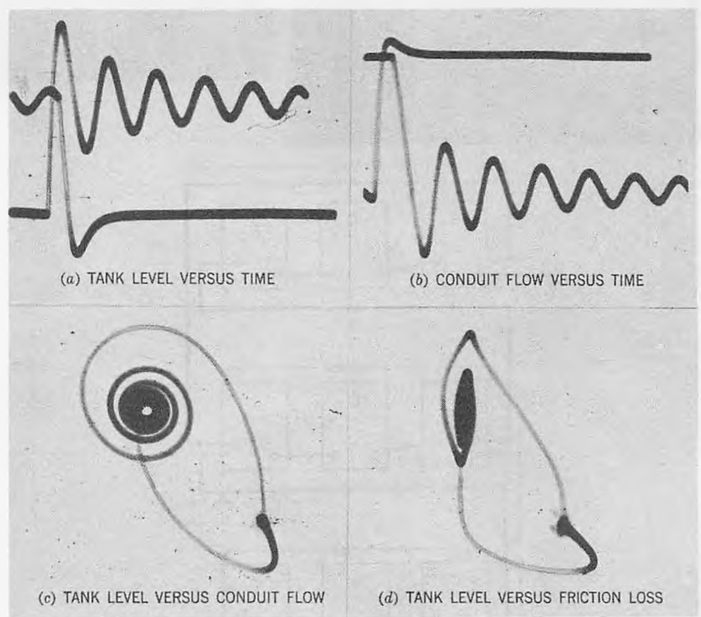


Fig. 11.—Representative Computer Solutions of Surge Tank Transients

might take the transient from an initial steady state $v_1 = R_1 = 1.7$ to a final steady state $v_2 = R_2 = 0$ that might be either full or partial rejection to zero for a great number of practical instances. This transient is indicated in Figs. 9 and 11.

Thus, any given plant can be specified by the value of the parameter R for rated conditions, and the particular transients are specified by the initial and final steady states $u_1 = v_1 = R_1$ and $u_2 = v_2 = R_2$, respectively.

The work of W. E. Milne in this field[41,42,43] involved a complete investigation by numerical methods of the system of Eqs. 43 and 44. Mr. Milne also described the general case of a quadratically damped vibration; but his solutions were only for a step change in demand flow, assumption 2. Nevertheless, his tables use the identical normalized variables as outlined in this paper and it is hoped that this paper may serve as another introduction to Mr. Milne's valuable work.

Nonlinear Features

The two principal nonlinearities of the simple surge tank equations, as outlined, lie in the demand flow characteristic v and the friction loss term $u^2/2$.

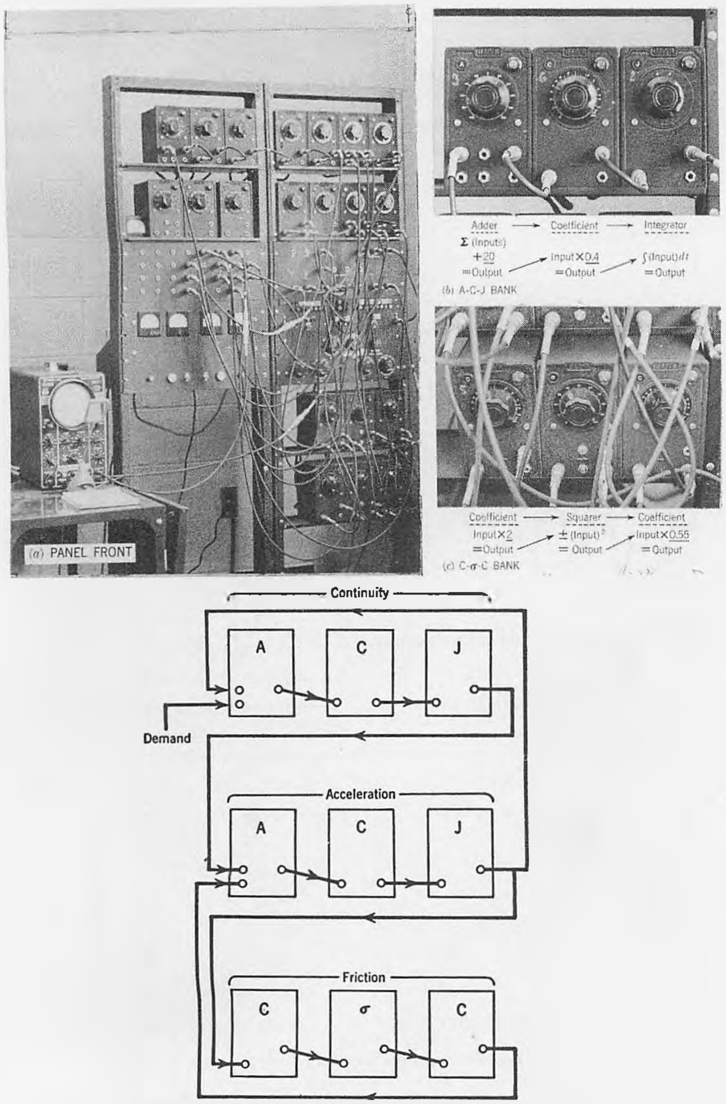
1. *The Demand Flow Characteristic v.*—This factor may have any of the three forms indicated in Fig. 10, and may vary with time as well because of the finite governor time.

2. *The Friction Loss Term 1/2 u^2.*—This term has been assumed to be quadratic since analysis is somewhat simpler for this case; but in general an exponent different from 2 may have to be taken into account. Although numerical or graphical solutions of particular instances may be desirable, it has been found possible, by means of the electronic analog computer, to calculate complete and universal solutions of the surge tank equations for all cases

normally arising in practice.

Computer Setup

The electronic computer used at the M.I.T. Hydrodynamics Laboratory for research in the regulation field is of the high-speed analog type, and is ideally suited to this type of analysis. The basic computing epoch is four milliseconds and the system variables are represented by analogous voltages functionally related through appropriate computer components that solve the fundamental equations.[44,45,46]



(d) COMPUTER BLOCK DIAGRAM

Fig. 12.—Arrangement of Computer

The general arrangement of the computer is illustrated in Fig. 12 along with operational details of representative banks of components. In particular, Fig. 12(*d*) shows the arrangement of components and the block diagram for simple tank transients following sudden changes in demand flow. It is of especial interest to note the $(C - \sigma - C)$-bank and its behavior as outlined in Fig. 12(*c*); it is this bank that represents the square-law friction.

Typical computer solutions, as photographed from the oscilloscope screen, are shown in Fig. 11. These may be compared to the solution diagrammed in Fig. 9. All problems may be set up in such a way that calibration and scaling can be made directly from the final photographic records. This arrangement has been found very successful in practice and constitutes another strong argument for the normalization procedures that make it possible.

Analysis and Results

It is worthwhile to present two examples of the many charts that have been assembled from the computer studies on simple surge tanks. Similar curves have been obtained for other types of tanks.

The first specimen, shown in Fig. 13, is a family of curves, plotted similar to the method indicated by Mr. Durand,[37] from which the maximum or minimum tank level resulting from any load acceptance or rejection transient with any simple tank may be computed, subject to the assumptions of constant demand flow and square-law friction. This plot may be compared with the earlier Johnson plot.[47]

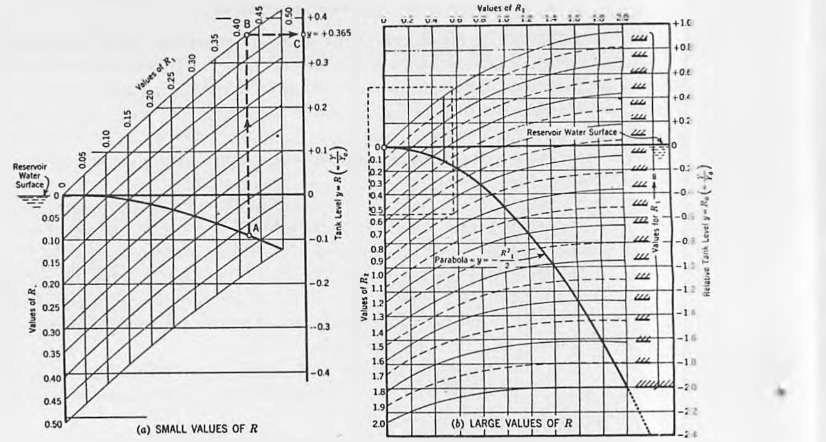


Fig. 13.—Simple Tank Surge Curves

The second curve, Fig. 14, has to do with surge tank stability for full-load acceptance and compares the conditions of the classical criterion[7] with actual computed solutions assuming constant input power, plotting the required minimum value of the simple, tank constant R for any given value of conduit friction loss relative to static head, $h_c = H_{f0}/H_0$. The curve is seen to be composed of three sections corresponding to the three types of instability indicated. There are, in order—the oscillatory region, the drainage region, and the region of unstable steady state.

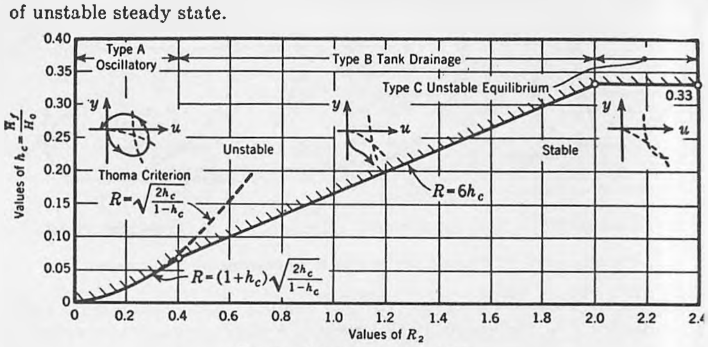


Fig. 14.—Surge Tank Stability

1. *Oscillatory Region* (*Small Frictional Effect*).—In this range stable response (below the curve) is indicated by damped oscillations and unstable behavior is evidenced by oscillations of growing magnitude.

2. *Drainage Region* (*Medium Frictional Effect*).—For this region satisfactory behavior results in rapidly damped return to full-load gradient conditions. However, if the tank is of insufficient size, the conduit flow would fail to accelerate as fast as the tank was being drained, and, in the practical case, the unit would evidence instability by failure to deliver the required power at the bottom of the swing. This might result in loss of synchronism.

3. *Region of Unstable Steady State* (*High Frictional Effect*).—When $h_c > 1/3$, the static equilibrium is unstable and the slightest disturbance would result in unstable response. Stating the fact another way, if the rated friction loss H_{f0} is greater than one third the static head H_0, stable regulation is impossible regardless of the presence or size of the tank.

Comments.—Of course, all these conclusions are based on the assumption of constant power input to the prime mover (case 3). However, the effect of the finite time taken by the governor to respond to the changes of head as well as the effects of interconnected generating units usually are such as to ameliorate these conditions. Nevertheless, the curves serve the useful purpose of design limitations and it should be noted that the conventional allowance for drooping efficiency curves applied to the Thoma criterion may still be inadequate for the larger values of h_c.

The use of these curves is shown in the Appendix for an actual surge tank installation in which field tests were made to check the design assumptions.[48]

Conclusion

It is hoped that this outline of the type of methods used in the research program at M.I.T. will stimulate new interest in the desirability of obtaining more complete and effective solutions of problems in the regulation field. Preliminary results have given ample evidence that much valuable information can be derived through use of appropriate analogies, whereby results in other fields (for instance, electrical engineering) may be profitably interpreted in an hydraulic context.

In addition, it is believed that adaptation of electronic computers to surge and water hammer studies offers new possibilities for obtaining results of general and permanent usefulness to problems whose solution is impractical by any other means.

ACKNOWLEDGMENT

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FROM THE AUTHORS CLOSING REMARKS . . .

In conclusion, the writer wishes to express his opinion as to the importance of finding and using practical methods of analysis and computation in solving engineering problems. In order for any method to qualify as practical for use by the engineer, it must be (*a*) simple and readily understood, (*b*) applicable to varied problems, (*c*) fast, (*d*) inexpensive, (*e*) available, (*f*) adaptable to the use of experimental data, (*g*) free as possible from manipulative errors, (*h*) easily checked, and (*i*) designed to give direct, easily-read results. Of the many tools to which these tests may be applied, three which pass with distinction are numerical methods, graphical methods, and analog methods.

Both Mr. Strowger and Mr. Rich have asserted in their discussions and have demonstrated in their many other writings that any problem involving hydraulic (or other) transients can be solved with simple techniques of arithmetic integration. For the same type of problem, the writer has developed what he considers to be an excellent method of graphical analysis. The paper attempted to outline the potentialities of analog concepts and analog computers, not only for schools and abstract analysts, but also for operating companies, equipment manufacturers, and consulting firms who have down-to-earth problems in design and development. In both these areas the commercially available, low-cost analog computer can play an important role. However, analog concepts, as distinguished from the actual computers, cost only the time to be learned but may reward the "investor" beyond his expectations.

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METHODS AND RESULTS FROM M.I.T. STUDIES IN UNSTEADY FLOW

By Henry M. Paynter,* Member

(Presented at a meeting of the Hydraulic Section of the Boston Society of Civil Engineers, held on November 7, 1951.)

General Introduction

With the publication during the summer of 1951 of the first American *treatise* on hydraulic transients by George R. Rich [1],† and with the welcome visit to M. I. T. during the following winter of Professor Giuseppe Evangelisti, of Bologna, Italy, a renowned authority on unsteady flow, it has seemed fitting to honor these occasions by presenting a brief survey of recent and current work of practical interest in the field of unsteady flow and hydraulic transients, carried on by the Hydrodynamics Laboratory of the M. I. T. Department of Civil and Sanitary Engineering.

This paper is divided into three parts, covering three phases of the program: Part 1—Regulation and Governing of Hydroelectric Plants; Part 2—Graphical Solutions of Transient Problems; Part 3— Flood Routing by Admittance Methods. It is hoped that each part is sufficiently complete in itself to provide information of value and interest to a broad section of engineering fields. However, it must be stated in all fairness that most of the results presented here can be taken only as samples of the material that has been or will be published later in greater detail; numerous references are given to the pertinent literature.

Part 1—Regulation and Governing of Hydroelectric Plants

Introductory

The research program in this field at M. I. T. places particular emphasis on the transient behavior of hydro plants subjected to both large and small changes in operating conditions, taking into account actual performance characteristics of the plant components. The purpose of these studies has been three-fold:

a) Development of an overall rational analysis of hydro plant transients with faithful attention to all important variables and characteristics;
b) Application and refinement of mathematical tools of analysis, including machine computation, with consideration of possible extension of these methods to related fields;
c) Execution of systematic general studies and presentation of results in readily available and generally useful form for application by engineers to design calculations.

As an example of the first objective, there is outlined in the paragraphs to follow the methods of analysis whereby general investigations of surge tank transients are made possible, with permanently useful results.

The second aim finds exemplification in Parts 2 and 3 of this paper, wherein concepts originating from the regulation studies have been applied to other unsteady flow problems.

As examples of the nature of the general results obtained and the form into which they have been put, a brief treatment is given of two topics of interest in hydro plant regulation:

a) A new surge formula for simple surge tank design.
b) Optimum governing of hydroelectric units.

Scope of Research on Regulation Problems

The ends of this program of study have required the consideration of the effects on transient behavior of the following plant and system components as indicated in Fig. 1:

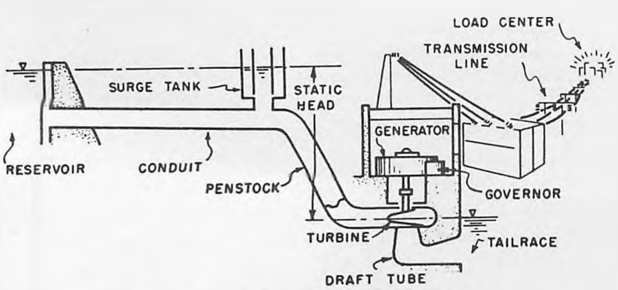

Fig. 1.—Representative Hydroelectric Plant.

I— FLOWLINE	II— GENERATING UNIT	III— SYSTEM NETWORK
Including:	*Including:*	*Including:*
Reservoir	Turbine	Transformers
Conduit	Governor	Transmission lines
Surge tank	Generator	System regulation
Penstock	Voltage regulator	Load characteristics
Scroll case		
Draft tube		
Tailrace		

Morever, it has been necessary to treat not only the case of a hydro plant operating alone, but also to investigate the transient behavior of plants operating in parallel with steam or other hydro stations. These analyses have been primarily concerned with:

a) Hydraulic transients: surges and water hammer
b) Machine transients: speed regulation and governing stability
c) System governing: prime mover paralleling and load-frequency control.

A hydroelectric plant under transient conditions following appreciable disturbances manifests a highly ***non-linear*** response (in the sense that the behavior depends on the magnitude of the changes). This feature, in addition to the overall complexity, makes general solutions of these problems intractable by ordinary mathematical means, and has led to the use of a wide variety of tools, among them:

a) Mathematical analysis
b) Field measurements and laboratory experiments
c) Electrical analogies
d) Electronic computer and differential analyser studies
e) Graphical and numerical solutions.

Application of these techniques has led to the successful general treatment of a substantial part of the problem of transient performance and operating stability of hydroelectric plants, thus extending the pioneer work of other American and European investigators in the field [2] [3] [4] [5] [6] [7] [8] [9]. Chief among the topics treated, the following might be listed:

I—Surge Tank Studies:
a) Normalization of equations
b) Simple tank transients
c) Throttled tank transients
d) Differential tank transients
e) Surge tank stability
f) Pulsing and resonance

II—Water Hammer Studies:
a) Normalization of equations
b) Effects of friction and gate stroke on pressures
c) Efflux energies
d) Resonance and stability problems
e) Surge tank and governor effects.

III—Machine and Governor Studies:
a) Normalization of equations
b) Stability and damping of governors
c) Optimum governor settings
d) Effects of surges and water hammer
e) Effects of turbine characteristics
f) Effects of system and load characteristics
g) Improvements in governing.

IV—System Governing and Load-Frequency Control:
a) Normalization of equations
b) Parallel operation of prime movers
c) Electro-mechanical transients
d) Load-frequency control
e) Improvements in system regulation.

These topics are all treated in some detail in other works and papers of this writer [10] [11] [12] [13] [14] [15].

Surge Tank Equations

In terms of the nomenclature of Fig. 2, the three fundamental equations describing the behavior of a cylindrical *differential* surge tank may be written as follows:

$$\text{Continuity:} \quad Q_c = A_t \frac{dY}{dt} + A_r \frac{dX}{dt} + Q_p \qquad (1.1)$$

$$\text{Acceleration:} \quad \frac{L_c}{gA_c}\frac{dQ_c}{dt} + K_c \mid Q_c \mid Q_c + X = 0 \qquad (1.2)$$

$$\text{Throttling:} \quad X = Y + K_t A_t^2 \left| \frac{dY}{dt} \right| \frac{dY}{dt} \qquad (1.3)$$

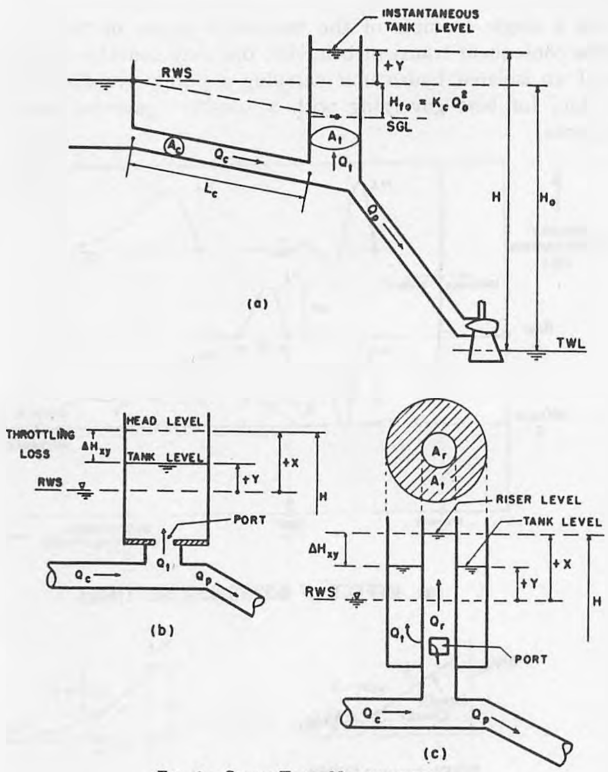

Fig. 2.—Surge Tank Nomenclature.

These equations are valid so long as neither tank nor riser spill, and so long as the port remains fully submerged. The restriction to "quadratic" conduit and throttling losses may be removed by considering the loss coefficients K_c and K_t as variables rather than constants. The purpose of the absolute value brackets on these loss terms is to indicate the necessary reversal of sign of the loss upon reversal of flow. The variables and constants might be listed as follows:

(A)—Variables

Flows:		Levels:	
Conduit Flow:	Q_c;	Tank Level:	Y
Penstock Flow:	Q_p;	Riser Level:	X

(B)—Constants

Areas:		Length:		Losses:	
Conduit Area:	A_c;	Conduit Length:	L_c;	Conduit Loss:	K_c
Tank Area:	A_t;			Tank Throttling:	K_t
Riser Area:	A_r;				

The simple and restricted-orifice tanks (the latter hereafter called a *throttled* tank) are merely special cases of the differential tank. This follows from the suppression of the appropriate constants in equations (1.1) (1.2) (1.3). In particular, if $A_r = 0$, then the tank would become a throttled tank since the internal riser would be absent; in this case the variable X becomes the *head level* at the base of the surge tank, and is above or below the tank level Y depending on the direction of flow through the port. If, in addition, the throttling coefficient K_t is made zero, then the tank becomes a *simple* tank, with the reduced equations:

$$Q_c = A_t \frac{dY}{dt} + Q_p \qquad (1.4)$$

$$\left(\frac{L_c}{gA_c}\right) \frac{dQ_c}{dt} + K_c \mid Q_c \mid Q_c + Y = 0 \qquad (1.5)$$

In all that follows the demand flow Q_p will be considered as the source of the disturbance to the equilibrium. If this flow is reduced suddenly from an initial value $Q_{p1} = Q_o$ to a final value $Q_{p2} = 0$, and if in equation (1.5) K_c is assumed for the moment to be zero, it is simple to show that Q_c and Y will oscillate in pure harmonic motion, with the following characteristics:

$$\text{Period} = 2\pi \sqrt{\frac{L_c}{g}\frac{A_t}{A_c}} = T_{ct} = \textit{Free Period} \qquad (1.6)$$

$$Y_{max} = +Q_o \sqrt{\frac{L_c}{gA_cA_t}} = +Y_o = \textit{Free Surge} \qquad (1.7)$$

$$Y_{min} = -Y_o$$

$$Q_{max} = +Q_o$$

$$Q_{min} = -Q_o$$

The oscillations would, of course, continue indefinitely in the absence of friction. In practice, such a frictionless condition can only be approached, but never reached, and it might at first be thought that the foregoing solution has no physical significance. However, this is not true since the free period and free surge can serve as basic reference values. If one defines a new set of dimensionless variables, including the time variable, for equations (1.1), (1.2), and (1.3), the equations may be put into a completely dimensionless, or ***normalized*** form, suitable for general investigation.

Such a set of transformed variables are the following:

Flows:

Dimensionless Conduit Flow Ratio: $\bar{u} = Q_c/Q_o$ (1.8)

Dimensionless Demand Flow Ratio: $\bar{v} = Q_p/Q_o$ (1.9)

Levels:

Dimensionless Tank Level: $\bar{y} = Y/Y_o$ (1.10)

Dimensionless Riser (or Head) Level: $\bar{x} = X/Y_o$ (1.11)

Time:

Dimensionless Time Variable: $\tau = 2\pi\,(t/T_{ct})$ (1.12)

In terms of these variables, the equations (1.1), (1.2), and (1.3) become:

$$\text{Continuity:} \quad \bar{u} = \frac{d\bar{y}}{d\tau} + B\frac{d\bar{x}}{d\tau} + \bar{v} \qquad (1.13)$$

$$\text{Acceleration:} \quad \frac{d\bar{u}}{d\tau} + \frac{R}{2} \mid \bar{u} \mid \bar{u} + \bar{x} = 0 \qquad (1.14)$$

$$\text{Throttling:} \quad \bar{x} = \bar{y} + F \left| \frac{d\bar{y}}{d\tau} \right| \frac{d\bar{y}}{d\tau} \qquad (1.15)$$

Thus the seven original constants (including the gravitational acceleration) are reduced to the three independent parameters:

Riser Area Parameter: $B = A_r/A_t$ (1.16)

Conduit Friction Parameter: $R = 2H_{fo}/Y_o$ (1.17)

Tank Throttling Parameter: $F = \Delta H_{xyo}/Y_o$ (1.18)

where H_{fo} and ΔH_{xyo} are the friction loss and throttling loss, respectively, corresponding to the index flow Q_o.

The principal benefits occuring from such normalization procedures are the following:

(a) All redundant constants have been eliminated;
(b) Similitude features are revealed;
(c) Greater speed and ease of solution is secured;
(d) Smaller numbers of solutions are required to obtain general conclusions.

The second feature permits the use of model studies and the subsequent extension of results to field installations. Moreover *generalized model* studies may be made, as at M. I. T., *wherein the* results are treated in the same way as numerical and analytical studies, ending with the construction of generalized charts obtained from actual water measurements. The third benefit is made evident in the studies of Part 2 of this paper. The last advantage suggests that more knowledge can be gained from a lesser number of solutions if the results are put in normalized form. The curves so obtained permit ready interpolation, and to some extent extrapolation, for particular instances.

It is finally of interest to note the following scheme of parameters:

I — Differential Tank
Three parameters: R, F, B

II — Throttled Tank (B = 0)
Two parameters: R, F

III — Simple Tank (B = 0) (F = 0)
One parameter: R

• • •

Optimum Governing of Hydroelectric Units

Another representative study undertaken at M. I. T. concerns the proper adjustment of the settings of hydro governors to provide optimum regulation in an interconnected system. In such studies the use of the electronic analog computer has been invaluable [12] [32].

The principles of stabilization of the governor of a hydroelectric unit are indicated in Fig. 4. The left diagram illustrates the behavior

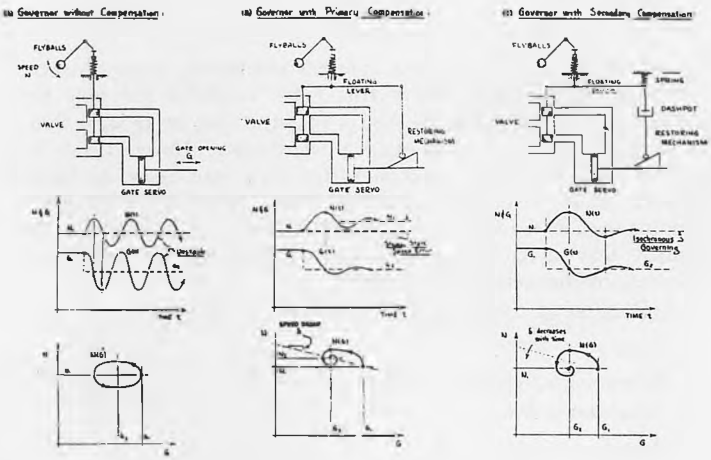

FIG. 4.—STABILIZATION OF GOVERNORS.

of the governed unit if no compensating mechanism is provided: a steady hunting would take place. The center panel, Fig. 4(b), demonstrates that provision of a restoring effect from gate servo back to the floating lever will stabilize the unit but will result in a *droop* between gate opening and speed such that the unit is left with a steady state speed error. This error may be removed and the governor made *isochronous* by putting a spring-dashpot mechanism in the restoring linkage as indicated in Fig. 4(c); in this case the droop may be said to "run out" in time at a rate depending on the dashpot needle setting. These two elements of stabilization are called first or *primary* compensation (restoring effect) and second or *secondary* compensation (dashpot effect). These settings for any particular unit, may be measured by the speed droop (δ) that would exist if the dashpot needle were fully closed (making the normally temporary droop fixed), and by the "relaxing" or recovery time of the dashpot (T_r) that could be observed if the dashpot pistons were given an initial displacement with normal needle setting. These constants and their adjustments may thus be related directly to the governor mechanism as shown in Fig. 5. It is important to note that the temporary droop (δ) is not the same as the adjustable but permanent droop (σ) which is required for paralleling prime movers in an interconnected system.

Also in Fig. 5, the effects of the adjustments of primary and secondary compensation are indicated by curves which were taken directly from the oscilloscope screen of the electronic computer and which have been amply confirmed by field tests. The speed transients indicated in the figure are those following sudden small load increments ΔP, with the instantaneous speed N plotted against time t. The *origin of coordinates* for each trace indicates the corresponding relative values of primary and secondary compensation. For best governing in hydro plants both settings will vary directly with the *water inertia or acceleration time* T_w, where:

$$T_w = \frac{\Sigma(LV)}{gH_o} \quad \text{(in } \textit{seconds}\text{)} \qquad (1.33)$$

and where $\Sigma(LV) = Q_o \int dL/A$ is the effective length-velocity (*square feet/second*) measuring the momentum of the entire water column from intake (or surge tank) to *draft tube exit*; and H_o is the rated head (*feet*) of the turbine. The best restoring effect (primary compensation) setting will depend also on the magnitude of the *machine inertia* or *acceleration time* T_m, which is defined as:

$$T_m = \frac{N_o^2\, WR^2}{1.6 \times 10^6\, P_o} \quad \text{(in } \textit{seconds}\text{)} \qquad (1.34)$$

where N_o is the prime mover synchronous speed (*rpm*); WR^2, the flywheel effect of runner and rotor (*lb-ft²*); and P_o, the rated capacity (*hp*).

As a single example of the numerical values of these settings and the consequent transient behavior, one may consider the extreme case of an isolated hydro unit carrying a purely resistive load. In this case, for best governing with a sensitive governor and small increments:

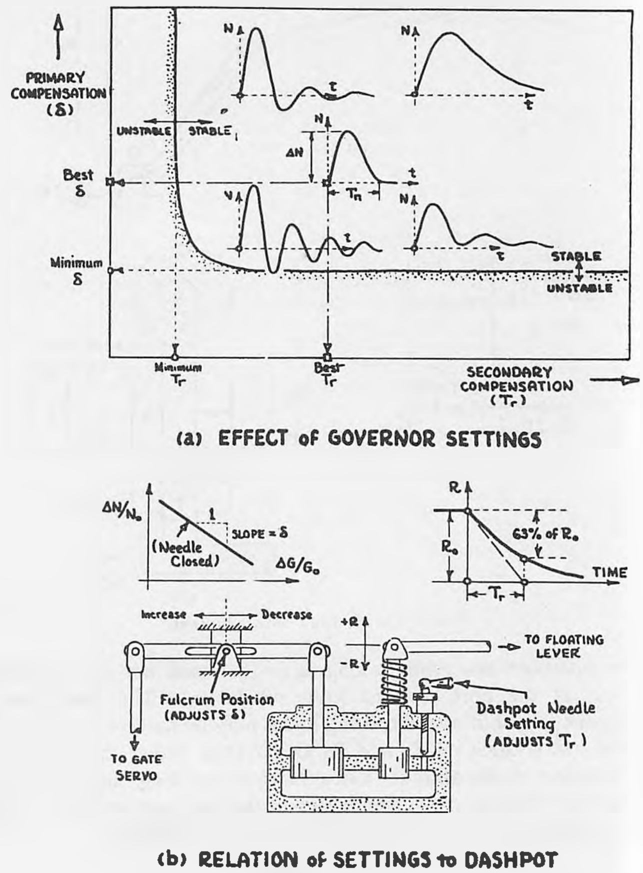

FIG. 5.—GOVERNOR SETTINGS.

Temporary Droop: (Needle closed) $\delta = 2 \cdot \left(\frac{T_w}{T_m}\right)$ [$\times$100 for percent]

Dashpot Time: (Needle normal) $T_r = 5 \cdot (T_w)$ [in *seconds*]

The characteristics of the step-load-change transient indicated in Fig. 5 that corresponds to these settings (again for small load changes ΔP) can be expressed as follows:

Relative Speed Change: $\left(\frac{\Delta N}{N_o}\right) = 2.5 \cdot \left(\frac{T_w}{T_m}\right) \cdot \left(\frac{\Delta P}{P_o}\right)$

Transient Time: $T_n = 8 \cdot (T_w)$

The water acceleration time (T_w) varies from less than 0.5 seconds for medium and high head plants with pressure regulation to more than 3 seconds for some large, low head plants. Thus the important need for sufficient machine WR^2 becomes apparent even with the best governing possible. Moreover, plants with high water inertia will necessarily be poorer regulators of system frequency than those with low water inertia, due to length of their transient intervals, which in extreme cases may be more than 20 seconds.

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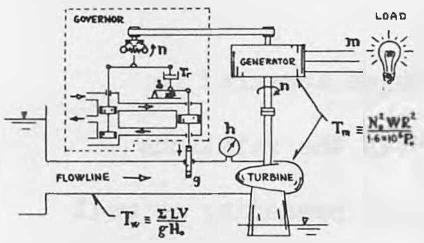
The following paper is a brief monograph which appeared in the LIGHTNING EMPIRICIST, v1, n1 June, 1952. Note the relations between this study and the BSCE paper results just preceding as well as the Koenig & Schultz paper following.

The Analog in Governor Design, I
A Restricted Problem

by Prof. Henry M. Paynter

A prime mover governor like other control mechanisms should possess the twin qualities of rapid response and stability. Ideal governors would operate in a power system so as to maintain constant frequency under changes in the electrical load. Of course this is not attained in practice, since all the governors operate from a frequency or speed *error* and it is only possible to adjust their settings so that the prime movers will reach a new equilibrium as rapidly as possible after a change in load. In the governing problem, then, the basic error signal is the change of prime mover speed which is detected by a flyball mechanism. This error signal is fed into a series of hydraulic valves and servos to produce a change in throttle valve or gate position of the prime mover. While for proper operation the governor assemblage must be sensitive to small changes in speed, and the control relays very quick, the response must not be such that the governor will overshoot the new equilibrium or even cause a steady hunting of the generating unit.

This tendency toward instability is remedied by introducing a stabilizing element into the governor, which in most American practice is a restoring mechanism. With no water inertia, in the case of hydro units, and without boiler lags in steam units, only slight restoration (feedback) between gate or throttle opening and pilot valve position is required for stability. Actually, however, in hydro units, when the turbine gates begin to close on a decrease of load, for example, the water inertia opposes this operation and produces a temporary *increase* in output due to the over-pressure or head rise. This unstabilizing effect may be overcome by an increase in the restoring action, which generally creates a speed droop so large that it must be made temporary by interposition of a dashpot. Thus one is confronted with the problem of establishing the proper settings of the governor stabilizing mechanism to insure a rapid return to equilibrium with stable and non-oscillatory response.



Schematic of Hydroelectric Plant

In the formulation of the simple governing equations outlined below, certain simplifying assumptions have been made which will be investigated in some detail in later instalments of this series. These may be listed as in Table I.

Subject to these assumptions, the *dimensionless* linearized equations for a *dashpot-compensated governor* may be written as follows, where $p = d/dt$:

MACHINE ACCELERATION: $$T_m pn = (g + 1.5h - m)$$

WATER ACCELERATION: $$-0.5T_w ph = (T_w pg + h)$$

GOVERNOR RESPONSE: $$-\delta T_r pg = (T_r pn + n)$$

TABLE I

	Initial Assumptions	Later Considerations
1.	Single hydro unit supplying an independent load	Parallel operation of both hydro and steam units
2.	Small disturbances, so that the response laws of all components can be assumed linear	Systematic investigation of the most important nonlinearities
3.	Electric load purely resistive with instantaneous voltage regulation, making the power independent of the speed and the torque vary inversely with speed, which is unfavorable for stability	Self-regulation effects of load on governing stability, including all coupling between the load and frequency
4.	Turbine efficiency constant for small variations in speed, head and gate opening	Effects on regulation of actual turbine characteristics, including gate limits
5.	Water and the flowline walls inelastic, so that only mass or inertia effects of the water are considered	Principal deviations in governor settings engendered by the elastic and resonant effects of the water and walls
6.	Governor with no bounds, lags, hysteresis, dead bands, etc., and with a very high gain between pilot valve position and gate servo velocity	Deviation from these assumptions encountered in actual governors

VARIABLES		PARAMETERS	
Relative Speed	n	Machine Inertia	T_m
Relative Head	h	Water Inertia	T_w
Relative Gate	g	Dashpot Time	T_r
Relative Load	m	Restoring Effect	δ

The governor is the "proportional plus integral" type with $1/\delta$ measuring the *sensitivity* and T_r the *reset* time. The feature which makes this problem different from conventional regulation problems is the effect of water inertia; the head change h counter to a change in gate opening g creates an unstabilizing term in the machine acceleration equation. The *relative stability* of any particular installation is therefore measured by the ratio of machine inertia to water inertia. Although the equations above are already dimensionless, it is possible to simplify them still further yielding a form in which the four defining parameters are reduced to two (independent) parameters.

Thus, with $s = T_w p$

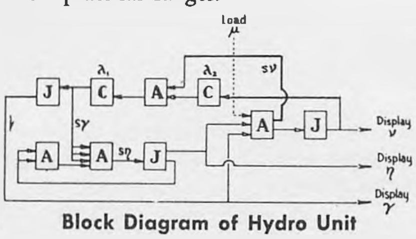
$$s\nu = (\gamma + \eta - \mu)$$

$$-s\eta = (3s\gamma + 2\eta)$$

$$-s\gamma = \lambda_1(1 + \lambda_2 s)\nu$$

VARIABLES	PARAMETERS
$\nu = (T_w/T_m)n$	
$\eta = 1.5h$	$\lambda_1 = T_w/\delta T_m$
$\gamma = g$	$\lambda_2 = T_w/T_r$
$\mu = m$	

In the above scheme the parameters λ_1 and λ_2 (which measure the two stabilizing components) completely specify the response of the governed unit to any particular load disturbance signified by μ. It is now possible to make a general analog study of this response merely by varying these two parameters over their practical ranges.



Block Diagram of Hydro Unit

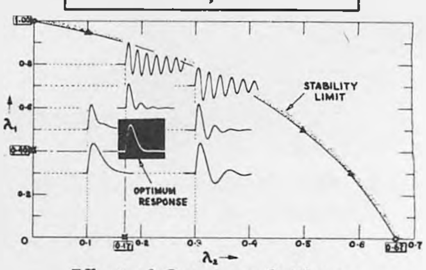
An effective block diagram is illustrated for this problem. Note that it is possible to multiply by integer coefficients using adding components.

As mentioned at the outset it is desired to make the governing as stable as possible and to return the machine to equilibrium at synchronous speed in the quickest time. Accordingly, it is of interest to investigate the present configuration for: *(1) stability limits; (2) optimum transient response.*

One finds by direct experimentation using the computor that there exist two extreme stability limits on λ_1 and λ_2 as indicated on the diagram; the parameter λ_1 (measuring the restoring effect or primary compensation) cannot exceed unity ($\lambda_1 < 1.0$) while λ_2 (measuring the dashpot effect or the secondary compensation) cannot be increased beyond the limit two-thirds ($\lambda_2 < 2/3$). Between these extremes, a smooth curve relates the values of λ_1 and λ_2 at the stability limit, which can be determined by direct adjustment and reading of the coefficient components.

The second question, that of the proper settings for optimum response, may also be found by simple experimentation with the computer settings (which correspond to the actual governor settings). The illustration indicates the effects of λ_1 and λ_2 by response curves which are pictures taken from the oscilloscope screen of the computor. The origin of coordinates for each trace, indicates the corresponding values of λ_1 and λ_2. Thus it may be found quite rapidly that for best governor response, and the shortest transient time (with the above assumptions) the following values are read directly from the coefficient settings:

$$\lambda_1 = 0.40, \quad \lambda_2 = 0.17$$



Effect of Governor Settings

A Discussion by H.M.Paynter
of AIEE Paper 53 - 172

"Tie-Line Power and Frequency Control of Electric Power Systems"

by C.Concordia and L.K.Kirchmayer

Henry M. Paynter (Massachusetts Institute of Technology, Cambridge, Mass.): The writer wishes to express his sincere appreciation of the authors' skill in selecting the most salient features of the system governing problem for particular study in this paper as well as in previous related papers. He would agree to the desirability of sufficient simplification of the actual problem to a point where at least a part of the system governing problem lends itself to quantitative analysis. However, it should continually be remembered that the original problem is fundamentally complex due to the multiplicity of generally disparate machines and regulators coupled together by a large number of "elastic" links. This thought serves to put analytic work into proper perspective alongside skilled observation and judgment based on system operating experience.

At the Massachusetts Institute of Technology (M.I.T.) a continuing program of researches, which began in 1947, is devoted to problems of transient response, speed governing, and load-frequency control in interconnected power systems. Certain phases of this work have been partially reported in two previous publications.[1,2] Appropriate to the present context are portions of these studies which are complementary to the authors' material on steam-steam interconnections. The writer has made very similar studies of this case as well as the case of steam-hydro, hydro-hydro, and mixed interconnections, where the regulation problem is considerably

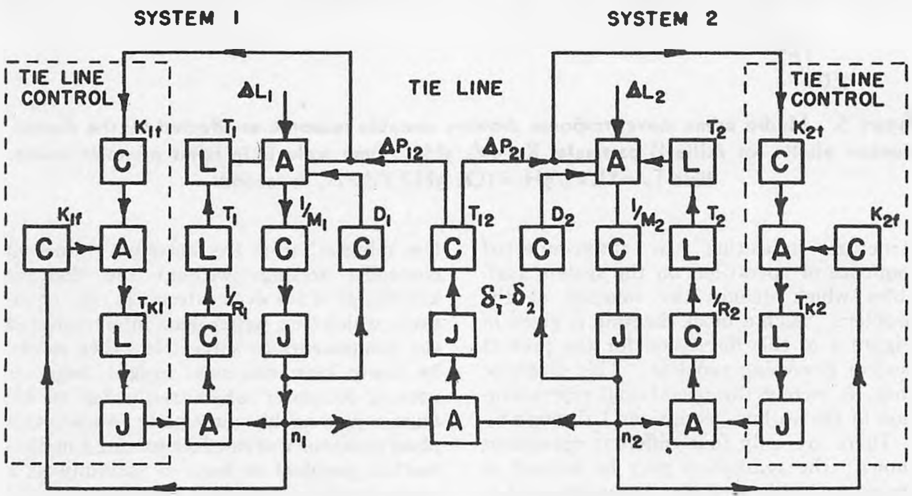


Figure 1. Block diagram for system governing. Operational structure representing equations 1 through 6

A = adding component
C = coefficient component
J = integrating component
L = lag component

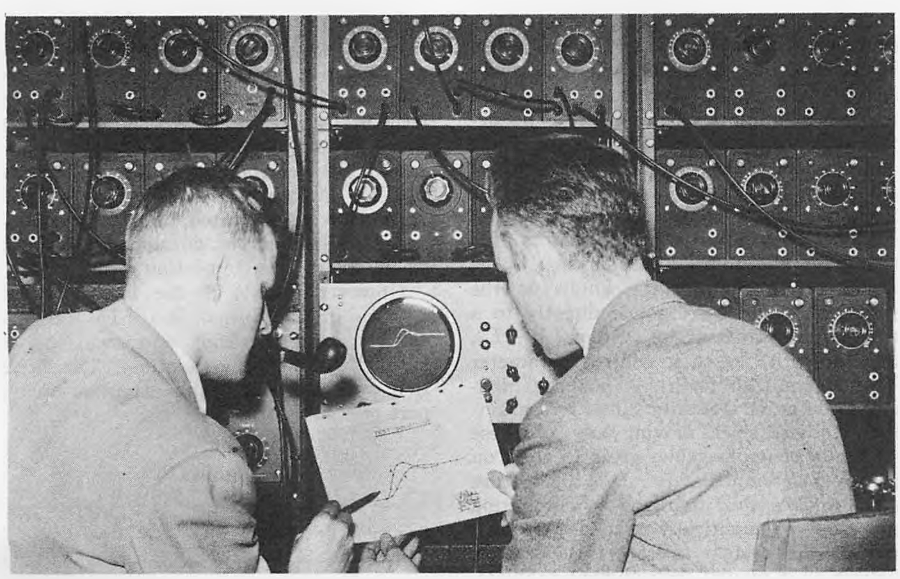


Figure 2. Philbrick electronic analogue computer. Each operational element is in a separate box Signal cables are visible in front, power cables are at rear. Display is customarily by cathode-ray oscilloscope as shown

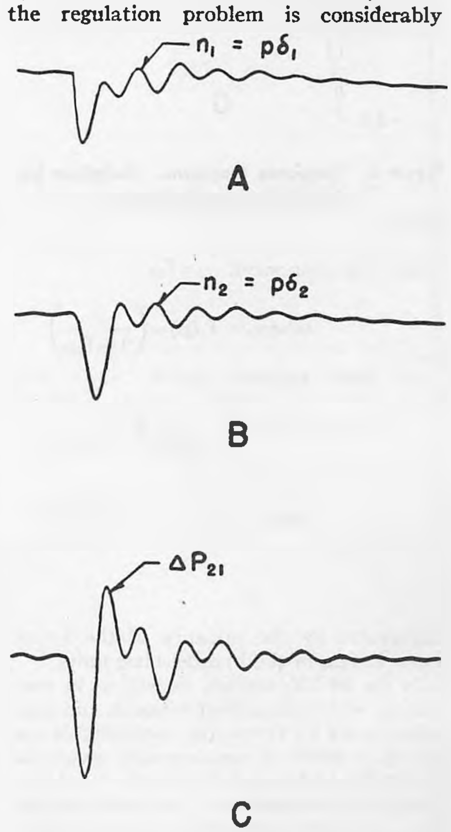


Figure 3. Electronic computer solution, comparable to Figure 4 of the paper

Curves:
- (a) frequency in area 1
- (b) frequency in area 2
- (c) tie-line power

Conditions:
- $R_1 = R_2 = 0.05$
- $D_1 = D_2 = 0.75$
- Load increase ΔL in area 1
- $K_{1t} = K_{2t} = 0.0036$
- $K_{1f} = K_{2f} = 0.0212$
- $T_{12} = 0.1$

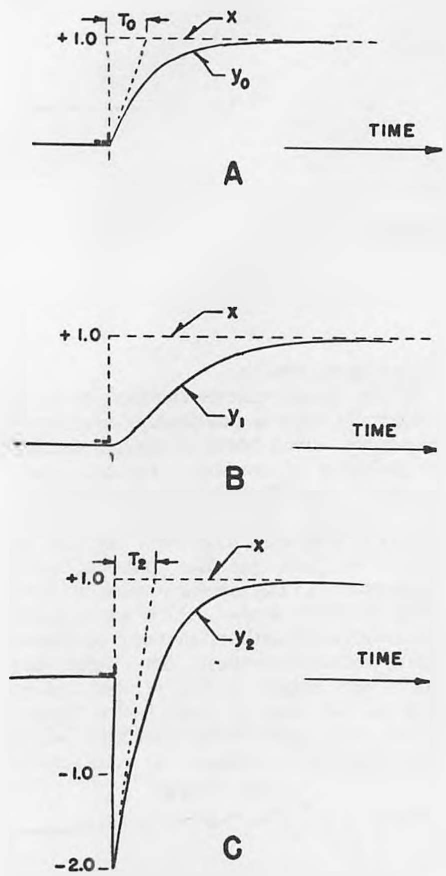


Figure 4. Response functions. Relations between input x and output y

Curves:

(a) Lag component: $y_0 = F_0 x$

where $F_0(p) = \left(\frac{1}{1+T_0 p}\right)$

(b) Steam response: $y_1 = F_1 x$

where $F_1(p) = \left(\frac{1}{1+T_1 p}\right)\left(\frac{1}{1+T_1 p}\right)$

(c) Hydro response: $y_2 = F_2 x$

where $F_2(p) = \left(\frac{1-2T_2 p}{1+T_2 p}\right)$

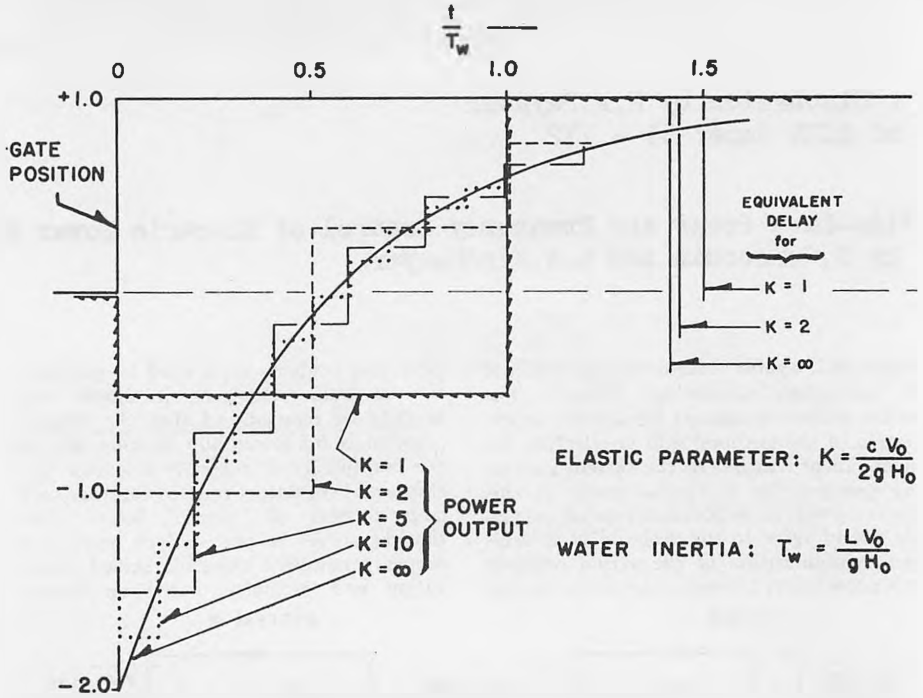


Figure 5. Hydro prime mover response showing unstable response as affected by the dimensionless elastic (or Allievi) parameter $K = cV_0/gH_0$. Time scale is in terms of water inertia time $T_w = \Sigma LV_0/gH_0 = (Q_0/gH_0)\int ds/A$, in seconds

aggravated by the presence of the latent water inertia in the hydroelectric units.

In the M.I.T. studies, as well as in connection with independent research and consulting work by the writer, considerable use has been made of commercially available (GAP/R) high-speed electronic analogue computing components.[3] For work of this type, where the primary task is to determine optimum and critical values of governor and controller settings as influenced by particular values of the system constants, high-speed repetitive computers of the Philbrick type, and its equivalents, have certain distinctive advantages, some of which are mentioned in the paragraphs to follow.

In order to use any automatic computing equipment, either of this sort or like the large differential analyzers of the Bush type used by the authors at General Electric and by the writer at M.I.T., it is necessary to translate the equations governing the system performance into an operational structure indicating the interconnected sequence of operations on the system variables which defines the solution of the problem. Such a block diagram is given in Figure 1 of this discussion for the present system governing problem. This diagram, then, is merely the operational representation of the authors' equations 1 through 6.

There are only four different operations shown; the symbolism may be defined as shown in Table I of the discussion, and is explained in greater detail by the writer and Mr. Philbrick in other publications.[3,4]

Even the rendering of such a block diagram can be of great assistance in "feeling out" a problem since it often has greater significance to practical-minded people than the differential equations it represents.

The translation of this operational diagram into computed results is indicated in Figure 2 of this discussion, which portrays the solution of an associated regulation problem on Philbrick equipment. The signal cables visible on the front are the direct counterparts of the directed lines joining the component elements shown on the block diagram. Thus it can be said that the block diagram in effect "comes to life" and gives the solution as shown on the oscilloscope. Each of the knobs on the coefficient boxes corresponds directly to a physical constant of the system (M_1, T_{12}, and so forth) or to a control parameter (K_{11}, R_2, and so forth). Thus the high-speed computer is ideally suited for parameter studies, since moving any of these system or control settings gives an effect on the response immediately. It is this feature which makes this approach to computing appealing to practically-minded personnel and to the system operators themselves. Indeed, in a related development problem, the writer had the rare privilege of witnessing a plant operator, who had almost no knowledge of mathematics but who knew his plant thoroughly, work out in a few minutes' time the optimum values of controller settings without the slightest knowledge of (or even interest in) the equations which the writer had interconnected the components to solve. In other words, by using keen empirical insight, together with a computer which responded to his fingers just as his own plant would, this plant operator was in effect solving a mathematical problem at least as skillfully as a trained researcher. For the base case of the present problem as indicated in Figure 1 of the discussion, this means for example, that the system load dispatchers might themselves solve an eleventh order differential equation. The significance of this statement should not be underestimated.

Figure 3 of the discussion depicts oscilloscopes of the electronic computer solutions corresponding to the case shown in Figure 4 of the paper. The agreement between the solutions is substantial and would have been better if greater care had been taken to calibrate all coefficient settings in the case of the electronic computer. This solution is, of course, representative of the variety of cases treated by Mr. Concordia and Mr. Kirchmayer, and measures typical steam-steam interconnection response.

However, many of the interconnected systems in the United States have hydroelectric units operating and regulating in parallel with steam generation. This situation warrants comparable attention on the part of analysis.

Table I

Symbol	Name	Operation
A	adding component	output = Σ(inputs)
C	coefficient component	output = $C \times$ input
J	integrating component	output = $\int$(input) dt
L	lag component	(See Figure 4 of this discussion)

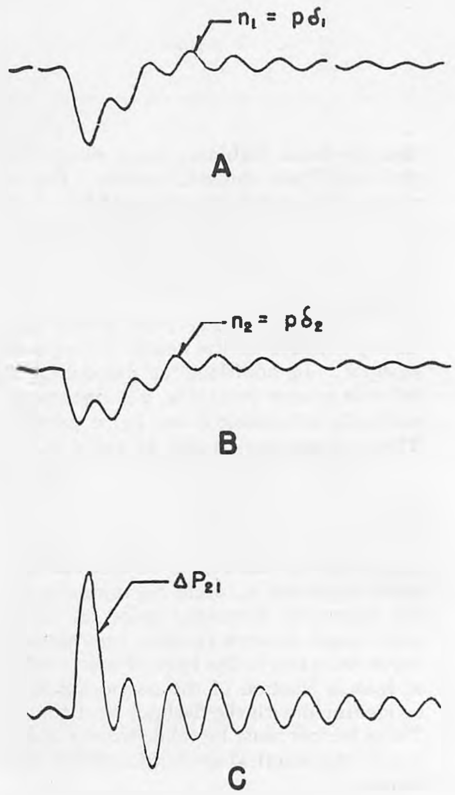


Figure 6. Effect of hydro prime mover in area 2. Steam prime mover in area 1, other conditions as below

Curves:
- (a) frequency in area 1
- (b) frequency in area 2
- (c) tie-line power

Conditions:

$R_1=0.05$; $R_2=0.25$
$D_1=0.75$; $D_2=0.75$
Load increase ΔL in area 2
$K_{1t}=0.0036$; $K_{2t}=0.00072$
$K_{1f}=0.0212$; $K_{2f}=0.00424$
$T_{12}=0.1$

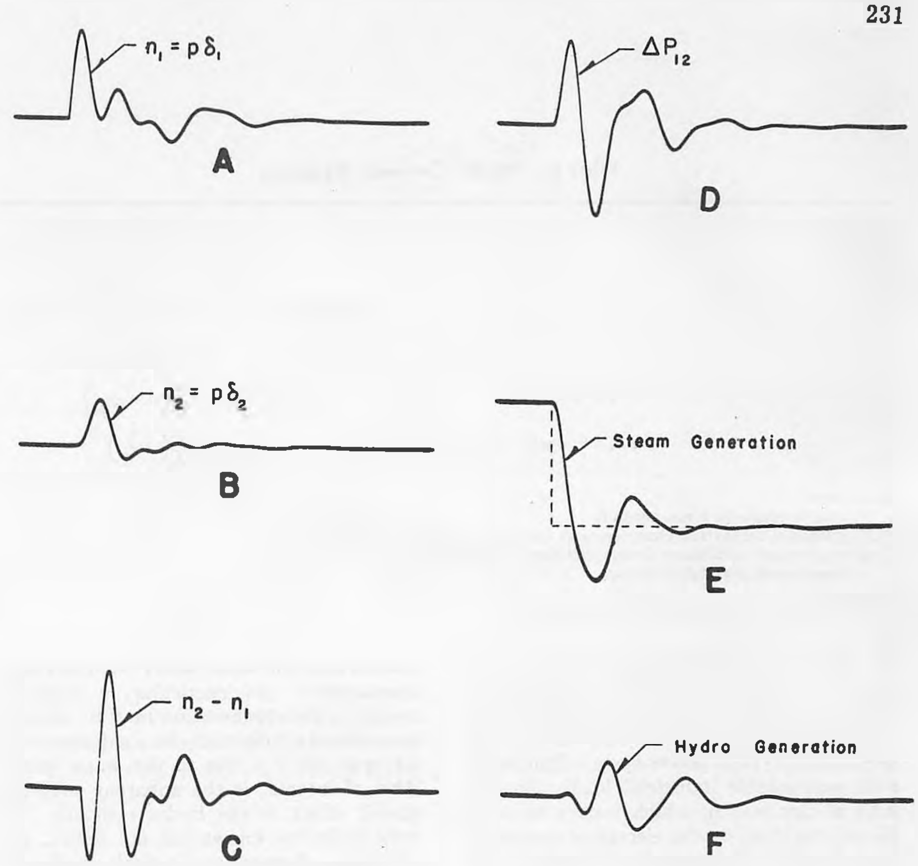


Figure 7. Near optimum system response. Steam generation in area 1, hydrogeneration in area 2, load change in steam area

Curves:
- (a) frequency of system 1
- (b) frequency of system 2
- (c) frequency difference
- (d) tie-line power
- (e) steam generation
- (f) hydrogeneration

Conditions:

$R_1=0.05$; $R_2=0.14$
$D_1=2.0$; $D_2=2.0$
$T_1=0.23$ second; $T_2=0.23$ second
$K_{t1}=0.0040$; $K_{t2}=0.0026$
$K_{f1}=0.0500$; $K_{f2}=0.0070$
Load decrement ΔL in area 1
$M_1=10.6$ seconds; $M_2=10.6$ seconds
$T_{12}=0.1$

Unlike the assumed response for steam generation, used by the authors and verified by the writer's information which is illustrated in Figure 4(B) of the discussion, hydraulic turbines have an inherently adverse response to changes in gate opening, arising from the inertia of the water in the flowline upstream and downstream of the turbine. Such response for low head plants, where elastic effects are negligible, is compared with steam response in Figure 4(C) of the discussion. This situation is outlined in more detail in Figure 5 of the discussion, where it is readily seen that for an initial time interval of approximately $0.5T_w$ second the power response is opposite to that desired. The water inertia or accelerating time constant T_w is related directly to flow geometry by the expression

$$T_w=\frac{Q_0}{gH_0}\int_0^L \frac{ds}{A} \text{ (in seconds)}$$

where the integral is taken over the entire enclosed flow path from intake (or surge tank) to draft tube exit. This time constant has the physical significance of measuring the time required to accelerate the discharge from rest up to its rated value under a constant accelerating head equal to the plant rated head. This time constant varies from less than 0.5 second in medium to high head plants with some form of pressure regulation, to more than 3.0 seconds for certain large modern low head installations.

It may also be seen from Figure 5 of the discussion, that the primary effect of flow compressibility and the flowline wall elasticity is to produce an oscillatory variation around the basically inertial phenomena. The relative elasticity is measured by the Allievi parameter K where

$$K=cV_0/2gH_0$$

and c is the speed of sound waves in the flowline. This factor measures the ratio of the elastic surge cV_0/g to the rated head H_0.

It can be shown that the relative regulation difficulty may be measured principally by the effective equivalent time delay of the prime mover response curve. The equivalent delays for hydro prime movers under various elastic conditions are also shown in Figure 5 of the discussion and are very nearly that for the inelastic or purely inertial case ($K=\infty$) for all finite values of K. This inertial value of equivalent delay is $1.42\ T_w$. For these reasons, it is found that elastic behavior—except in cases of resonance—has but a slight effect on the system governing problem.

Accordingly the authors' system as indicated by equations 1 through 6 and by Figure 1 of the discussion, can be modified to show the effects of steam-hydro governing by replacing the response function of Figure 4(B) of the discussion by that of Figure 4(C) of the discussion, where $T_2=\frac{1}{2}T_w$, in order to represent hydrogeneration in area 2.

Figure 6 of the discussion represents a typical solution of this sort which corresponds to Figure 3 of the discussion (and to Figure 4 of the paper) except that hydrogeneration is regulating in area 2, which requires that the regulation must be increased to a value of R_2 near 25 per cent merely to maintain operating stability; attempting to regulate with $R_2=0.05$ would now result in *instability and a desynchronization* of the system. More will be said on this point.

It is of further interest to investigate this modified steam-hydro interconnection for optimum controller and governor settings.

Table II. Hydro Governor Response

Number	Case	Response Function
1	$T_s,\ T_r,\ \delta_t,\ \sigma \neq 0$	$F_g = \frac{T_r p+1}{T_s T_r p^2 + (\sigma+\delta_t) T_r p + \sigma}$
2	$T_s = 0$	$F_g = \frac{T_r p+1}{(\sigma+\delta_t) T_r p + \sigma}$
3	$T_s = 0,\ \sigma = 0$	$F_g = \frac{1}{\delta_t}\left(1+\frac{}{T_s p}\right)$
4	$T_s = 0,\ \sigma = 0,\ T_r = \infty$	$F_g = \frac{1}{\delta_t}\left(=\frac{1}{R}\right)$

Constants:
T_s = servo response time, seconds
T_r = dashpot relaxation time, seconds (Secondary Compensation)
δ_t = temporary stabilizing droop (Primary Compensation)
σ = permanent paralleling droop

Equation:
Gate position = (response F_g) (speed error)

In order that these results be more representative of actual system operating conditions, the self-regulation in each area was increased to the value $D_1 = D_2 = 2.0$, corresponding to typical power system conditions with considerable industrial load. The results of this search, which took a total of 15 minutes' trial on the electronic computer (including the time required to take Land camera photographs), are shown in Figure 7 of the discussion, which represents near optimum conditions. It should be noted that since the load change occurred in the steam area, only the steam generation has a steady-state change in level. Thus all the variation in hydrogeneration is essentially undesirable and arises solely due to dynamic interactions. In particular, a slight increase in hydrogeneration may be observed immediately following the load drop; this adverse effect is due to the water inertia. Also of interest is the apparent swell-and-shrink effect in the hydro response. The time scales are known but not shown, since all these figures are actual oscilloscope photographs; the relative timing may be judged from the knowledge that the fastest natural period, that of the tie-line, is 2.3 seconds.

As mentioned earlier, hydro units require for stable speed governing materially higher values of instantaneous governor regulation than do steam turbines; these values often rise to 100 per cent and greater. For this reason, the stabilizing droop (δ_t) in hydro units is made temporary by interposing a spring dashpot system in the feedback path.[1] This dashpot effect in conjunction with a modern actuator governor results in an effective integrating or reset action whose timing depends on the needle setting of the dashpot. In addition, for paralleling stability in system governing, a permanent but manually adjustable droop (σ) is provided. These effects may be seen in Table II.

From this table it can be seen that the actual cases studied here and presented in Figures 6 and 7 of the discussion may be interpreted either as case 4, in which the regulation R_2 would be considered as measuring the droop δ_t, while K_{2f} measures only the frequency controller gain, or, on the other hand, a more realistic interpretation might be taken in the light of case 3, where at least a fraction of the nominal gain K_{2f} be identified with the dashpot reset time T_r. These factors must be taken into consideration in the practical application of the above results.

In conclusion, the writer wishes to acknowledge the help of his assistant, E. N. Rein, in the preparation of this discussion.

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3. Catalogue and Manual. G. A. Philbrick Researches, Inc. (Boston, Mass.).

4. The Electronic Analog Computor as a Laboratory Tool, G. A. Philbrick, H. M. Paynter. *Industrial Laboratories*, May 1952.

The paper that follows represents an outstanding contribution of a Jugoslav pioneer who has constructed his own general purpose electronic analog computor. However, the field of application is power system governing, and he has included a problem of a regulating hydro plant connected to a large power system.

Conférence Internationale des Grands Réseaux Electriques a Haute Tension
112, Boulevard Haussmann, Paris
Session 1952. — 28 mai - 7 juin

320. — ELECTRONIC COMPUTER FOR RESOLUTION OF STEADY-STATE STABILITY PROBLEMS AND PARTICULARLY FOR AUTOMATIC CONTROL STUDIES

by I. OBRADOVIC,
Engineer in the Electrotechnical Institute
of the Serbian Academy of Science in Belgrade (Yugoslavia).

SUMMARY

Description of an electronic computer specially adapted for the solution of linear differential equations of regulation. It differs from previous designs in that it measures in universal units, which, with a minimum number of connections simulate all fundamental transfer functions in which the behaviour of control circuit elements may be expressed. At present the computer consists of 12 such units and is adapted for studies of problems of complex regulation of large interconnected power systems, especially for direct control of frequency, load and time. Up to now studies have been limited to linear problems with the object of determining the accuracy of this computer.

REPORT

The problem of load, frequency and time control in interconnected systems has not yet been definitely solved. In order to satisfy the requirements placed on the performance of these systems, a supplementary control has up to now been used [1, 2, 3]. In this case each individual prime mover is controlled by its speed governor in the usual way while the desired load division is obtained by the supplementary control. It seems that among several posible methods, the best results have been obtained by the method of Darrieus [4] and Fallou [5], known under the name of frequency-load control. This method has been also proposed for international interconnections.

All these combined methods delay the regulating action and necessitate a great number of movements of the speed governor. In addition to the wear of governor parts, they also endanger the stability of the operation. It is therefore clear that direct control of interconnected systems is desirable, so that as soon as variations occur in the power system, only those governors of prime movers, designed to control such variation should come into action. Governors of those machines which are distant from the place where the variation took place or governors of prime movers that according to the plan, were designed to operate with constant load, should make no movement or reduce their movements to a minimum. One proposal of such a direct control of prime movers has been submitted to the last meeting of C.I.G.R.E. [6]. This was only a proposal, and to realise it, it will be necessary to solve a great number of problems which are related to the operating stability of this type of automatic control. It will be necessary to analyse the performance of this control by a study of differential equations by which movements of all governor parts may be expressed. It will be necessary to make a detailed analysis of the influence on the control action of the variation in the characteristic values of individual governor elements. Differential equations are linear equations of higher order and cannot be practically solved by mathematical tools. For their solution differential analysers must be used.

For these automatic control studies it is preferable to use differential analysers with electronic integrators, known in literature as Electric Analogue Computers [7]. They are more convenient because results may be obtained very quickly thus facilitating direct observation of the influence on the control action of the variation in the individual characteristic values of the governor. Besides for such automatic control problems, great accuracy is not required.

In the Electrotechnical Institute of the Serbian Academy of Science in Belgrade an electronic differential analyser has been designed which is especially adaptable for theoretical studies of complex automatic control problems. But this electronic computer can also be used for the solution of other problems which can be expressed by linear differential equations of higher order with constant coefficients. The means at our disposal for designing this computer were very limited, so that in its construction it differs greatly from American Large-Scale Electric Analogue Computers [7, 8].

FUNDAMENTAL PRINCIPLES OF OPERATION

The well known [8] feedback connection of electronic amplifiers for the solution of fundamental mathematical operations is indicated in the top row (*I*) of figure 1. The functional relationship obtainable between output voltage e and input voltages $e_1, e_2, \ldots e_n$ is indicated

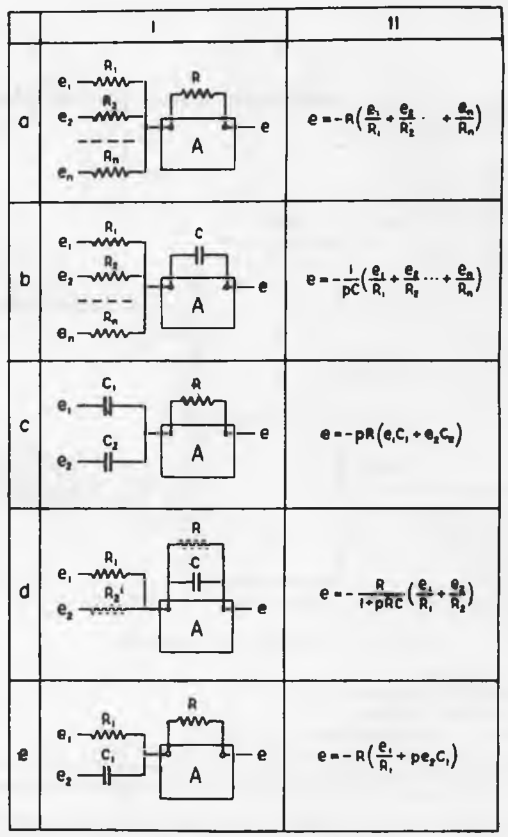


FIG. 1. — Connections of electronic amplifiers for solution of fundamental mathematical operations.

A - high-gain direct-coupled electronic amplifier;
$e_1, e_2, \ldots e_n$ - input voltages;
e - output voltages;
p - differentiation;
$1/p$ - integration;
I - amplifier connection with resistances and capacitors;
II - functional relationship between output and input voltages.

in the row (*II*) of the same figure. The operator $p = d/dt$ is used to symbolize differentiation and $1/p$ integration. Amplifiers are used so that, even at the smallest variations of the amplifier's input voltage with respect to the ground they give sufficiently great output voltages without being affected by the load impedance. On the assumption that the amplifier input potential is always aproximately equal to ground and that the current cannot flow into the amplifier, we can be certain that the written equations in row (*II*) on figure 1 are exact.

UNIVERSAL UNITS

The chief aim of this electronic computer was to realise a universal unit capable of performing all mathematical operations represented in figure 1 but without any supplementary connections of resistances, capacitors, attenuators, sign reversers etc.

Figure 2 shows the wiring of a universal unit. It consists of a high-gain direct-coupled amplifier (A), described in the next chapter, with a symmetrical (push-pull) output. By means of a selector switch (1) it is possible to insert a feedback resistance of the unit value ($R = 2\ M\Omega$), feed-back capacitors, or a combination of capacitors and unit resistance. Both push and pull outputs may be reduced through a low-resistance potentiometer with a large scale graduation in degrees of 10 per cent, using 11 precision wire-wound resistors with switch (2), and also with another potentiometer (3) equipped with small scale graduation. Resistances (4) which are used for connection with the next unit are so designed that adjusted outputs on the potentiometer may, if necessary, be increased or decreased in fix relations. With respect to the unit resistance (R) the resistances (4) have the values 5, 2, 1, 0,5, 0,2, 0,1 and 0,05 so that on the potentiometer adjusted outputs may be multiplied by coefficients 0,2, 0,5, 1, 2, 5, 10 and 20.

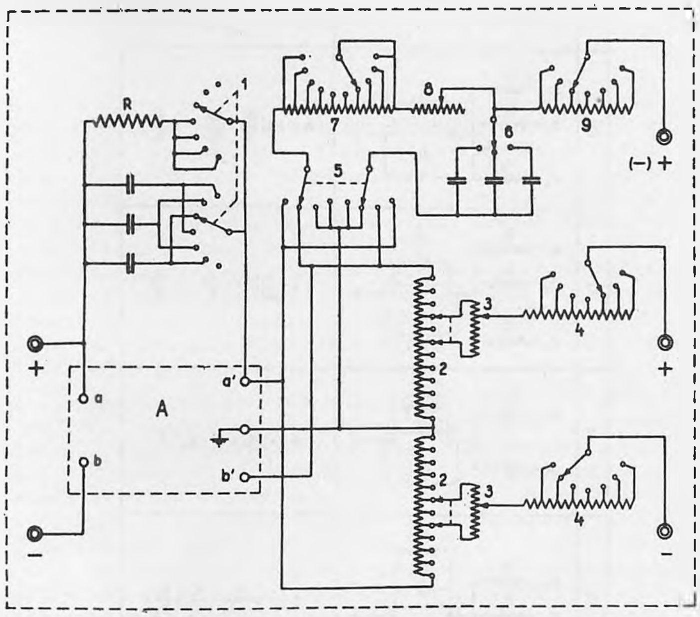


Fig. 2. — Universal unit connections.

A - amplifier;
R - feed-back resistance 2 MΩ;
1 - feed-back selector;
2 and 3 - output adjustement in per cent;
4 and 9 - resistances for connection with the next unit;
5 - selector for transfer functions $1/(1 + Tp)$ or $Tp/(1 + Tp)$;
6, 7 and 8 - capacitors and resistances for time constant adjustement.

In order to avoid any mistake which may take place because the amplifiers change sign during computing action, the outputs in this case are reversed so that in the universal units signs are not changed.

On each side of the symmetrical output there is an output with transfer functions $1/(1 + Tp)$ or $Tp/(1 + Tp)$. By means of the switch (5) it is possible to make a choice of one or the other function and of one or the other sign. The time constant T is calibrated directly in relative values of the fundamental time base (100 ms). By means of the switch (6), switch (7) and calibrated resistance (8) the time constant can be adjusted within wide limits. The high-resistance output (9) for connection with the next unit is similar for both symmetrical outputs.

AMPLIFIERS

Figure 3 shows a simplified plan of the wiring of the amplifiers. This is a usual two-stage direct coupled amplifier with differential input and symmetrical push-pull cathode follower output. The gain of the amplifier is about 2 000 and its performance is entirely stable both if used as an integrator in figure 1 (b) and as addition unit in figure 1 (a). The novelty in the amplifier is that it possesses an additional two-stage amplifier designed for very accurate output zero adjustement in order that output should always be symmetrical, i.e. that all positive and negative output values should be mutually equal. The assymmetry is measured by two equal resistances (r). The result is amplified by tubes (5) and (5′) in differential connection and then brought to the grid of the tube (6) which regulates plate voltage levels of tubes (3) and (3′). By this control, the outputs (a') and (b') are always equal with respect to the ground, and with opposite signs. The results obtained with this connection are very good and deviations between the two values can be observed only by very precise instruments.

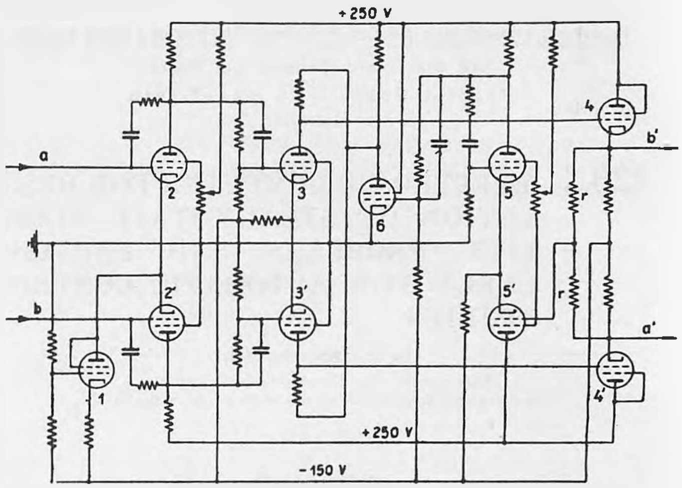


Fig. 3. — Simplified connection of hight-gain direct-coupled amplifier.

1 - constant-curent tube;
2 and 2′ - first stage;
3 and 3′ - second stage;
4 and 4′ - follower output;
5, 5′ and 6 - symmetry adjusting amplifier.

As addition unit the amplifier works with an equivalent delay with a time constant of about 10 μs while the symmetry adjusting circuit works with equivalent time constant of only 4 μs. Output voltages are symmetrical up to $\pm$ 60 V but they are used only in the range of $\pm$ 50 V, while the input difference varies not more than 50 mV.

The + 250 V and — 150 V supplies are common for all amplifiers and are obtained by electronically regulated rotative machines. The control problem is facilitated by the fact that the amplifiers are supplied with pratically constant current. The tubes are heated by a separate d.c. source, electronically regulated.

REPRESENTATION OF TRANSFER FUNCTIONS WITH UNIVERSAL UNITS

Each closed control circuit consists of complete or partial units each of which can be represented by certain transfer functions indicated in row (I) figure 4. For a better comparison, row (II) represents the corresponding transient response of the system when subjected to a step input, and row (III) the transfer function as a polar plot with the driving frequency ω as a parameter. The arrows indicate the direction in which the locus is traced as ω goes from 0 to $+\infty$. Row (IV) on the same figure shows how the corresponding transfer functions can be formed by universal units. Resistances and capacitors represented in the universal unit show symbolically the internal connection which is made by the selector-switches (1) and (5). One can see that very few connections are necessary in order to realise desired transfer functions. The last transfer functions (j) in figure 4 on the analogue computer is produced by the combination of inductances and capacitances. The advantage of this connection lies in the fact that the transfer function may be effected with any time base and with any damping factor.

Up to the present, only 12 universal units for the formation of transfer functions have been built. Later on, provision is made for the construction of new special units, such as non-linear units, units for dead band simulation etc. All requirements for theoretical research in the field of complex automatic control in power systems can, however, be dealt with by the present universal units.

OBSERVATION OF THE CONTROL ACTION

The formation of unbalance in the control circuit is effected in the usual way as in other analogue computers, by the inclusion of a square-wave function, produced by means of an electrical relay supplied across an electronic trigger. The frequency can be adjusted from 1 to 5 cycles per second. In order to simulate special initial conditions (example in Figure 7), 8 square-wave functions, independently adjustable in their amplitudes and entirely synchronised are generated by an electronic device.

The observation is carried out on an " Cossor " double beam oscillograph Model 1049, which can be connected to the output of any universal unit by means of a selector.

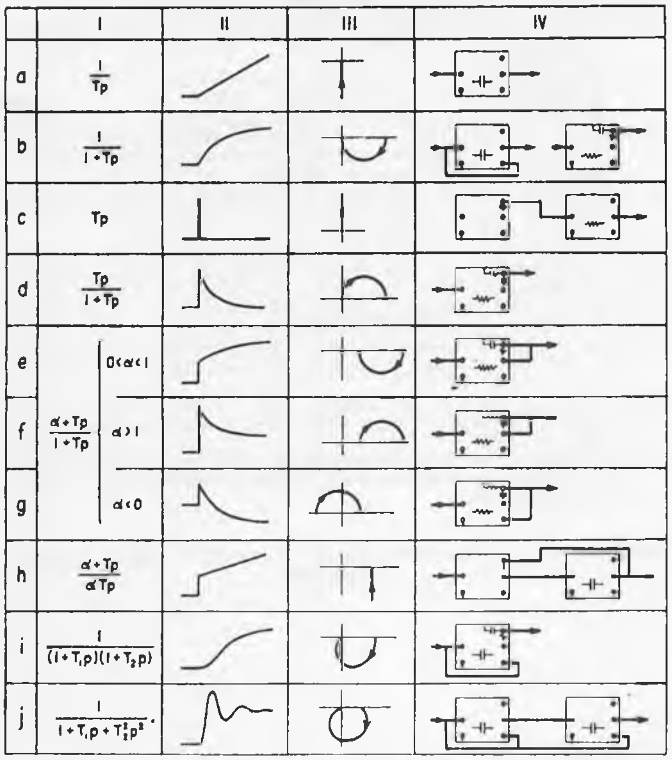


FIG 4. — Representation of certain transfer functions by individual units.

I - transfer function ;
II - transient response when subjected to a step input ;
III - transfer functions as polar plot with the driving frequency as parameter. The arrows indicate the direction of frequency increasing ;
IV - connections of universal units.

EXAMPLE OF SETTING UP THE PROBLEM

As the first example of setting up the system for study, let us take the load control of a tie-line between a prime mover (or a system) and an infinitely large power system. The measurement of the load in the tie-line, its first derivative and the integral of its deviation from the value which must be maintained by the governor, brings into action the pilot valve and thereby the piston with which we vary the torque. Figure 5 represents the connection of universal units for this study.

The load variation in the tie-line during the variation of the torque on the end of the line, where the regulated prime mover is, may be expressed by a differential equation of second order [9], which by Laplace transform may be brought to the form (*j*) in figure 4 and be represented by the corresponding connection of the two units (1) and (2) in figure 5. In the output of unit (2), we obtain a tension representing the angle between the prime mover rotor and the large network which, in this case, may be considered as the load to be transmitted. In this way we shall obtain on the output of the first unit a tension representing the speed of the rotor i.e. the first derivative of the load. Unit (3) computes the integral of the load deviation and so all these three values may be added together by the horizontal connection (*P*) representing the pilot valve lift. Unit (4) expresses the law of the torque variation with respect to the given signal, i.e. the pilot valve tift. The equation by which the piston of the servomotor follows the movement of its pilot valve may be brought to the form (*b*) in

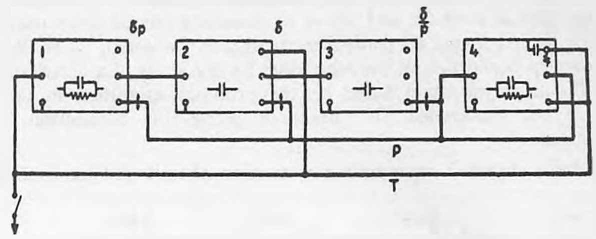


FIG. 5. — Example for load control of a tie-line between a prime mover and an infinitely large power system.

δ - the effective prime mover angle (or transmitted power) ;
$p\delta$ - prime mover speed ;
δ/p - integral of angle deviation ;
P - connection for pilot valve lift ;
T - connection for torque additions.

figure 4 where *T* represents the time constant of the servomotor. The variation of the torque, which in fact acts on the prime mover, lags behind the variation of the piston position, owing to the storage of steam or water. For steam turbines, this lagging may be expressed by the equation which by Laplace transform may be brought to the form (*b*) in figure 4, while for water turbines the action of the water impulse must be taken into consideration, so that the equation takes the form (*g*) in figure 4. The output of the unit (4) in figure 5 relates to the study of the performance of water turbines and represents the torque which actually acts on the rotor of machine being controlled. The horizontal connection (*T*) in figure 5 represents the sum of torques. In order to observe the action of the automatic control, an impulse (square-wave function) is added to this connection line (*T*) expressing the variation of the torque, i.e. the variation of the load in the region supplied by the prime movers whose control is being studied.

With the connection in figure 5, studies have been made up to now to determine the accuracy of performance of this computer. Only such characteristic cases have been chosen which could easily be proved by computation. The most characteristic case is when the system becomes unstable and the movement is transformed into oscillation with constant amplitude. For this case the Hurwitz determinant [10] must be equal to zero. The results thus obtained are satisfactory because deviations from zero are of the order of 1 per cent with respect to the values arrived at by computation. The next characteristic value is the area described by the controlled variable round its prescribed value during the regulation [11]. Here, no deviation could be proved with the oscillograph used.

Figure 6 represents the setting up of the problem of the relative angle delay control in the regulation of two systems in parallel operation. The main difference from the previous case is that the synchronisation torque is not proportional to the angle δ but is proportional to the difference between the angles of both systems ($\delta_1 - \delta_2$). This difference is measured by the unit (5) in figure 6. By adding an exterior resistance (*r*) the unit is able to measure the difference on any potential with respect to the ground. Units (3) and (8) repre-

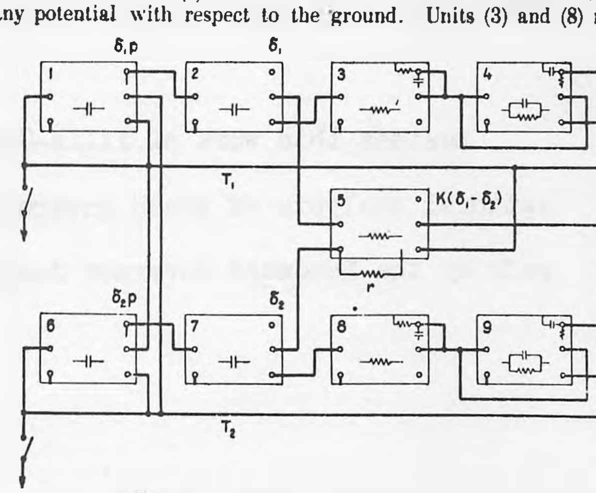
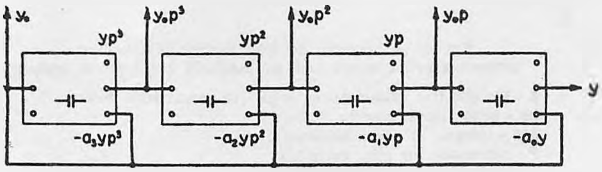


FIG. 6. — Example for angle delay or time control of two systems in parallel operation.

$K(\delta_1 - \delta_2)$, synchronising torque ;
r - resistance for measurement of angle difference. Other definitions as in fig. 5.

sent the regulators themselves. The output tension of these units represents the position of the servomotor piston. By varying the properties of units (3) and (8) or by inserting one or more universal units behind them, all control methods may be tested. The figure 6 shows the connection of the regulators for the study of a direct control of the angle (or time) based on the proposal submitted to the last C.I.G.R.E. conference [6]. Research using this connection is in progress.

Finally, figure 7 represents a connection of units for the functional



$$yp^4 + a_3 yp^3 + a_2 yp^2 + a_1 yp + a_0 y = 0$$

Fig. 7. — Example for direct (functional) solution of differential equations.

solution of linear differential equations. As an example, differential equation of the fourth order has been chosen with its prescribed initial conditions. They are inserted by an electronic device so that all constant currents with which the condensers of the integrators have been charged up to $t = 0$, are simultaneously cut off at $t = 0$, and from then on the solution of the differential equation is observed. Regulation problems may also be studied by using this connection, but owing to the variation of individual characteristic values it is not advisable.

CONCLUSION

This paper deals with a new design for an electronic differential analyser for the solution of automatic control problems. It consists of 12 universal units on which transfer functions of individual parts of the control circuit may be directly set up. The time constants are set up directly on time base and the other characteristic values of individual regulator parts on percentages of total movement. In this way it is easy to set up problems without much previous calculation and without connecting combinations of resistances, capacitors, etc. In the same way it is easy to set up differential equations of the regulation functionally if they are given in this form.

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Extrait de la *Conférence Internationale des Grands Réseaux Electriques.* Session 1952.

The next contribution extends the general studies of Paynter and others into specific problems of governor design and adjustment.

Besides this work of Allis-Chalmers, similar progress in rational analysis of speed governors using analog computors has been made by the Woodward Governor Company.

How to Select Governor Parameters with Analog Computers

ELDO C. KOENIG and WILLIAM C. SCHULTZ, Allis Chalmers Mfg. Co.

The governor system shown in Figure 1 is unstable for certain values of spring constant, K_1, and viscous damping coefficients, K_2. For other values, load-disturbance correction is slow. The problem: find the optimum values of K_1 and K_2. The attack illustrates one method of systems analysis. The sequence:

- ▶ Write descriptive equations.
- ▶ Insert the operators from the equations into a block diagram.
- ▶ Make a direct transformation from the block diagram to a computer circuit.
- ▶ Vary the regulator's parameters and observe the solution on an oscilloscope or on a recorder.

Mathematical Relations

Relative flyball displacement, Δy, is proportional to speed, or $\Delta y = \frac{1}{K_4}\Delta n$.

Pilot valve motion, $\Delta x_2 = \frac{a+b}{a}\Delta y - \frac{b}{a}\Delta x_1$,

where x_1 is the spring displacement.

Gate velocity, $\frac{dg}{dt} = K_3 \Delta x_2$.

The turbine is a single one supplying an independent load. Because the load is purely resistive and has instantaneous voltage regulation, power is independent of speed, and the turbine torque varies inversely with speed. The turbine efficiency is constant for small speed, head, and gate-opening variations. Because the water and piping are inelastic, only the water's inertia need be considered.

The equation for the water acceleration is

$$-0.5T_w\frac{dh}{dt} = T_w\frac{dg}{dt} + h,$$

where T_w = penstock time constant and the equation describing the machine acceleration is

$$T_m\frac{dn}{dt} = g + 1.5h - L,$$

where T_m = machine inertia
h = relative water head = $\Delta h/h_o$
L = relative load = $\Delta L/L_o$

The equation for the feedback mechanism is

$$\frac{\Delta x_1 - \Delta g}{\Delta g - \Delta x_4} = \frac{d}{e}$$

and

$$K_1 \Delta x_1 d = eK_2\frac{d\Delta x_4}{dt},$$

describing the force balance between the spring and the dashpot.

Several manipulations are performed on the equations before the computer is set up. The first three are used as they stand. In the equations for machine and water acceleration, eliminate h and solve for n. Then eliminate x_4 and solve for Δx_1 in the equations for the feedback mechanism. The equations then have the form:

$$\Delta x_1 = \frac{f_e K_2}{d^2 K_1}\left[\frac{p}{1 + pT_a}\right]\Delta g,$$

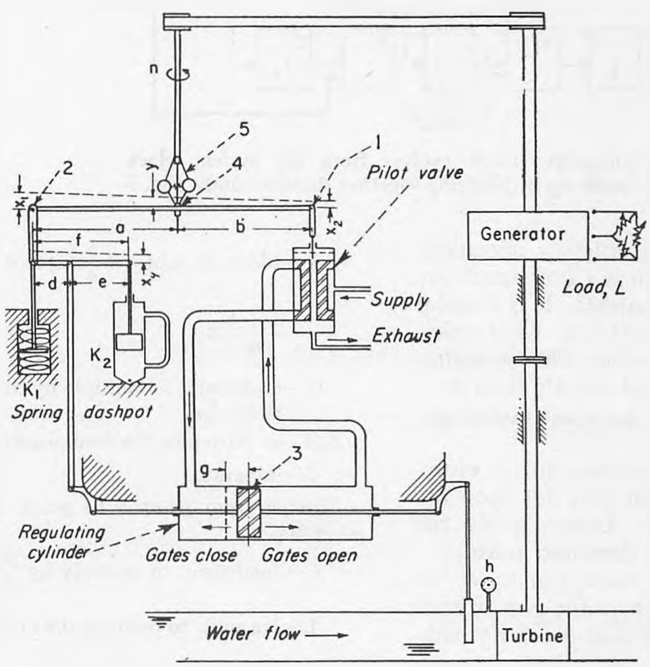
where p is the operator $\frac{d}{dt}$,

$$T_a = \frac{e^2 K_2}{d^2 K_1},$$

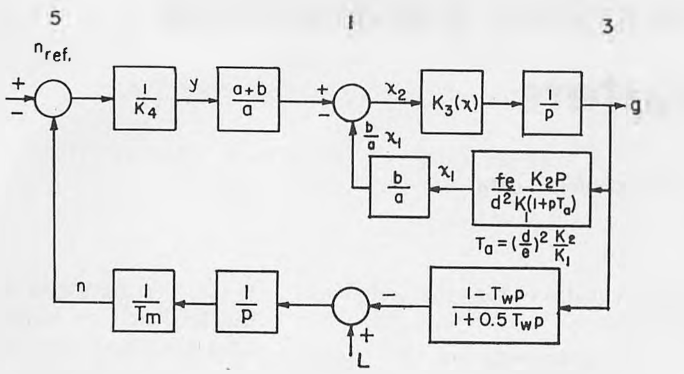
and

$$n = \frac{1}{pT_m}\left\{\left[\frac{1 - pT_w}{1 + 0.5pT_w}\right]g - L\right\}$$

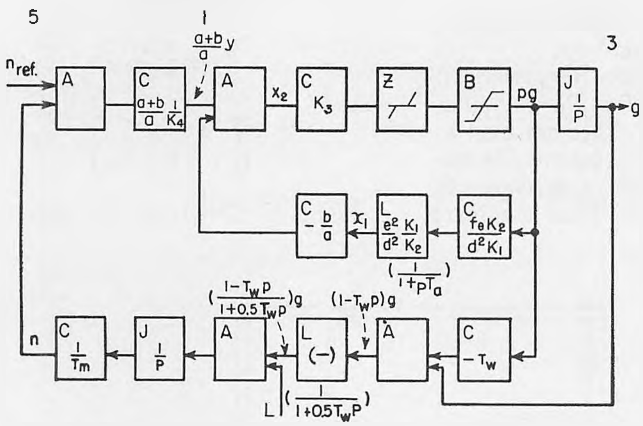
Figure 1 illustrates the governor.



Following an increase in generator load
▶ Speed, n, decreases.
▶ Flyballs move in. Points 1 and 2 and attached pilot valve move down.
▶ Piston movement directs oil flow as shown by arrows.
▶ Regulating cylinder piston moves to right, opening gates.
▶ Vertical rod connected to lever arm f moves down, dropping point 2.
▶ Because point 4 canont move until speed accelerates, point 1 and pilot valve rise.
▶ Pilot valve closes, stopping oil flow. Regulating cylinder and gate stop. Fig. 1



Block diagram of turbine governing system. The mathematical operation indicated in each block is performed on the input to that block. Fig. 2



The analog computer circuit evolves from the system block diagram. It is made up by plugging together standard units. Fig. 3

The equations describe its operation. The block diagram is a functional way of writing the equations. It is a useful roadmap for visualizing what takes place mathematically. The computer diagram is obtained directly from it.

The pilot valve has some nonlinearities. They are:

(1) Valve lap—the initial movement of the piston does not open the port to flow oil. Consequently the regulating piston does not move.

(2) The pilot valve can move beyond the point where the port is completely open. At this point no additional oil flow can occur with an increase in pilot valve displacement. This limits the velocity of gate movement. It is analogued with a bounder (B) unit. The lap is simulated with a dead-zone (Z) unit.

Setting Up The Computer

Starting at point 5, we use the following computer units:

A—adder, to compare speed with reference,

C—coefficient, to multiply by

$$\frac{a+b}{a}\frac{1}{K_4},$$

A—adder, to subtract $\frac{b}{a}\,\Delta x_1$ from

$$\frac{a+b}{a}\,\Delta y,$$

C—coefficient, to multiply by the constant K_3,

Z,B—to introduce the nonlinearities,

J—integrate.

Moving from point 3 to point 1, the units are:

C—coefficient, to multiply by $\frac{fe K_2}{d^2 K_1}$,

L—lag unit, to perform the operation

$$\frac{1}{(1+pT_a)},$$

C—coefficient, to multiply by $-\frac{b}{a}$.

Tracing out the block diagram from point 3 to point 5:

C—coefficient, to multiply pg (taken from ahead of the integrator) by $-T_w$,

A—to add the quantity g (the output is $(1-pT_w)g$),

L—lag unit, to simulate $\frac{1}{(1+0.5pT_w)}$,

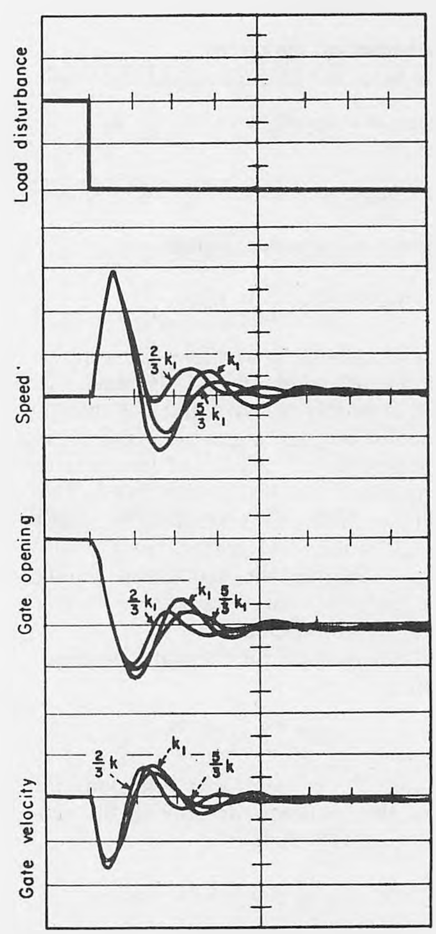
A—adder, to add in the load disturbance,

J—integrator,

C—coefficient, to multiply by $\frac{1}{T_m}$.

The complete computer circuit is shown in Figure 3.

The constants K_1 and K_2 appear in the feedback from point 3 to point 1 in a C unit and also in the setting of an L unit. Actually the effects of varying any other parameters can be studied—for instance the lever arm lengths a, b, d, e, f, the nonlinear characteristics, or the time constants T_m and T_w. Figure 4 shows reproductions of analog solutions of gate opening gate velocity and regulated turbine speed for several settings of K_1.



The computer solution is displayed on an oscilloscope. Display tracings show how the system's response to load disturbance changes when K_1 is varied. Fig. 4

COMPUTOR TECHNIQUES IN HYDROLOGY

A promising field of application for electronic computors involves the dynamic routing of water flow in drainage basins from the first raindrops to the final torrents.

Not only can analog machines aid in the generalized study of such problems, but more dramatically they permit simulation of vast river systems on a real time or accelerated time base, for flood prediction and water control purposes. Alternative storage operations may then be explored at will (especially on fast time) and the most effective, yet safe, operation can be undertaken. Other uses of computors for hydraulic design and flood damage-frequency analyses are also being explored.

On the following pages we reproduce another portion of the Paynter BSCE paper. Besides the hydrologic interpretation an important consequence of the linear techniques suggested therein has been a renewed interest in the general analysis and control field in the use of convolution methods of computation (for which see the related papers of Janssen and Ensing as well as Medkeff and Mathews in this collection).

Such computor approaches have resulted in the significant thesis work of Peralta at M.I.T., from which some diagrams are included at the close.

Flood Routing by Admittance Methods

Introductory

The use of impedance and admittance functions to characterize the behavior of transients and oscillations in a linear system is an old, "tried and true" method; for example, it forms the basis of all electrical network theory. However, the direct and conscious application of these concepts to hydraulic transient problems has been little appreciated to date, although such methods seem promising. Outlined in the following paragraphs is a preliminary accounting of some current studies underway at M. I. T. to explore the possibilities of improved flood routing procedures making use of these concepts.

Admittances and Flood Routing

In the absence of simple physical laws describing the behavior of changing flows in a reach of a river, it is not reasonable to abandon too hastily the useful concepts of linearity and superposition. While all real phenomena are non-linear to a greater or lesser degree (in the sense that the governing relationships are not just simple proportionalities involving the variables and their rates of change), nevertheless, the use of linear tools with appropriate caution and modification can lead to many valuable methods and results. This concept has been at the heart of all rational physical and engineering analysis.

Accordingly, it might be asked: what is the most general assumption of linear response that one could make? The answer is simple: subject to the single assumption that superposition is valid, the response of any system can always be represented by the Duhamel superposition theorem:

$$y(t) = x_0 A(t) + \int_0^t A(t-\tau) \cdot d[x(\tau)] \qquad (3.1)$$

where x (t) is the INPUT or DISTURBANCE; y (t) the OUTPUT or RESPONSE; and A (t), called the INDICIAL RESPONSE or ADMITTANCE, represents the relation between input x and output y, and is the response of the system to a unit step input as shown in Fig. 13. This relationship is capable of describing very complex systems either with lumped elements and a finite number of degrees of freedom (such as vibratory systems and electrical networks) or with continuous elements and an infinite number of degrees of freedom (such as flood flows in rivers, elastic waves, long transmission lines, and heat flow problems).

An excellent introductory treatment of admittances and the superposition theorem is given in the book of von Karman and Biot [25]. However, a brief derivation of the basic formula is in order.

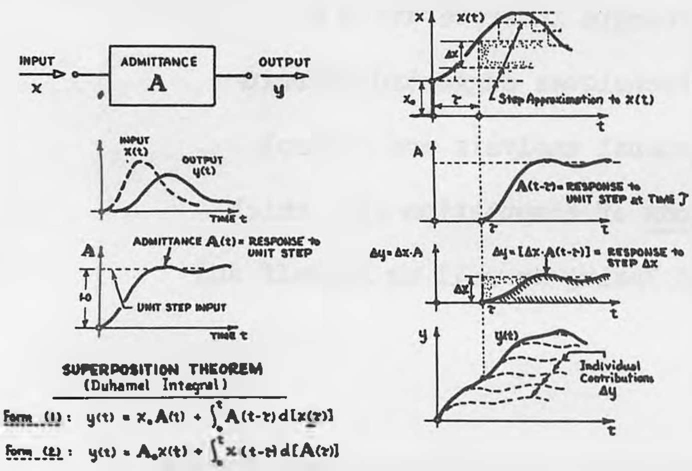


Fig. 13.—Significance of Admittance Function.

Referring to Fig. 13, A (t) is shown to be the output resulting from a unit step input. The general input x(t) may be considered to be the sum of a series of small steps. The height of the step at a certain time τ is shown as Δx. The result of applying such an input, Δx, to the system is to give rise to an output which has the same shape as A(t), but which begins at time τ and has a magnitude proportional to the *input* Δx, *thus:*

	INPUT	OUTPUT
Unit Step:	1	$A(t-\tau)$
Step:	Δx	$\Delta x \quad A(t-\tau)$

If the results of all such steps are added together, with the height of each step becoming infinitesimally small, the result is the Duhamel integral.

There are a number of marked advantages in considering flood routing problems in this light; among them one might list the following:

(1) It is the most general and universally applicable linear method.
(2) All existing linear techniques of routing may be considered as special instances of this method.
(3) No additional assumptions or equations are required.
(4) The relationship between x (t) and y (t), as characterized by A (t), is always possible to find and is unique.
(5) The admittance function A (t) can be determined from past flood records.
(6) On the other hand, A (t) can also be related to the characteristics and real properties of the stream.
(7) The method is amenable to rapid manual computation, as well as machine solutions and analog representations.
(8) It is possible to work with gage heights directly and use stage-routing methods.
(9) Floods may be routed successfully and simply through long channel reaches.
(10) It is not absolutely necessary to correct initially for ground-water increases, tributary or local inflows; these may always be investigated and corrections made after the analysis.

Such methods are not new to the hydraulic field; the unit hydrograph method is an outstanding example of this approach applied to the relation between rainfall and runoff in a stream. However, the conventional practice in this case of using only a "unit storm" is not a necessary part of the linear treatment. Rather, the above formulation suggests that considerable knowledge is derivable from every past storm, regardless of the space-wise or time-wise distribution of rainfall over the area.

Significant Properties of Admittances

Just as in any AC or DC electrical network one may write:

$$\begin{aligned} e_{input} &= \widetilde{Z} \cdot i_{output} \\ i_{output} &= \widetilde{A} \cdot e_{input} \end{aligned} \qquad (3.2)$$

where $\widetilde{Z}$ = Impedance, and $\widetilde{A} = 1\ \widetilde{Z}$ = Admittance, so one can write for any linear system:

$$\underbrace{\begin{bmatrix} \text{output} \\ \text{or} \\ \text{effect} \\ \text{or} \\ \text{response} \end{bmatrix}}_{y} = \underbrace{\begin{bmatrix} \text{Admittance} \end{bmatrix}}_{\bar{A}} \cdot \underbrace{\begin{bmatrix} \text{input} \\ \text{or} \\ \text{cause} \\ \text{or} \\ \text{disturbance} \end{bmatrix}}_{x} \qquad (3.3)$$

However, one must consider the admittance $\bar{A}$ as an *operator*, or operation, which when applied to the input x gives the output y. Every linear system may be characterized by a unique operator $\bar{A}$. If a series of such systems $\bar{A}_1$, $\bar{A}_2$, etc., are arranged in tandem such that the output from the first becomes the input to the second, and so forth, then the results may be expressed as follows:

For three such systems, there would result:

$$x \rightarrow [\widetilde{A}_1] \rightarrow [\widetilde{A}_2] \rightarrow [\widetilde{A}_3] \rightarrow y$$

$$y = [\widetilde{A}_1 \cdot \widetilde{A}_2 \cdot \widetilde{A}_3]\ x$$

$$y = \widetilde{A}_s \cdot x$$

and for n systems:

$$x \rightarrow [\widetilde{A}_1] \rightarrow [\widetilde{A}_2] \rightarrow \ldots \rightarrow [\widetilde{A}_n] \rightarrow y$$

$$y = [\widetilde{A}_1 \cdot \widetilde{A}_2 \cdot \ldots \cdot \widetilde{A}_n]\, x$$

$$y = [\Pi \widetilde{A}_k]\, x$$

$$y = \widetilde{A}_s \cdot x$$

In short, any sequence, finite or infinite, of linear admittances will result in an effective resultant admittance $\bar{A}_s$, which is the product of all the admittances in the chain.

This resultant admittance $\bar{A}_s$, relating output y to input x depends on the particular nature of the system and completely determines the response of the system to any particular disturbance. As just mentioned $\bar{A}$ is not generally a simple constant but is rather an operator. By means of various integral transformations, such as the Laplace or Fourier transformations, this function may be expressed as an *algebraic function,* such as:

Laplace Representation: $\bar{A} = \bar{A}(p)$ where p = Complex number

Fourier Representation: $\bar{A} = \bar{A}(\omega)$ where ω = Angular frequency

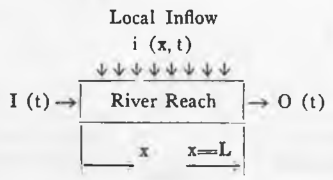
For details of such representations, see, for example, the works of Gardner and Barnes [26], Churchill [27], Jaeger [28], as well as the Karman-Biot text previously cited.

However, as a much more practical expedient, for the present purpose, it is of interest to consider the representation of such operators in the *time domain*; in this case A becomes A (t), the INDICIAL RESPONSE OR ADMITTANCE FUNCTION, which is the response of the particular system to a *unit step input* as indicated in Fig. 13. Application of A to the input x becomes *convolution* as represented by the Duhamel integral.

As applied to the present problem, one may consider the input x to be the *inflow I,* and the output y, as the *outflow O,* of a particular river reach. The admittance function A(t) will be considered as characterizing the behavior of the reach.

While the generality of the admittance concepts includes the possibilities of propagation and diffusion as the flood wave moves downstream, together with continuous variation of the stream properties along the length of the reach, yet it is important to mention at the outset a serious difficulty to be overcome in the practical application of this method, as well as all other methods: the problem of continuous local inflow along the length of the reach, which is in the last analysis at least partly unknown and unknowable. Moreover, this "inflow" may even in extreme cases become *negative* (outflow) due to evaporation and to leakage *into* the groundwater table.

As a model for this process, one may consider the following scheme:



At least conceptually, this situation may be considered "lumped" into the following admittance scheme:

$$I \rightarrow [A_1] \xrightarrow{i_1} [A_2] \xrightarrow{i_2} \ldots \xrightarrow{i_{n-1}} [A_n] \rightarrow O$$

Even so, the errors in determining or estimating the local inflows i_1, i_2, etc., will in turn cause corresponding errors in the determination of the admittances A_1, A_2, etc.

A possibly fruitful model for a drainage basin is shown in Fig. 14.

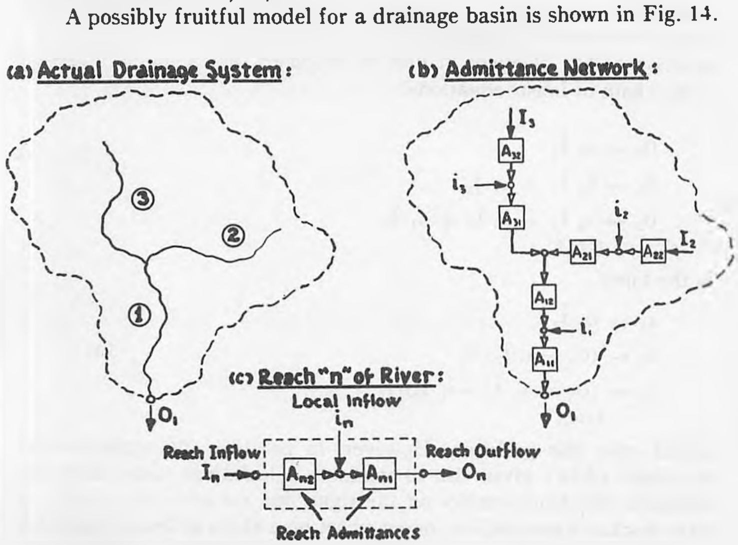


FIG. 14.—FLOOD ROUTING WITH ADMITTANCE NETWORK

The physical streams are replaced by a network of admittances which are here shown to have but one intermediate junction for concentrating all local inflow.

Admittances of Existing Methods

In Fig. 15 are shown the admittance functions of several of the more common routing procedures. These will now be discussed.

The upper part of the figure, Fig. 15(a), shows the admittance corresponding to a pure travel delay of a flood wave, with no distor-

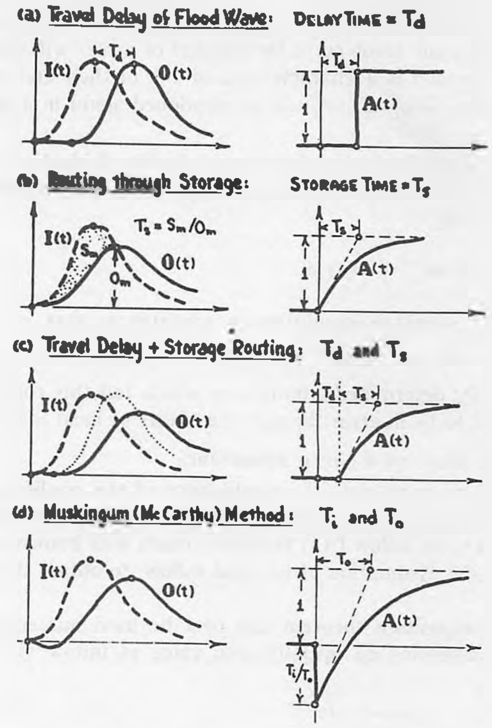


FIG. 15.—ADMITTANCE FUNCTIONS OF EXISTING ROUTING METHODS.

tion. It is clear that A(t) is merely another unit step displaced in time by the delay T_d.

A common source of routing techniques has been the lumped storage or continuity equation:

$$\frac{dS}{dt} = I - O \qquad (3.4)$$

previously employed in Part 2 under reservoir routing. If only prismatic storage is considered, and the relationships to depth are considered *linear,* then storage S may be considered as related to outflow O by the relation:

$$S = T_s \cdot O \qquad (3.5)$$

where T_s is the storage time constant and is related to both the general flood and to the admittance as indicated in Fig. 15(b).

The resulting differential equation is first order, and has the form:

$$T_s \frac{dO}{dt} + O = I \qquad (3.6)$$

The admittance function in this case is given by:

$$A(t) = 1 - e^{-t/T_s} \qquad (3.7)$$

The combination of travel delay with prismatic storage is shown in Fig. 15(c). The result follows directly from (a) and (b) above.

The widely and successfully used "Muskingum Method" outlined by G. T. McCarthy in 1938 [29] is based upon an assumed storage function taking into account both prismatic and wedge storage in the form:

$$S = T_i I + T_o O \qquad (3.8)$$

for use with equation (3.4). In conventional notation the storage time constants T_i and T_o are given by:

$$T_i = Kx$$
$$T_o = K(1-x) \qquad (3.9)$$

The resulting first order differential equation has the form:

$$T_o \frac{dO}{dt} + O = I - T_i \frac{dI}{dt} \qquad (3.10)$$

A general flood and corresponding admittance for this equation are shown in Fig. 15(d). In this case the admittance $A(t)$ is given by:

$$A(t) = 1 - \left(\frac{T_1 + T_o}{T_o}\right) e^{-t/T_o} \qquad (3.11)$$

It is found that for most reaches T_1 approximately equals T_o. The physical interpretation of the initial negative values resulting from this method is difficult to find: any investigation of the actual continuous mechanisms involved in the passage of waves will suggest that such a phenomenon is a characteristic of the method and not of the natural stream. This subject will be mentioned again in a later paragraph.

Note that in all the preceding cases, as with all idealized instances where there is no net local inflow (or outflow), the volumetric continuity condition

$$\int_0^\infty I(t)\, dt = \int_0^\infty O(t)\, dt \qquad (3.12)$$

results in the *necessary* condition on admittances that

$$A(t) \rightarrow 1.00 \text{ as } t \rightarrow \infty \qquad (3.13)$$

Experimentally determined admittances which fail this condition may be considered to be in error through the effects of local inflow.

Routing Procedure with Given Admittance

In order to appreciate the significance of the application of admittance concepts to flood routing, one may consider the simplest case of routing a known inflow $I(t)$ through a reach with known admittance $A(t)$, under the assumption of no local inflow, to obtain the final outflow $O(t)$.

The superposition theorem can best be used numerically in its finite form, assuming an initially zero value of inflow (i.e. $I_o = 0$):

$$O_n = \sum_{k=1}^{n} a_{n+1-k} \cdot \bar{I}_k \qquad (3.14)$$

which becomes, on expansion:

$$O_1 = a_1 \cdot \bar{I}_1$$

$$O_2 = a_2 \cdot \bar{I}_1 + a_1 \cdot \bar{I}_2$$

$$\vdots$$

$$O_n = a_n \cdot \bar{I}_1 + a_{n-1} \cdot \bar{I}_2 + \ldots + a_1 \cdot \bar{I}_n$$

The distribution coefficients a_k are merely the changes in admittance ΔA_k corresponding to the interval used, as shown in Fig. 16. The values of inflow $\bar{I}$ are taken as the mean values of inflow I over the interval of integration.

It may be seen from Fig. 16 that the continuity condition of

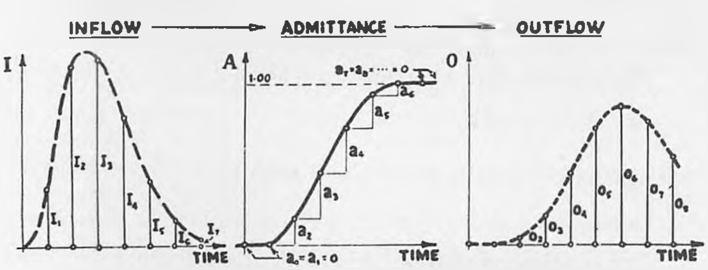


FIG. 16.—NUMERICAL ROUTING WITH GIVEN ADMITTANCE.

equation (3.13) imposes a condition on the distribution coefficients a_k which represent the admittance in this finite form, namely:

$$\sum_k a_k = 1.0 \qquad (3.14)$$

In the case of oscillating admittance functions (not shown) the coefficients a_k might be both positive and negative. It should be noted in passing that the distribution coefficients here are analogous to the distribution coefficients of the unit hydrograph method.

Table C details the calculation procedure corresponding to Fig. 16. The resemblance to the unit hydrograph procedure becomes clear at this point. Indeed, in this form, the admittance method becomes the generalization of the method proposed by Tatum [30], and the so-called "non-storage" routing procedures discussed in the book of Linsley, Kohler, and Paulhus [31].

TABLE C

Example of Admittance Routing

		k	0	1	2	3	4	5	6	7	8	...
		a_k	0	0	0.16	0.29	0.28	0.20	0.07	0	$\Sigma = 1.00$	---
k	I_k	$\bar{I}_k$ \ O	0	0	14	84	221	362	428	385	271	..
0	0	90	0	0	14	26	25	18	6	0	0	...
1	180	365	—→	0	0	58	106	102	73	26	0	...
2	550	565	—	—→	0	0	90	164	159	113	40	...
3	580	490	—	—	—→	0	0	78	142	137	98	...
4	400	300	—	—	—	—→	0	0	48	87	84	...
5	200	140	—	—	—	—	—→	0	0	22	41	...
6	80	50	—	—	—	—	—	—→	0	0	8	...
7	20	...	—	—	—	—	—	—	—→	0	0	...

It is altogether possible to perform the routing procedure using automatic computing equipment, either in step-form using equation (3.14) or in continuous form using equation (3.1), which may be rewritten for flood applications in the form:

$$O(t) = \int_0^t A(t-\tau)\, dI(\tau) = \int_0^t I(t-\tau)\, dA(\tau) \qquad (3.16)$$

There now exists high-speed electronic computing equipment of commercial make [32], which will permit the continuous solution of both the direct problem (Given I and A to find O), as well as the inverse problem (Given I and O to find A), which will be discussed in the next article. Fig. 17 shows schematically the nature of such computer components for a single river reach.

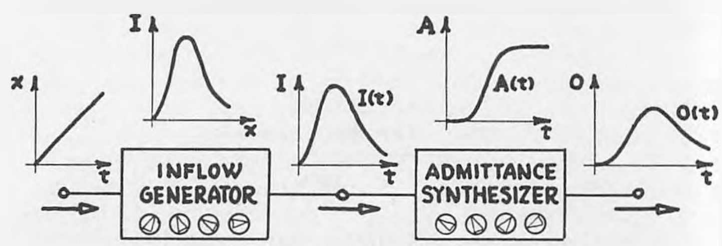


FIG. 17.—AUTOMATIC ADMITTANCE ROUTER.

Determination of Flood Admittances

As mentioned at the outset of this presentation, the flood admittances of natural channels may be determined both on the basis of past flood records and from rational considerations.

The determination of $A(t)$ where $I(t)$ and $O(t)$ are known functions is essentially a so-called "integral equation" problem. For manual computation, using the finite difference form of the integral equation (3.14), it might at first be supposed that a simple inversion of the chain of linear equations:

$$O_1 = a_1 \bar{I}_1$$

$$O_2 = a_2 \bar{I}_1 + a_1 \bar{I}_2 \qquad (3.17)$$

$$O_3 = a_3 \bar{I}_1 + a_2 \bar{I}_2 + a_1 \bar{I}_3$$

(etc.)

in the form:

$$a_1 = O_1/\bar{I}_1$$

$$a_2 = (O_2 - a_1 \bar{I}_2)/\bar{I}_1 \qquad (3.18)$$

$$a_3 = (O_3 - a_2 \bar{I}_2 - a_1 \bar{I}_3)/\bar{I}_1$$

(etc.)

would solve this problem. However, in practice, the application of equations (3.18) gives rise to unstable calculations, since both numerators and denominators on the right side are generally small. As more workable alternatives, one may set up a chain of linear regression equations for the a_k and use successive approximation techniques for solutions, or one may use a series of convergent approximations to the distribution coefficients a directly.

Two items of existing automatic computing equipment offer possibilities of considerable assistance for this inversion problem. The first is an integral equation analyser at M. I. T., upon which solutions of admittance functions have been successfully obtained. The second machine is the linear synthesizer previously referred to and sketched in Fig. 17, which is capable of solving rapidly both the direct and inverse problem. As pictured, a first element is capable of generating an arbitrary wave form representing I(t), while the second component, capable of either direct or trial adjustment, synthesizes the admittance A(t). The output from this component therefore represents the outflow O(t). It would be possible to set up a network of these components to simulate the drainage system of Fig. 14. With such a computer assemblage, one would be able to route and predict floods continuously, and continually revise the estimates of reach inflows and local inflows, as well as the admittance functions themselves. A flood analyser in this form would be more versatile and flexible than the existing machines developed for this task [33] [34] [35].

As a result of preliminary studies using the manual computation and machine techniques outlined above, it has been found that in many cases, the admittances of river reaches may be represented with

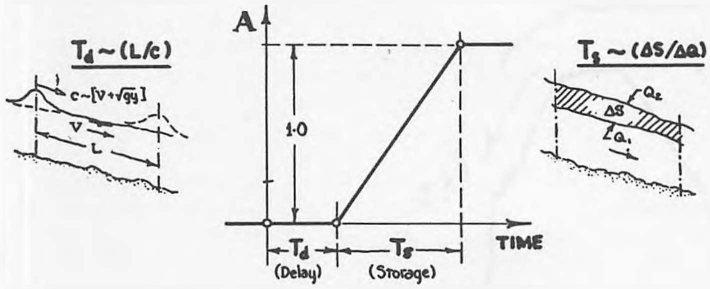


FIG. 18.—APPROXIMATE FORM OF ADMITTANCE FUNCTION FOR MOST NATURAL STREAMS.

surprising accuracy by the simple function shown in Fig. 18. This function can be described by just two constants: the travel delay T_d and the storage-distortion time T_s. As indicated, the travel delay may be associated with the time of wave propagation down the reach, while the storage time is related to the ratio of the valley storage increment ΔS to the equilibrium flow increment $\Delta Q = \Delta I = \Delta O$. Since this admittance resembles that of Fig. 15(c), it might be stated that the effect of routing through a normal river reach is roughly equivalent to the combined effect of a pure delay and a single storage lag; this feature has been suspected for some time, being mentioned, for example, in the paper of Clark [36].

Another rich source of qualitive and quantitive information in flood admittances comes through an analogy existing between the equations for unsteady flow in a uniform channel and those describing transients in an electrical transmission line; this analogy is indicated in Table D. For relatively small flow changes (small floods) the analogy is quantitatively valid, while for major floods the parallelism becomes merely qualitative. The advantage of such an analogy lies in the fact that there is a large body of electrical experience which may be applied, with appropriate limitations, to the flood routing problem. For example, one significant conclusion that has been drawn from theoretical and experimental studies of the transmission line problem.

TABLE D
Flood Wave — Electrical Analogy

	Uniform Water Channel		Uniform Electrical Line	
A — Equations:	$-\frac{\partial y}{\partial x} = \frac{1}{g}\frac{\partial v}{\partial t} + mv$		$-\frac{\partial e}{\partial x} = L\frac{\partial i}{\partial t} + Ri$	
	$-\frac{\partial v}{\partial x} = \frac{1}{d}\frac{\partial y}{\partial t}$		$-\frac{\partial i}{\partial x} = C\frac{\partial e}{\partial t}$	
B — Variables:	Depth Change	y	Voltage	e
	Velocity Change	v	Current	i
	Time	t	Time	t
	Distance	x	Distance	x
C — Constants	Unit Inertia	(1/g)	Unit Inductance	(L)
	Unit Storage	(1/d)	Unit Capacitance	(C)
	Unit Resistance	$(m = V_o/C_o^2 R_o)$	Unit Resistance	(R)
D — Properties:	Wave Celerity:	$c = \sqrt{gd}$	Wave Celerity:	$c = 1/\sqrt{LC}$
	Surge Impedance:	$Z_o = \sqrt{d/g}$	Surge Impedance:	$Z_o = \sqrt{L/C}$
	Travel Delay:	$T_d = l/c$	Travel Delay:	$T_d = l/c$
	Diffusion Time:	$T_s = (m/d)l^2$	Diffusion Time:	$T_s = (RC)l^2$

and which is directly applicable to flood admittances, might be stated as follows (with the hydraulic interpretation given parenthetically). If the transmission line (or ***natural channel***) is of sufficient length, regardless of the original frequencies or intensities of the input signal to the line (or ***flood inflow***), the resistance term Ri (or ***mV***) will eventually become predominant compared to the inductive term $L\frac{\partial i}{\partial t}$ (or ***inertia*** $\frac{1}{g}\frac{\partial v}{\partial t}$). This means that at a point sufficiently removed from the input or sending end of the transmission line, the admittance function relating the signal at this point to the input signal closely resembles Fig. 18 and is of a pure delay plus diffusion form. Knowledge of this sort is very helpful when applied to the problem of determining flood admittances from past records, making suitable corrections and allowances for unknown local inflows.

Another clue to the general significance of the admittance function shown in Fig. 18 comes from the fact that if the Muskingum equation (3.10) is applied to a channel in such a way that the reach is subdivided into many small parts, all with the same time constant $T_1 = T_o$, the resulting admittance resembles Fig. 18 if the number of subdivisions is very large. However, even with many such sections, the admittance still has an initial sudden jump to —1 (for an ***odd*** number) or +1 (for an *even* number). These jumps cannot be physically meaningful, due to the finite time for a disturbance to traverse a reach of a natural stream.

Non-Linear Effects

Attempts have been made in recent years to introduce non-linear storage equations, such as:

$$S = aI^m + bO^n \qquad (3.19)$$

into the "lumped" continuity equation

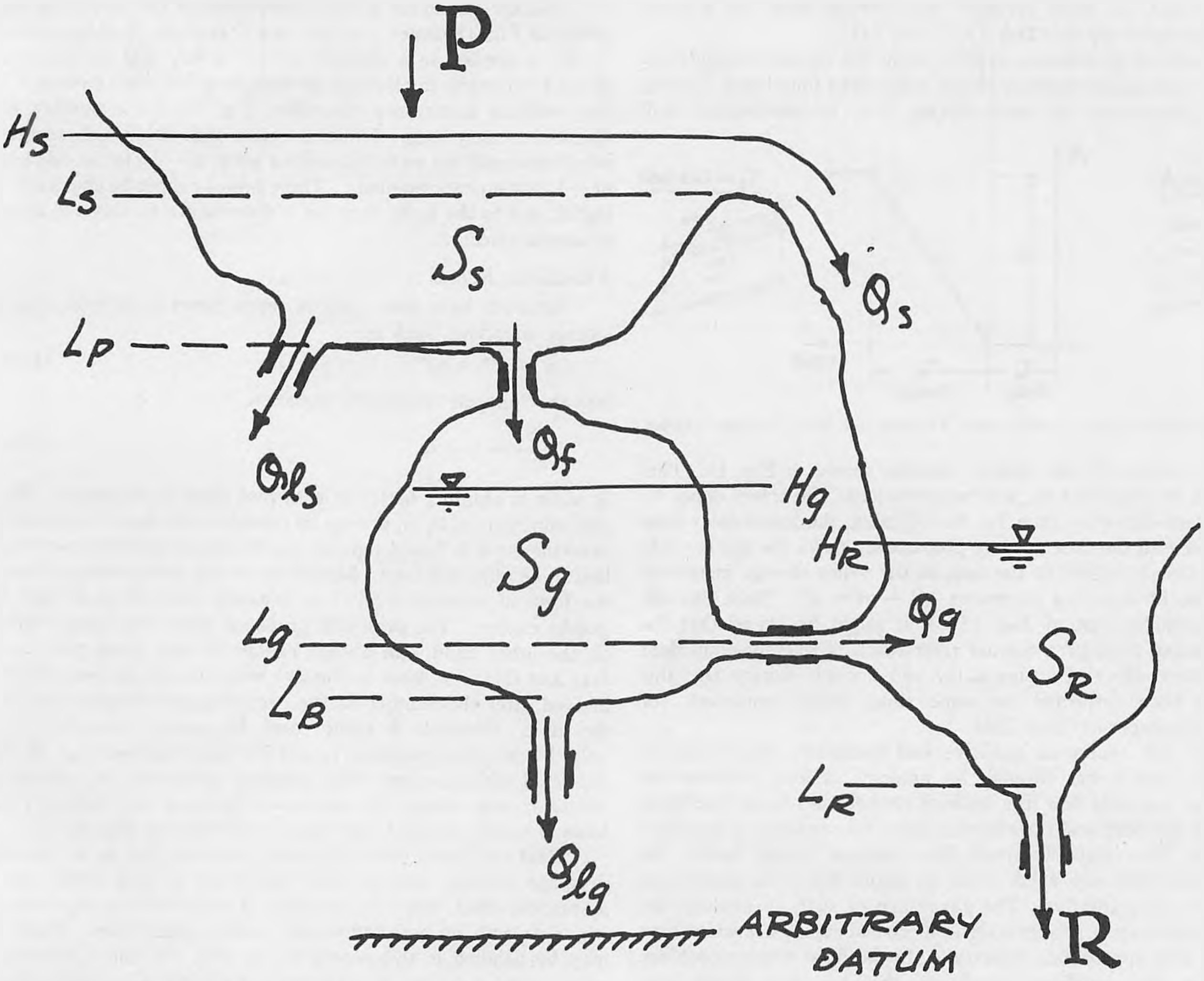
$$\frac{dS}{dt} = I - O \qquad (3.20)$$

in order to obtain a better fit with past flood hydrographs. While the four constants a, b, m, n, may be considered to have a rational origin, nevertheless it is found that for no two floods will the coefficients for best fit be quite the same. Moreover, for any given storage function of the form of equation (3.19), it is rarely possible to fit past hydrographs exactly. The presently proposed linear admittance technique, on the other hand, can always exactly fit any given past record of I(t) and O(t) (at least in theory) even though no two admittances, derived after the manner of the preceding paragraphs, will be quite the same. However, in many cases, by using appropriate values of only the two time constants T_d and T_s, flood outflows may be fitted or predicted with ease and with adequate precision; in addition, these constants may always be estimated (lacking past records) on the basis of simple physical reasoning as indicated on Fig. 18.

Real non-linear effects do exist, however, just as in the unit hydrograph method. The two most significant of these seem to be: (1) a seasonal effect, and (2) an order of magnitude or flood-size effect which depends on both inflow and outflow magnitudes. These effects may be handled in this procedure, as with the unit hydrograph, by constructing a family of admittance functions, all of which resemble one another qualitatively, but which are quantitively slightly different (i.e. the values of T_d and T_s are different). Sufficient experience with this method is not yet available to estimate the number of such admittances necessary to characterize a channel reach adequately and precisely; however, with an automatic computer technique after the fashion of Figs. 14 and 17, it would always be possible to vary the admittances by trial for best fit with a flood *during its occurrence*.

Routing from Point Rainfall to Runoff

General Situation



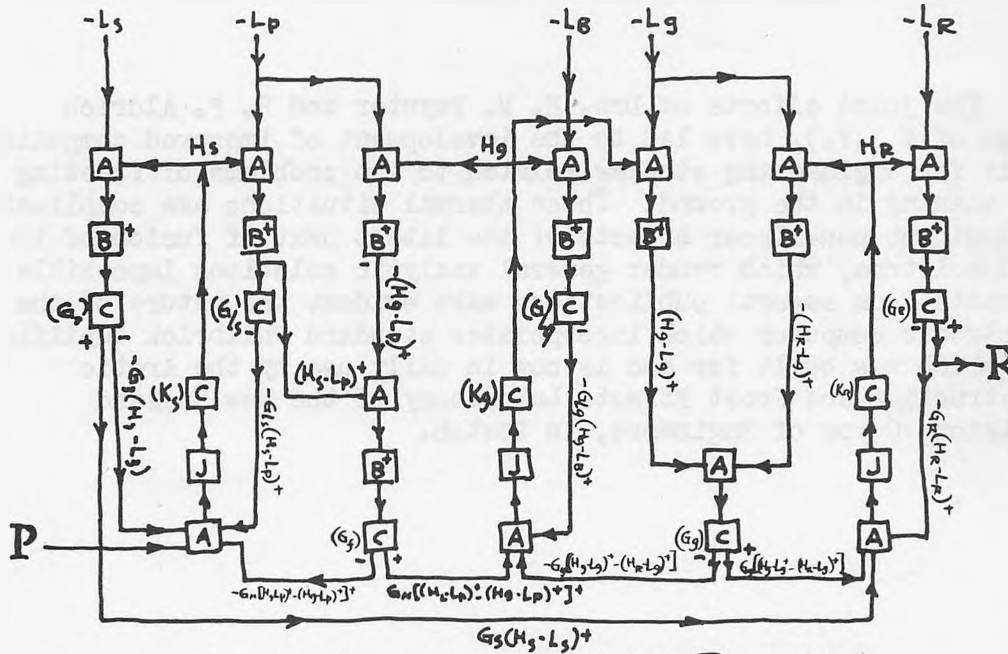
P, Q, R = Rates of flow

S = Volumes

H, L = Levels

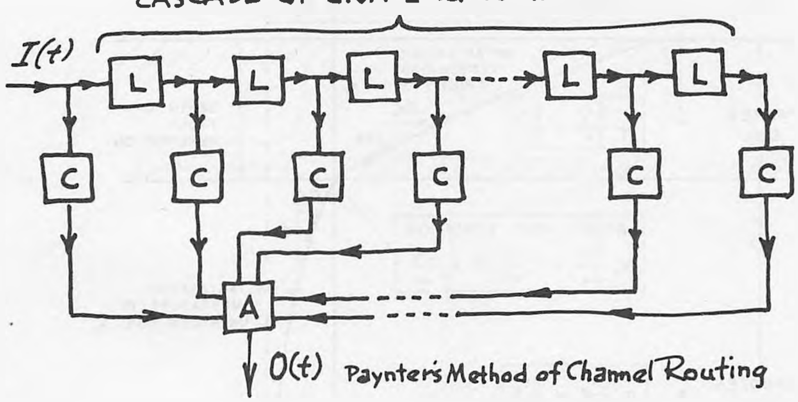
P and **R** are measurable quantities

from H.M. Paynters Research Notes



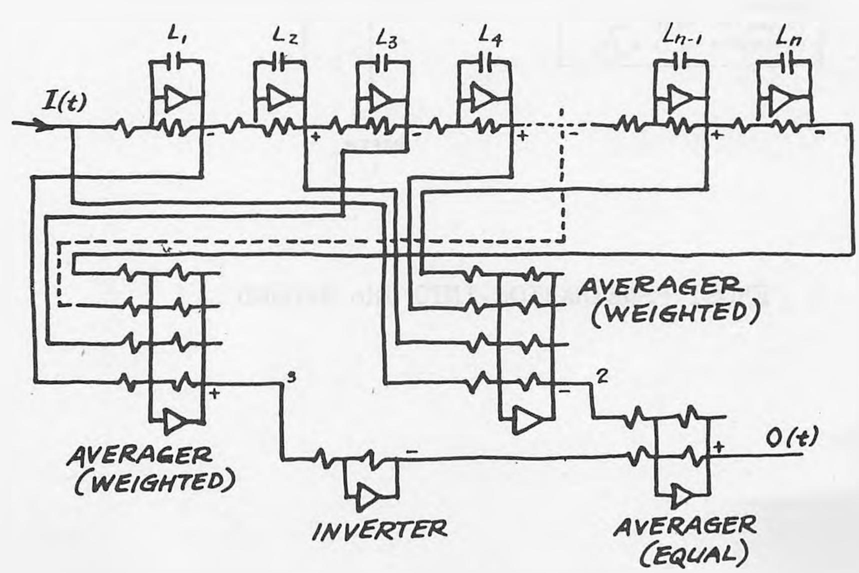
Routing from Point Rainfall to Runoff

CASCADE OF UNIT LAG COMPONENTS



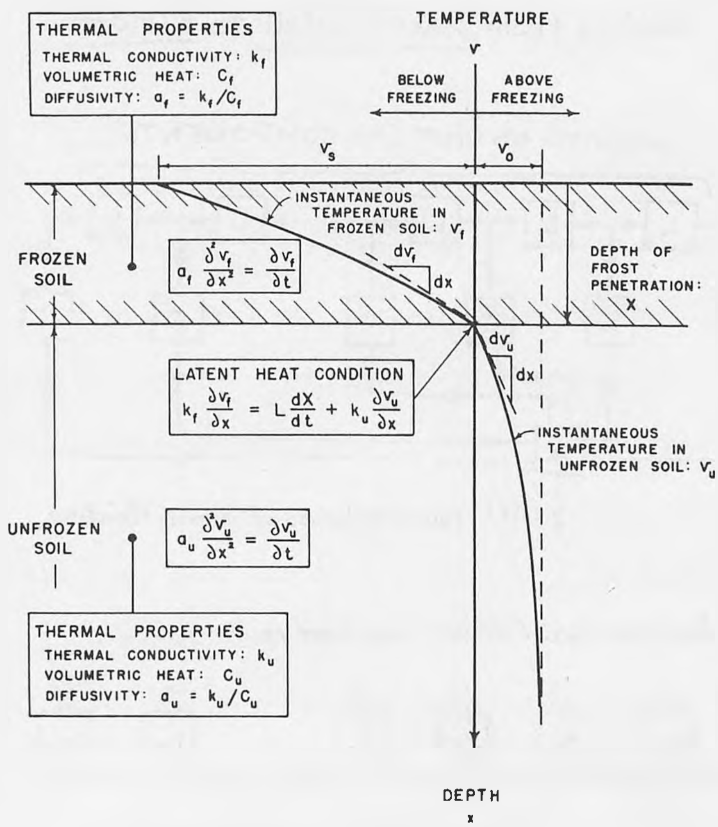
Paynter's Method of Channel Routing

The ELECTRIC CIRCUIT involved is:



ANALOG SOLUTIONS OF THERMAL PROBLEMS

The joint efforts of Drs. H. M. Paynter and H. P. Aldrich (also of M.I.T.), have led to the development of improved computing tools for engineering studies related to the problems of freezing and thawing in the ground. These thermal situations are complicated by dominant non-linear effects of the latent heat of fusion of the soil moisture, which render general analytic solutions impossible. Extracts from several publications make evident the nature of the electronic computor which incorporates standard Philbrick amplifiers and which was built for and is now in daily use by the Arctic Construction and Frost Effects Laboratory of the New England Division, Corps of Engineers, in Boston.



FROST PENETRATION INTO THE GROUND

Excerpts from:

FIRST INTEREM REPORT
Analytic Studies of Freezing and Thawing of Soils
By H.P.Aldrich,Jr. and H,M.Paynter

for the
Arctic Construction and Frost Effects Laboratory
New England Division, Corps of Engineers

Boston, Mass.,June 1953

PART II. ANALYTICAL STUDIES 2-01. MATHEMATICAL FORMULATION. -

a. Synopsis. - This article treats the mathematical basis for studies of thermal diffusion, depth of frost and thaw penetration, as well as the seepage diffusion concepts required for analytical studies of the thaw-consolidation problem.

After stating the first principles from which all developments begin, both exact differential equations and approximate difference equations are given which embody these principles. Moreover, practical and fruitful fluid and electric analogies to the thermal problem are outlined.

In order to overcome the handicap to analysis imposed by the complex latent heat conditions, several practical machine and hand methods of computation are presented.

b. Fundamental Principles. - At any point and instant in a thermal medium, such as soil, the local rate of change of total heat content (or internal energy) u must exactly balance the net efflux of heat. This amounts to a statement that heat is conserved and in the absence of heat sources, its flow must satisfy the continuity equation:

$$\frac{\partial u}{\partial t} + \operatorname{div} \vec{q} = 0 \qquad \ldots. (2 - 1)$$

Moreover, this directed heat flow $\vec{q}$ is experimentally found to be proportional to the negative temperature gradient, with the factor of proportionality defined as the thermal conductivity k of the substance. This heat flow law thus yields the second fundamental equation:

$$\vec{q} = -k \; \overrightarrow{\text{grad}} \; v \qquad \text{.... (2 - 2)}$$

where v is the instantaneous temperature field in the material.

These two principles were first set forth by J. B. J. Fourier in his classic treatise on the "Theorie Analytique de Chaleur" in 1822, and they form the basis of all rational investigations of heat conduction phenomena.

The heat content u of a substance is found experimentally to depend primarily upon its temperature v, and one finds under constant volume conditions, at least, that from a consideration of energy conservation, it must vary only with the temperature. For example, the experimental curve of the internal energy u of water as a function of temperature v is shown as Figure 1. Three distinct, effectively linear segments are distinguished, as marked by the numbers I, II and III.

The region I denotes the fluid state of water in which the temperature v increases approximately in proportion to the increase in stored heat u. This proportionality or slope we may define as the volumetric specific heat (under constant volume) C_u in the unfrozen or liquid phase, or:

$$C_u = \left.\frac{\partial u}{\partial v}\right]_{\text{unfrozen}} \qquad \text{....(2 - 3)}$$

In the region marked III, the substance (water) is in the frozen or solid phase and the slope of the curve may be measured by the frozen volumetric specific heat C_f, where:

$$C_f = \left.\frac{\partial u}{\partial v}\right]_{\text{unfrozen}} \qquad \text{....(2 - 4)}$$

The flat or horizontal portion II of the internal energy curve displays the heat content which must be released or absorbed to produce the phase change, and which is defined as the latent heat of fusion L of the material.

As stated, the curves I and III are not exactly straight, but the deviations are usually negligible such that the small errors introduced by assuming C_u and C_f both constant are consistent with the assumption of negligible unit volume change, upon change of phase.

Under these conditions, except in the region of the fusion (or freezing) temperature, the variation in u with time may be directly related to the variation in v, as the following:

$$\frac{\partial u}{\partial t} = C \frac{\partial v}{\partial t} \qquad \text{....(2 - 5)}$$

where C becomes C_u or C_f depending on whether v is above or below freezing, respectively.

Substituting this last expression, together with Equation (2 - 2), into Equation (2 - 1) yields:

$$C \frac{\partial v}{\partial t} + \text{div}\,(-k\ \overrightarrow{\text{grad}}\ v) = 0 \qquad \text{....(2 - 6)}$$

c. Differential Equations. - If the substance is homogeneous such that C and k are constant throughout a region, and is entirely either above or below the fusion temperature, then the following differential equation holds for the temperature field v:

$$C\frac{\partial v}{\partial t} = k\ \nabla^2 v \qquad \text{....(2 - 7)}$$

This last expression may be re-arranged to give the thermal diffusion equation:

$$\frac{\partial v}{\partial t} = a \nabla^2 v \qquad \text{....(2 - 8)}$$

in which $a = k/C$ is called the thermal diffusivity. For three-dimensional (x, y, z) problems, this equation would be written:

$$\frac{\partial v}{\partial t} = a \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) \qquad \text{....(2 - 8a)}$$

For two-dimensional (x, y) situations it becomes:

$$\frac{\partial v}{\partial t} = a \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \qquad \text{....(2 - 8b)}$$

while for one-dimensional (x) problems there remains:

$$\frac{\partial v}{\partial t} = a \frac{\partial^2 v}{\partial x^2} \qquad \text{....(2 - 8c)}$$

d. Approximate Equations. - While the expressions in the previous sections describe the frost penetration problem in a mathematically correct fashion, exact solutions can be found only for a small number of idealized cases, due to the complex conditions of the latent heat transfer and other effects.

When such conditions arise in engineering problems, it is the customary practice to resort to largely empirical techniques to obtain reliable prediction formulas. However, a useful alternative follows from the substitution of an approximate system of simpler mathematical equations, which may either be solved directly in closed form, or which may be solved by various forms of computation or analogies.

In particular, it is often found effective to replace the continuous conduction equation (written in one-dimensional form for simplicity):

$$q = -k\frac{\partial v}{\partial x} \qquad \ldots\ldots(2 - 9)$$

by a "lumped" approximation for a finite layer n of thickness $\Delta x = d_n$, in the form:

$$q_n = -k_n \frac{(v_{n+1} - v_n)}{d_n} \qquad \ldots\ldots(2 - 10)$$

or, by defining a thermal resistance $R_n = d_n/k_n$, as:

$$q_n = +\frac{(v_n - v_{n+1})}{R_n} \qquad \ldots\ldots(2 - 11)$$

The thermal continuity equation is here considered to hold for lumps of material concentrated at the terminals or end points of a series of such resistors; the relation between the net heat input and the stored heat for the n-th such lump would then be given by the equation:

$$\frac{du_n}{dt} = q_{n-1} - q_n \qquad \ldots\ldots(2 - 12)$$

and the relation between the temperature v_n and heat u_n, is indicated by the form:

$$v_n = f_n(u_n) \qquad \ldots\ldots(2 - 13)$$

where the function f_n resembles the curve of Figure 1.

For hand computations and digital computing machines the continuous thermal continuity Equation (2 - 12) may be expressed discretely in terms of the finite difference approximation:

$$\frac{du_n}{dt} \approx \frac{u_n(t + \Delta t) - u_n(t)}{\Delta t} \qquad \ldots\ldots(2 - 14)$$

to give:

$$u_n(t+\Delta t) - u_n(t) = \Delta t \left[q_{n-1}(t) - q_n(t)\right] \qquad \ldots\ldots(2 - 15)$$

In terms of a three-dimensional space lattice, the differential Equation (2-8a) may be approximately represented by the finite difference equation:

$$\frac{v_{\Delta t} - v_o}{\Delta t} = a\left[\frac{v_1 + v_2 - 2v_o}{(\Delta x)^2} + \frac{v_3 + v_4 - 2v_o}{(\Delta y)^2} + \frac{v_5 + v_6 - 2v_o}{(\Delta z)^2}\right] \quad \text{....(2 - 16)}$$

where:

$v_{\Delta t} = v(x, y, z, t + \Delta t)$

$v_o = v(x, y, z, t)$

$v_1 = v(x + \Delta x, y, z, t)$

$v_2 = v(x - \Delta x, y, z, t)$

$v_3 = v(x, y + \Delta y, z, t)$

$v_4 = v(x, y - \Delta y, z, t)$

$v_5 = v(x, y, z + \Delta z, t)$

$v_6 = v(x, y, z - \Delta z, t)$

If $\Delta x = \Delta y = \Delta z = \Delta s$, this may be abbreviated to the form:

$$v_{\Delta t} - v_o = \left(\frac{a\,\Delta t}{\Delta s^2}\right)(v_1 + v_2 + v_3 + v_4 + v_5 + v_6 - 6v_o) \quad \text{....(2 - 17)}$$

or, with $\beta \equiv a\Delta t/\Delta s^2$, to the expression:

$$v_{\Delta t} = \beta \sum_{k=1}^{6} v_k + (1-6\beta)v_o \quad \text{....(2 - 17a)}$$

In the same way, Equation (2-8b) becomes:

$$v_{\Delta t} = \beta \sum_{k=1}^{4} v_k + (1-4\beta)v_o \quad \text{....(2 - 17b)}$$

and the one dimensional Equation (2-8c) becomes:

$$v_{\Delta t} = \beta \sum_{k=1}^{2} v_k + (1-2\beta)v_o \qquad \ldots.(2 - 17c)$$

$$= \beta(v_1 + v_2) + (1-2\beta)v_o$$

It has been demonstrated that certain stability limits impose conditions on the maximum values of β which can be used for computation (1) (2).*

e. Fluid and Electric Analogies. - It is of significant interest to compare the thermal principles and equations to the analogous set of equations governing the diffusion of water pressures in a confined fluid medium. This analogy gives rise to a simply realizeable physical model as described in Sections 2-01-f-(1) and 4-04-a below as well as in References 3 and 4.

Moreover, the same equations hold for the diffusion of currents and voltages in an electrical conducting medium which has a fixed capacitance to ground. This, too, may be realized in simple one and two-dimensional passive electrical circuits, as described in References 5, 6 and 10 in addition to the lumped equivalent as represented by the electronic analog computor outlined in Section 2-01-f-(2).

These analogies may best be summarized in tabular form as shown in Table I. The practical value of such analogies becomes obvious when the much greater manipulative flexibility of the fluid and electrical systems are considered. Moreover, instrumentation is significantly simpler and more precise in the analog media than in the prototype thermal system. However, one feature is distinctively absent, under normal conditions, in the fluid and electrical systems that indeed characterizes the present thermal problem and makes studies difficult: the presence of state changes

*Numbers in parenthesis refer to Bibliography

TABLE I

THERMAL - FLUID - ELECTRIC ANALOGIES

ITEM	MEDIUM		
	THERMAL	FLUID	ELECTRIC
A-Variables (1)	Heat u	Volume S	Charge Q
(2)	Heat Flow q	Flow Q	Current i
(3)	Temperature v	Heat H	Voltage e
B-Principles:			
Continuity (1)	$\frac{\partial u}{\partial t} + \nabla \cdot q = 0$	$\frac{\partial S}{\partial t} + \nabla \cdot Q = 0$	$\frac{\partial Q}{\partial t} + \nabla \cdot i = 0$
Conductivity (2)	$q = -k\nabla v$	$Q = -k\nabla H$	$i = -\sigma \nabla e$
Capacitance (3)	$eu = Cdv$	$dS = AdH$	$dQ = Cde$

and the concurrent latent heat effects. The analogy to this situation would require the equivalent capacitances to remain temporarily infinite until a corresponding amount of fluid volume or electrical charge had been transferred to or from the element. How this feature is secured in the hydraulic analog and in the electronic analog is described briefly in Sections 2-01-f-(1) and 2-01-f-(2), respectively.

f. Practical Methods of Computation. - Due to the complexity of the latent heat conditions in addition to the normally heterogeneous nature of the soil, and the randomness in the surface temperature distributions, it is not possible to obtain very many practically significant solutions to these problems in closed, mathematical form.

In order to overcome this handicap to rational analysis, several useful machine and hand computation methods are presented in the following section which are based on the principles outlined in the previous sections.

(1) Hydraulic Model. - By applying the analogy relationships of Section 2-01-e and the "lumping" approximations of Section 2-01-d, one

finds that it is possible to solve a wide variety of thermal problems to an excellent approximation by means of a simple hydraulic model.

Basically, the hydraulic analog would consist of a series of small vertical chambers or wells connected to each other at the bottom through laminar tubes or orifices. Each chamber represents the heat content, both volumetric and latent, of a lump or layer of soil at a particular depth. Distance along the model corresponds to depth in the soil. The surface level of the liquid in the well is analogous to the temperature. To simulate the latent heat content of the layer an expansion chamber of suitable area would be provided at a level representing the freezing temperature of the soil moisture. The size of this expansion, as well as the size of the vertical chambers, would be adjustable to correspond to the variations in latent heat and volumetric heat, respectively. More specifically, the sectional areas of the chambers are directly analogous to the C and L values, since storage volume S represents heat u and level H represents temperature v.

The interconnecting capillaries or laminar orifices are directly analogous to the conductive paths with the fluid conductance representing the thermal conductance. These conductances can be made adjustable in a simple fashion so as to readily permit wide variations in the values of k between layers.

The surface temperature variations are directly represented by variations in the level of an overflow-source tank at the end of the model which corresponds to the ground surface.

The flexibility of such a scheme is directly evident; moreover, the same model can be used to represent consolidation, seepage flow and other diffusion problems. Such a model had been proposed several years ago by Barron (4) and has been successfully used for latent heat problems by Bäckström (7). For diffusion problems associated with consolidation phenomena such a model has been effectively employed by one of the authors of this report (3).

(2) Electronic Analog Computor. - The hydraulic model just described becomes complex for 2-dimensional problems and 3-dimensional representations seem entirely impractical, due principally to the difficulties of interconnecting the chambers in a manner which will still permit direct observation.

These disadvantages are not present in an electronic computor operating on essentially analogous principles. A 2-dimensional array of elements can be mounted on relay racks covering a relatively small amount of wall space.

The basic computing elements can be designed to solve the "lumped" approximate equations - the same as those which govern the behavior of the hydraulic model. For the sake of simplicity, only the circuits for a one-dimensional homogeneous computor will be outlined here, although, as stated, this method readily generalizes to the 2 and 3-dimensional cases.

In terms of operational blocks, the basic structure of the computor components representing one lump or layer of soil is shown in Figure 2A where the letters stand for the operations indicated beside the drawing. In particular, the "Z" - component represents the latent heat effect and has an input - output characteristic similar to Figure 1.

Thus each layer of soil is represented by an assemblage of computing components F which are interconnected as shown in Figure 2B to represent the entire soil mass.

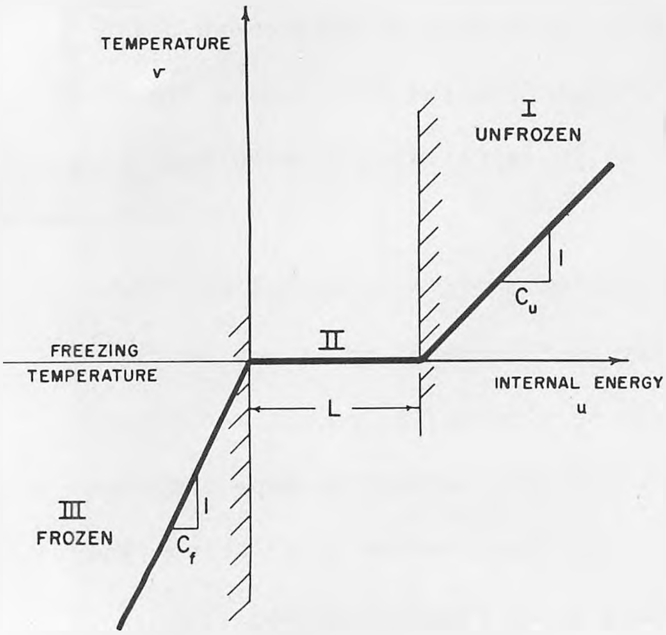
This particular type of electronic computor is referred to as an active representation in which all variables are represented by analogous voltages in order to expedite the instrumentation and interconnection. Thus ground surface temperature fluctuations are represented by corresponding input voltage variations applied to the elements representing the top layer of soil.

This particular representation is therefore a so-called "active" analog, in which all variables are voltages; in the absence of latent heat components, such computors are discussed in References 8 and 9. This type of analog computor is distinct from the more common "passive" analogs (5) (6), in which temperature (v) is represented by a voltage (e), but heat flow (q), by a current (i).

Such a computor can be made to operate on so-called "fast-time" to permit display on a cathode ray oscilloscope, or it may be run at "slow-time" (over a period of seconds or minutes) to permit recording on a single or multi-channel chart oscillograph. Actual surface temperature observations may be traced by a stylus input device which drives the computor, permitting direct check against field temperature and frost - thaw depth measurements.

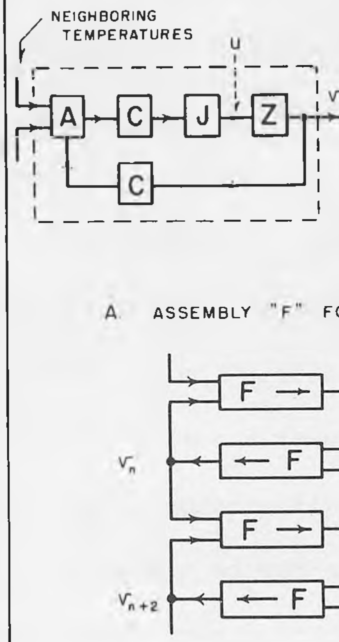
The internal circuitry of a single "F" component is indicated in Figure 3. The input resistors (R_i) and the feedback resistor (R_f) correspond to the thermal resistance. The feedback capacitor (C) corresponds to the volumetric heat of the soil layer. The dropping resistor (R_d) determines the ratio of latent heat to volumetric heat. Since the computor may be arranged so that these circuit elements can be varied, it is possible to represent variations in soil thermal properties in a routine fashion.

The only changes in the above circuit required for 2 and 3-dimensional representation would be to provide four and six input resistors (R_i) respectively



INTERNAL ENERGY DIAGRAM FOR WATER

FIGURE 1



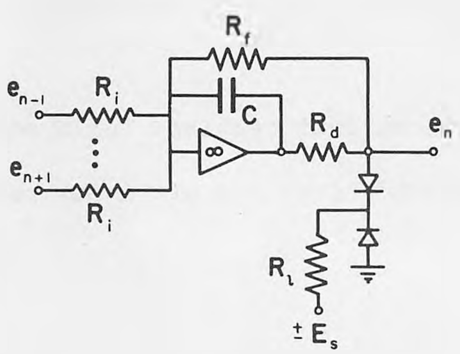
SYMBOL	NAME	OPERATION
A	Adding Component	Output $= \Sigma$ (Inputs)
C	Coefficient Component	Output $= C \cdot$ (Input)
J	Integrating Component	Output $= \int$ (Input) dt
Z	Inert Zone Component	(Similar to Figure 1)

A. ASSEMBLY "F" FOR ONE LAYER

B. INTERCONNECTION OF "F" COMPONENTS

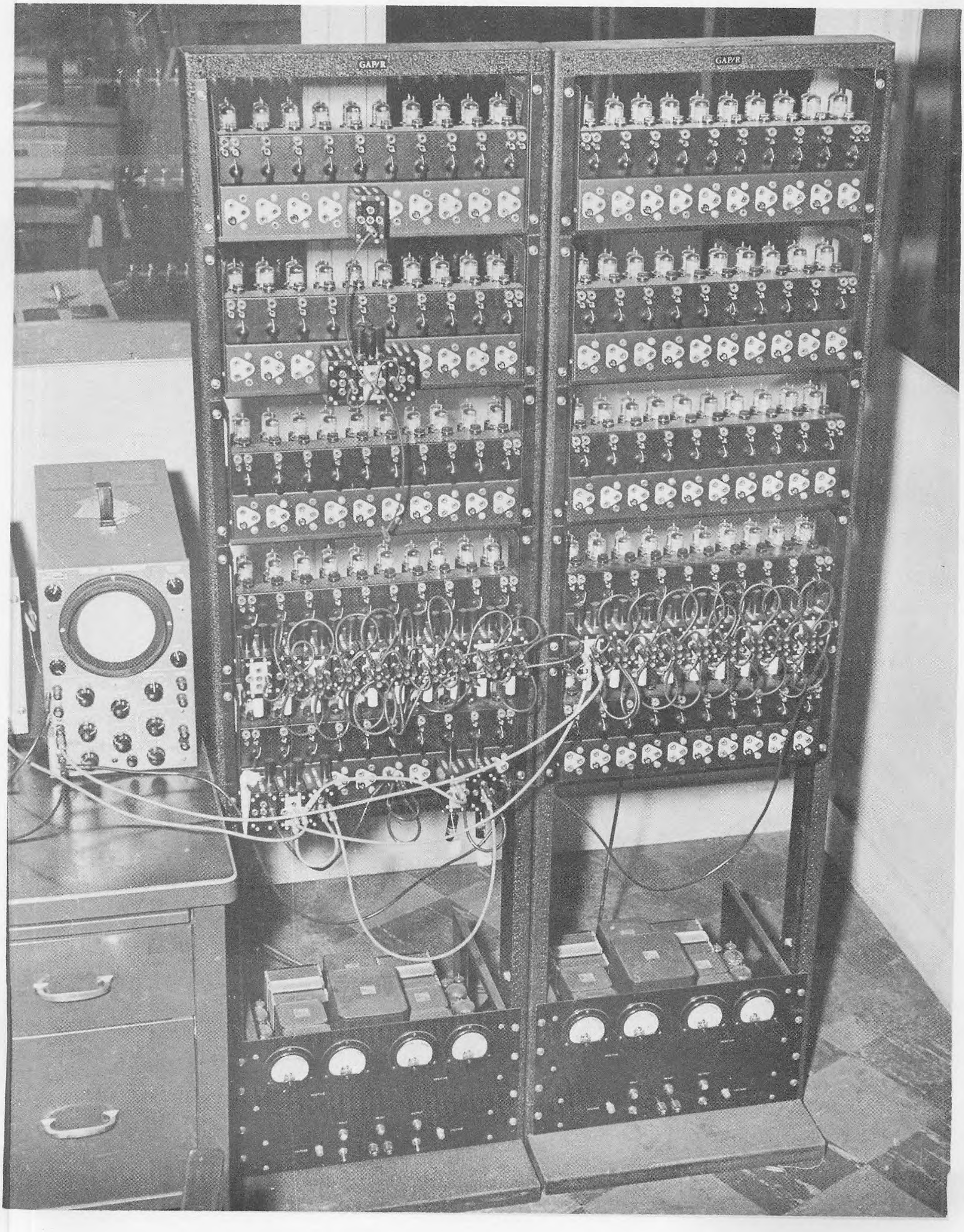
SCHEMATIC BLOCK DIAGRAM
ELECTRONIC ANALOG FROST COMPUTER

FIGURE 2

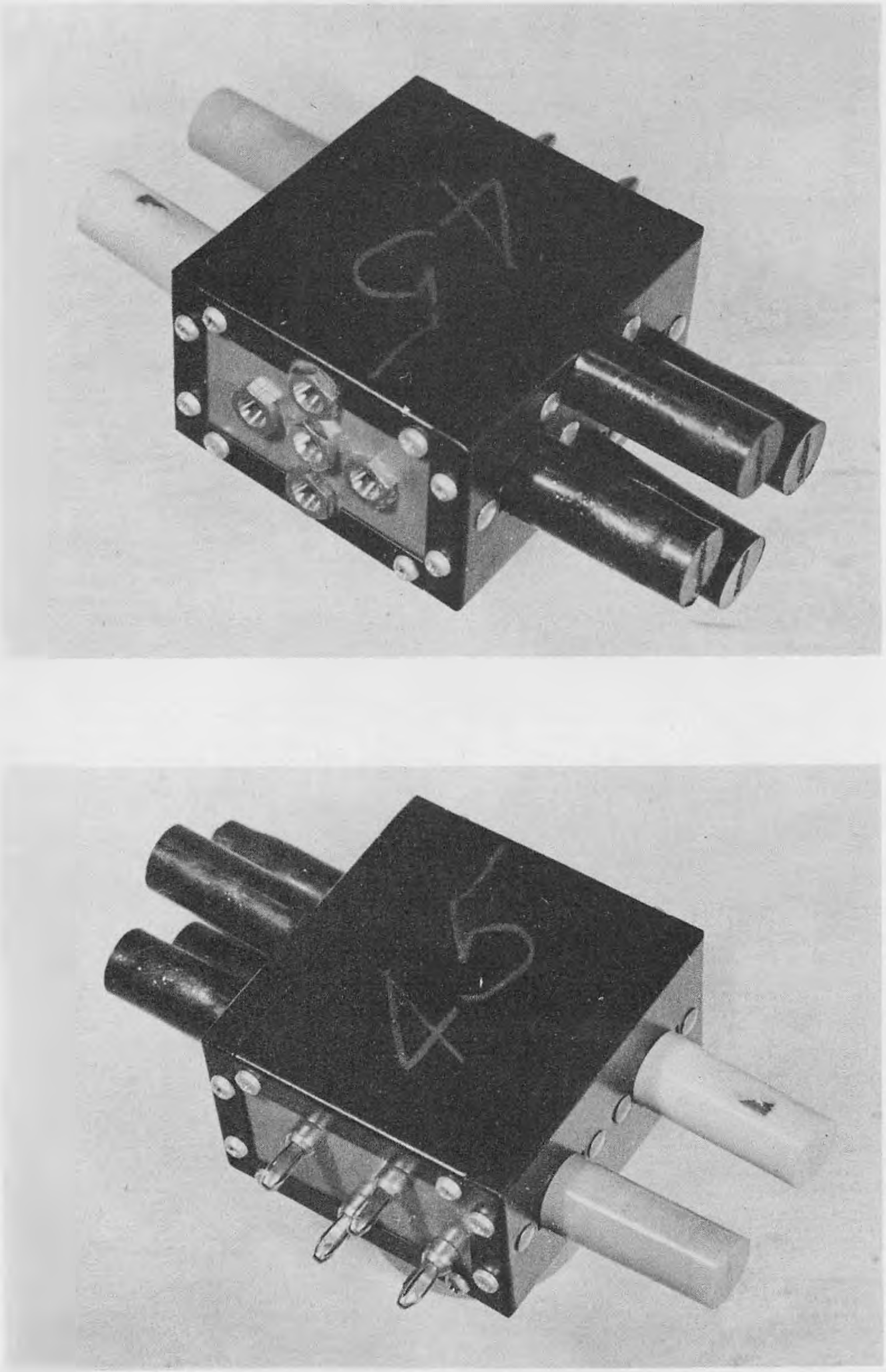


CIRCUIT DIAGRAM
SINGLE "F" COMPONENT
ELECTRONIC ANALOG FROST COMPUTER

FIGURE 3



1.) The Electronic Analog Frost Computor (EAFCOM)
of the NED, Corps of Engineers



2.) Plug-in Components Representing Thermal
Properties of the Ground

THE ANALOGUE COMPUTER and Automatic Control Applications

BY ROBERT J. BIBBERO, *Technical Editor*
AUTOMATIC CONTROL NOVEMBER 1954

■ Whether or not the automatic factory of the future will be controlled by a digital computer is still undecided. Whatever happens the analogue computer will surely be an integral part of the assembly.

In the broadest sense, any essentially continuous transducer, such as a feedback potentiometer, is an analogue machine, in that it converts a mechanical variable such as linear or rotational position of a tool or other measurable quantity, into an electrical, pneumatic, or hydraulic signal. In the same sense, any means by which this signal is subsequently altered or modified can be called analogue computation.

Anyone not familiar with analogue computation need only examine a slide rule to grasp its principle. The slide rule is actually an analogue computer, consisting of two calibrated rods which can be adjusted lengthwise relative to each other. The length of the calibrations is an analogue of number, or more exactly, the logarithm of the numbers to be multiplied. Adding one calibrated length to another is the analogue of adding the logarithms, hence of multiplying the two numbers together. In the same way, a rotatable potentiometer can be used as an electrical analogue computer. A voltage placed across the potentiometer terminals is the analogue of one number, and the degree of rotation of the slider the analogue of the other.

Early Developments — The earliest general purpose analogue computers were mechanical devices, in which addition was performed by mechanical differentials (similar to the differential in an automobile), integration by ball-and-disk mechanisms, multiplication by linkages, and so forth. The mechanical analogue computer was highly developed during World War I, as well as subsequently in the form of the Naval fire-control computer, and later, by Vannevar Bush at M.I.T., as a large scale general-purpose differential analyzer.

The electrical analogue computer was first used in the solution of power network problems. The DC network analyzer used combinations of adjustable resistances similar to the potentiometer example described above. Later the AC network analyzer appeared, permitting calculation of complex load problems involving load factor as well as wattage.

The present-day general purpose analogue computer is neither electrical in the sense of the network analyzer nor mechanical, but can be classified as electronic and electromechanical. The heart of the modern computer is the DC operational amplifier, which can be used to sum several variables as a function of time.

The Operational Amplifier — Essentially an operational amplifier is a very high gain amplifier (over 50 million in most cases) with a uniform response to frequencies from zero up to the kilocycle range. The output of the amplifier is connected back to the input through a *feedback* element, which, if the amplifier is to be used for summing, is a resistor. When input voltages are applied to the grid of the first tube through resistances equal in magnitude to the feedback, the output is the sum of the inputs (reversed in sign) and the overall gain is unity.

The high internal gain and heavy feedback results in unusual stability and broad frequency response, in addition to complete decoupling of input from output loads. The *transfer function* (dynamic variation of output with respect to input) is determined almost entirely by the feedback elements and not by the amplifier characteristics, owing to this stability and broad frequency band.

If the input resistors of the operational amplifier are a fraction of the feedback resistance, the outputs are multiplied by the inverse of that fraction. Hence a 1 megohm feedback resistor in combination with a 0.25 megohm input resistor will multiply that input by four. Substitution of a calibrated potentiometer for the input resistor thus allows considerable freedom in multiplication by a constant coefficient. Such potentiometers are thus known as *coefficient potentiometers*, and are widely used in all general purpose DC computers.

Integration — If a capacitor is substituted for the feedback resistor in the operational amplifier, it will integrate the sum of the input voltages. Essentially the current through a capacitor is a function of the rate of change of the voltage applied across it. The high gain of the amplifier tends to keep its input grid at ground, hence the output voltage must rise continuously to balance the input and keep the sum of the voltages at the grid close to zero. Thus within the limits of amplifier output (set by the power supply), the output is the integral of the input taken with respect to time.

Approximate integration is an inherent property of any capacitor fed through a resistor, but the operational integrating amplifier is far more accurate and has less attenuation of voltage than the simple RC quasi-integrating network. In addition, thanks to the decoupling action of the amplifier, it can operate at much higher power levels and is relatively unaffected by the load of the following stage.

Operational amplifiers can also be made to differentiate their inputs, but because of noise sensitivity and other practical difficulties, this is seldom done. Rather than use an electronic differentiator, it is better to use a passive RC network or to integrate through the entire equation to be solved.

The electronic DC operational amplifier provides an extremely versatile basis for analogue computation. There are certain limitations—power supplies must be stable to prevent drift, output is limited (usually to plus or minus 100 volts) as is integration time, and accuracy is a function of the feedback and input elements; about 0.1 percent in most cases. Within these limits, however, the DC analogue computer provides one of the most powerful tools of modern engineering science. Today's designer of plants and processes is fortunate to have these machines available to him in a convenient package and at reasonable cost. The use to which they can be put is limited only by the imagination and knowledge of the systems engineer and executive.

Putting a Computer to Work — General-purpose analogue computers find their application to industrial automatic control in the design and development of equipment, and may be expected to play an increasing role in the operation of the automatic plant. Since analogue computers are ideally suited to the simulation, analysis and synthesis of dynamic systems of all types, they naturally provide rapid solution of the many design problems involved in automatic control. Thus the earliest and best known application is in the design phase, and this provides the greatest number of examples.

The analysis of a linear servomechanism is the classic example of analogue computer use in design. It can be safely stated that the aircraft and guided missile autopilots would never have reached their present high state of development without the aid of the analogue computer. The mathematic and functional similarity between the linear servomechanism and the computer is so complete that the application is almost straightforward, except for the linearization process itself.

Unfortunately any real servomechanism (or any physical system for that matter) is linear only over a limited range of inputs, hence a great deal of ingenuity must be exercised in determining the appropriate transfer function of each component. Once this is done, however, the operational amplifiers of the computer can be easily connected to simulate the gain (multiplication) and integrations of the servo system. The performance of the servo can then be determined by introducing input functions, or commands, either of a stylized and arbitrary nature or, if known, simulations of the inputs that will be applied in actual operation.

WHERE ARE THEY USED?

Analogue computers discussed in this article are not only the special purpose "lead" and "lag" controllers of servomechanisms or controlled processes, but the general purpose machines found in nearly every laboratory where automatic control is studied. Major use of these machines today lies in mathematical computation, primarily in the aeronautical field. Nevertheless, general purpose analogue computers are being widely applied to industrial automation. The general purpose analogue computer has a future as an integral part of the automatic factory. Specific details of these plans are for the most part industrial secrets. However, an idea of the direction they're taking can be gleaned from this report on the present role of analogue computers in process study and simulation.

Use in Servo Design—The advantage of the analogue computer in linearized servo system design is not immediately obvious, since once a relatively simple equation is derived (in the linearization process) for the closed loop dynamics, the response to a step function (sudden rise of input) or to a sinusoidal input can be readily calculated by a number of algebraic and graphical techniques.

The contribution of the computer in this case lies in its ability to: (a) handle a relatively complex multi-loop situation with the same degree of facility as a simple one; (b) permit introduction of completely arbitrary input functions rather than the stylized step or sinusoid, which may require considerable interpretation in terms of the real inputs; and (c) permit *rapid* changes of transfer function, corresponding to change in operating environment or to adjustments of the design itself, thus producing the performance characteristics of a whole family of designs from which the most suitable may be chosen.

The technique used to solve the aircraft servo design problem can also be applied to industrial servos and controls, since the aircraft servo must operate under a far wider set of conditions and must (in general) have a higher degree of performance and safety margin. Industrial design has its own set of problems, however. In the case of a machine tool servo, the load on the tool may vary erratically because of the variations in depth of cut, hardness of material, and condition of the tool. Thus while the input to the servo may be regular and predictable, performance and stability must be adjusted to a wide range of loads.

Non-Linear Operation— It is hardly necessary to dwell on how the computer can be applied to linearized system design. The design of systems which operate entirely in the non-linear condition are more interesting. Owing to the lack of information available to the average designer and to the limitations of mathematical techniques, the analogue computer often provides the only means of predicting the performance of non-linear systems.

The importance of non-linear system design is emphasized by the known simplicity and reliability of such control devices as the relay and the on-off valve, both inherently non-linear. If such simple elements can be made to yield the same system performance as the more precise, delicate (and expensive) linear devices, the technology of control is so much advanced. Since it is basic in feedback theory that nonlinearities can be tolerated to a large extent in the forward or amplifying path, this goal can often be achieved with proper systems design, and, in some cases, performance can be even improved over the linear systems.

It is difficult to draw a sharp line between design and development. The latter is, roughly, the activity which seeks to define or improve the performance of an existing piece of equipment. This equipment is functionally complete or part of a larger system. Design, on the other hand, seeks to define specifications for an entirely new piece of equipment and to insure adequate performance for a given task, drawing upon information about the known behavior of components. In relation to the general purpose analogue computer, design is accomplished by complete simulation of an equipment or system. The simulation is built up from known component transfer functions which are synthesized by the computer elements, and by testing of the *simulated* equipment. In development, only a portion of the system is simulated, either that portion which has not yet been reduced to hardware or which is inconvenient or impossible to bring into the laboratory.

Missile Development— In the development of aircraft autopilot components and guided missile servomechanisms, the airplane or missile airframe is "flown" in the laboratory through the agency of simulation on the analogue computer. The autopilot or servomechanism components are present "in the flesh" and are connected to the computer in such manner as to close the system loop.

It is noteworthy that in the case of the airplane, a human being can be introduced into the loop and in fact may be the principle component "under development", which is true of a pilot being trained in a flight simulator.

Systems testing of autopilot components often involved directional or motion-sensitive elements, such as gyros and accelerometers. If the transfer functions of these elements are both important and unknown, or difficult to simulate, it is necessary to mount them on a "flight table" or moving platform which, under the guidance of the simulated airframe analogue computer, physically duplicates the motions of the real airframe.

Construction of a flight table may well prove a difficult task if it is to carry any significant weight of equipment, since its dynamic performance should better that of the airframe it is to simulate. If it is not to inject its own dynamic characteristics into the equipment tests, it must have a transfer function of unity.

Testing Components— In the same way, elements or pickups which are sensitive to altitude, temperature. or other functions of spatial position and motion can be placed in pressure chambers, ovens, refrigerators, etc., all servo-controlled and under the command of a "space simulation" portion of the analogue computer set-up which continuously calculates the position of the "aircraft" from its heading and speed.

The "muscles" of the autopilot, that is the servo actuators, should also be put under the proper load, since servo performance in aircraft, at least, is usually sensitive to the load. Either an average load can be computed for the ranges of speed, altitude and angle of attack of the aircraft and duplicated by a spring, or the instantaneous load can be computed by another section of the analogue computer and used to control a counter-acting servo actuator of greater power than the one under test.

Industrial Servo System—The above description of aircraft autopilot development can be applied directly to the industrial servo system, with as much or as little elaboration as may be appropriate to the case. In general, any servo or regulator system which forms the so-called "minor loop" in a larger feedback control system, can be tested and evaluated in its "major loop" environment by means of analogue computer simulation or partial systems test. It is very difficult to predict the behavior of the input signal to a minor loop element unless the entire feedback system is tested as a unit.

As a specific example, consider a tape-controlled machine tool system composed of the stored motion command (tape), a tool servo operating from the command transducer through a summing device and provided with an internal command follow-up feedback loop, a measuring device recording the workpiece dimensions, and an outer feedback loop from the dimension gauge back to the tape transducer summing point (the servo input).

This is a completely automatic system which should reproduce the tape orders regardless of the load on the tool, as long as the linear range of operation of the system is not exceeded. However, the tool servo does not merely follow the tape orders, but rather the *difference* between the tape order and the actual piece dimension as measured by the gauge. If the servo falls behind the tape owing to a hard spot on the work or to a sudden command for a deep cut, the servo order will rise to a maximum, because the difference between the command and outer loop feedback will be great. In responding to such a maximum input, the servo may overshoot and receive a large signal in the other direction from its own feedback loop.

Such a servo cannot be adequately tested by itself, unless we are willing to overdesign the dynamic performance to anticipate the worst possible contingency, and even in this case we have no assurance that the system will be stable. As performance specifications of the automatic factory become tighter, either through demands for higher production speed or better accuracy, overdesign becomes impossible and instability intolerable, so we must resort to systems simulation and partial systems test to obtain the most efficient and economical design.

Simulating Work Resistance — In making such a test in the case cited above, the servo would be loaded by a device which simulated the resistance of the work to fabrication—this resistance being a function of the workpiece position relative to the tool, the characteristics of the work such as hardness and springiness, the speed of cut, temperature, and many other factors. Conceivably these could be expressed in equation form and simulated on the analogue computer.

The servo ean then be tested under the conditions of its working environment, not merely by an empirical specification. The tape commands are introduced to the servo as before, the servo responds by acting against the simulated load, with the analogue computer determining both the servo load and the dimensional change resulting from the tool action. The dimensional change is then fed back to the tape order summation device, either through an analogue of the feedback gauge dynamics or through a small instrument servo which actuates the real gauge.

The Proportional Valve — Although the above may seem somewhat far-fetched considering the present state of the machine-tool art, the same principles can be fruitfully applied to the problem of a high-speed proportional valve in a fluid flow system subject to high pressures, such as may be found in petroleum refining practice. Although the valve servo load is a known function of valve opening and of the flow rate, the valve dynamics may influence the servo command through an outer feedback loop including a regulator. In a multi-loop system of this nature, instability and destructive oscillations are possible and even probable if proper design, development, and systems tests through simulation or other adequate techniques are not carried out.

PROCESS REGULATION WITH ANALOGUE CONTROL

BY ROBERT J. BIBBERO

Technical Editor, AUTOMATIC CONTROL

AUTOMATIC CONTROL FEBRUARY 1955

■ Industrial control system schematics now being drawn behind closed doors may mean that this year general-purpose analogue computers will be used as process controllers in at least two major plants.

Automatic controls in the rudimentary sense have been used in the process industries for many years. Even the most elementary form of processing requires *instrumentation, indicators,* and *recorders* to inform an operator of the temperature, pressure, rate of flow, or other important process variable. The operator reads the indicators, compares their measurement with the desired value of *set point,* and himself provides the feedback loop by opening or closing valves, rheostats, and so forth until the desired reading is obtained and held. By his knowledge of the process and his skill in operating, he provides *anticipation* or *lead* to overcome lags in the process. By reference to the recorder he can *integrate* the process variable to the required area of variable *vs* time or can smooth out fluctuations in his control.

The automatic controllers or regulators in wide-spread use today are simple extensions of the manual feedback process. They operate by detecting the difference between the indicator reading and the set-point, then controlling an amplified source of power according to the sense of the error. Thus, the valves, rheostats, switches, etc. which directly operate on the process variable may be manipulated without the intervention of the operator. His role is then modified to that of *judgement* (in determining the set-point) and *supervision* in the event of emergency.

The power-amplifying and detecting controllers are usually pneumatic or electrical in nature. In rare cases are they hydraulic. A rapidly increasing number of controllers utilize simple electronic means for error-detection, initial signal amplification, and modification.

Electrical controllers detect and amplify errors by means of motor-driven bridge circuits, such as the well-known Wheatstone bridge. The traditional pneumatic controller detects and modifies the error signal by various ingenious combinations of bellows, nozzles, and flappers which act on a low pressure source of air. These controllers are essentially simple, special-purpose analog computers. They convert the measurement of the primary instrumentation or sensing device into an error signal, most usually a mechanical displacement, change it to an easily handled pneumatic or electrical variable, operate on the variable and transmit it to a *power unit* which controls the process through the valve or other device.

***Actually,* there is no real reason why general-purpose analogue computers cannot be used today in the modern industrial plant. In fact, there are strong indications that experimental installations of this nature do exist.**

Why isn't the analogue computer being used extensively in today's factories? Certainly not for economy's sake. One analogue computer could conceivably replace a number of expensive conventional controllers. Convenience is not the reason. The present emphasis upon centralized location of control rooms makes the compact form of the analogue computer a distinct advantage.

The analogue computer is also a highly-developed example of a reliable control tool. Several complex installations (Goodyear Aircraft's GEDA, for example has 420 amplifiers, multipliers, function generators, etc.) are operating at 98% efficiency with the services of only one maintenance repair technician.

If industry has not taken advantage of the extensive research, development and application engineering that has gone into the many electronic analogue computer installations all over the country, it is because, conservatism, lack of knowledge or fear of "electronics" have stood in the way.

You may already have recognized the similarity between the process controller and a position servomechanism but, in spite of what has been said, you may balk at the claimed likeness to the analogue computer. If we neglect the aspect of power amplification, it is true that any signal modifying or *compensated* servo, controller, or regulator is an analogue computer, and that any confusion which exists is entirely one of terminology

Consequently, there should be no reason why the general purpose electronic analogue computer cannot be connected to the proper sensing instruments on the one hand and the power units on the other, and installed in the plant as a sophisticated controller, or indeed as a number of controllers. Its operational amplifiers can provide all the functions of the controller with complete versatility and, no doubt, better accuracy. Since the usual analogue computer package has approximately twenty amplifiers, and no more than three to six of these are required for a controller problem, the computer may substitute for four to seven conventional controllers. If the analogue computer has not yet been widely used as a direct process controller, its advantage as a simulator of both the controller and the process has already been recognized by forward-looking engineering activities.

To understand the applications potentialities that they foresee, its first important to understand how the set-point of the controller is chosen, and by what method differential and integral ("rate" and "reset") adjustment are determined.

The analogue computer is a logical extension of the familiar process controller. It can be applied now to replace many controllers by a single-package unit, or to supervise the operation of many controllers in an integrated automatic plant.

It is compatible with modern graphic panels and the rigid demands of industrial realiability standards.

We can use existing general-purpose electronic analogue computers to control complex processes by automatic solution of spectroscopic or infra-red sensory data, by process simulation and for the testing of automatic products.

In the past, the adjustment of a controller to a particular kettle or furnace was done largely by "cut-and-try." Even today, this is probably still the most common way of compensating for process lags and individualities. In this writer's own experience as a pilot plant engineer, the installation of a new controller called for many hours of "knob-twiddling" by the instrument manufacturer's representative, and when the correct adjustments were found, a small change in the process called for a repeat session.

Yet today there are more scientific approaches to process compensation, involving the "oscillation" of the entire controlled process by pneumatic or electrical low-frequency sine wave generators acting through the controller or power unit, making it possible to find the frequency response and hence the transfer function of the process. Even these modern methods, which are useful and necessary in research and development still consume much engineering and production time. Also may harm equipment if not properly applied.

The general purpose analogue computer can be useful in eleminating hours of down time needed to establish the proper setting for a controller. It can be useful in determining the feasibility of new control loops, or studying the effects on the entire plant of new equipment or emergency conditions.

Such useful applications are possible because the computer can be set up not only to reproduce the action of the controllers, but also the action of the process itself. Since the process variable's response to a change in controlled variable is apt to be non-linear, the analogue computer must incorporate non-linear function generators and multipliers to accomplish this task.

Solving Equations Automatically—Analogue computers can solve complicated sets of equations involving many variables. In general, the analogue machine is faster than the digital. In many industries, the nature of the product quality must be expressed by a large number of variables, such as those which comprise the chemical analysis of a steel.

Another example is found in spectrographic analysis, where the intensity of a number of distinct spectral lines must be correlated with each other, and with standard lines, to arrive at a quantitative determination of a single element or compound. Additional examples of computations of this type are found in the use of the mass spectrograph, photronic colorimeter, polarograph, x-ray spectrometer, fractional distillation column and many others.

The interpretation of raw results from analyzers of this type have been speeded in many instances by the construction of resistive analogue computers of the *adjuster* type, consisting of a large number of potentiometers, a voltage source, and a null indicator. These solve an array of simultaneous algebraic equations of the form:

$$AX_1 + BX_2 + CX_3 + \ldots\ldots\ldots = 0$$

The procedure is to set the constants A,B,C, etc. which represent, for example, the intensity of each line on the spectrograph, as constant multipliers on the potentiometers. The voltages

X_1, X_2, X_3, etc. are applied across the pots and the summation of all the pot voltages applied to the null indicator. The values of the X's are systematically varied until all the meters are nulled at which time the roots X_1, X_2, etc. are found. One or more of these roots is the desired variable.

If the spectrograph or other device can be instrumented (by photocells, for example) to continuously feed A,B,C,—values into the computer, and the computation itself is made automatic, the assembly can be made to control a process by holding the desired composition of the product or an intermediate stage in its manufacture. The general purpose analogue computer can easily be set up to solve simultaneous algebraic as well as integro-differential equations even more accurately than the resistive analogue, because the amplifiers can be used to prevent pot loading errors. The iterative process of root solving can be automatized by sweeping through the range of input voltages and detecting the value when the null is reached.

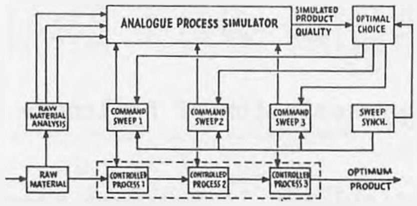
When values of voltage must be varied to obtain the solution, repetitive type computers would seem to be advantageous for this type of work, owing to their greater speed.

Overall Control of Many Processes — The type of automatic control application discussed heretofore is in the nature of regulation, that is, maintaining a predetermined equilibrium of a particular part of a process. This is the type of automatic control that has been used in the past and will continue to be used in the future. But a more interesting type which is just now beginning to evolve: the overall supervision of many linked processes to create an optimum production environment. This higher kind of control, which is usually linked with the concept of automation, is considered to require the use of *experience* and *judgement*.

Analogue Computers Have Memory, Too —For some reason, when the average control engineer thinks of a machine substitution for human supervision, he thinks in terms of the digital computer, probably because of the importance of memory and logical circuitry in these machines. However, the analogue computer, too, possesses memory, even though it is not always thought of in this way. An analogue computer cannot store tabular data as such, but in its *function generators* it can and does store the same kind of data in continuous curve form. The difference is primarily one of accuracy and speed, with the digital slower but more accurate, and the analogue faster but less accurate. In process control, initial data is normally in analogue form. Most analogue computers have the required accuracy for this application. Insofar as judgement is concerned, the digital machine can only act in accordance with predetermined instructions provided by the human operator. These instructions guide the machine toward a goal. But any closed-loop device is also a goal-seeker, and has the capability of adjusting itself according to circumstances in order to reach that goal.

I believe that the analogue approach is as capable of achieving overall supervision functions as is the digital. The approach to be taken depends entirely on the nature of the task at hand.

To demonstrate how analogue simulation may be used to supervise a controlled series of processes, a hypothetical case is shown(See above).



The production flow line consists of a number of processes, in this case three (shown in the dashed box), each of which acts in turn on the raw material and intermediate stages to produce the end product. Each of the three processes is controlled in a *minor loop,* not shown, which drives its set point from the box labeled *command sweep.*

It is assumed that the quality and/or quantity of the product is a known function of the individual process controllers set points and of the raw material itself. If this function is known, it can be set up and simulated by means of an analogue computer, shown in the upper box. Furthermore, the analogue simulation is not limited to operation in *real time,* but may be immensely speeded up to reproduce a process in seconds or minutes that in actuality may take hours.

The simulator is furnished continuously with the analysis of the raw material, perhaps by some of the means discussed under "automatic equation solution." In addition, the instantaneous value of the set point of each controlled process is likewise furnished to the simulator. Under these conditions, the simulator will indicate a simulated product *quality* (yield, purity . . . almost any characteristic which can be detected and measured) or *quantity,* resulting from the action of each process on the raw material.

Now suppose that the simulated set point of each process in turn is swept very rapidly over a range of values under the control of the *sweep synchronizer* block. As the set points are varied, the quality of the simulated product also varies. The optimal choice block freezes each set point where it will result in the optimum product quality. In doing so, it must measure this quality variable (which is only an analogue computer voltage) and "remember" the point at which it reaches its highest value.

This easily can be done by some form of peak voltage detector and by measurement of the sweep synchronizer's phase at the time the peak occurs. As each optimum set point is detected, the optional choice block opens an electronic gate in the command sweep channel, permitting this set point to be transmitted to the real process controller.

Thus, the operation of the system is to set each controller in turn to its optimum value with respect to the particular raw material being processed. If there is some interaction between processes, the entire searching process can be repeated many times in a short interval until the "best product" is found. The meaning of "best product" must, of course, be built into the optimal choice block. If the analysis of raw material changes, the entire procedure can be repeated and, in fact, can be made to be continuously repetitive.

The system just described can be instrumented in any continuous or batch process for which an analogue can be constructed. Conceivably, the analogue need not be electrical, but could even be a miniature high-speed version of the manufacturing process. Electronic analogues provide the greatest versatility, however, and high-speed repetitive equipment is readily available today.

Because the system will respond to any change in raw material, cheaper material, requiring less rigid control of quality, can be used. The setting of the optimal choice block could be easily changed to produce a variety of products, or grades of product, as the demand changes. The utility of this feature can be seen readily in the case of a petroleum refinery where the same equipment can produce many different grades of product, depending upon the temperature, pressure, and other process variables.

Automatic Production Testing — Analogue simulation in the operation of an automatic factory can be used for production testing. Analogue computers can simulate a very complicated environment such as that experienced by an aircraft autopilot servo. In the production of this type of equipment, whether by automation or by conventional means, the computer is already widely used as a means of production testing.

If the production process is automatic, there is no reason why the testing and acceptance or rejection should not be automated. It is only necessary to stylize the test procedure in such a manner that rejection or acceptance becomes a matter of "go-no-go" judgement by the machine. In the event that the desired product performance is a function of a variable, rather than a single determination, this function can be compared with a similar function stored in the computer, and the degree of match between the two curves used as the factor of acceptance.

Taken in a narrow sense, the "environment" of a simple product such as a printed electronic circuit, can be considered as the voltage applied to its imput terminals, and the "response" is the indication of continuity, output signal, or other measurable quantity. Environmental testing in this limited sense is just what has been done in Project Tinkertoy. Although a simple circuit tester such as is used in Tinkertoy can hardly be classed an analogue computer, it can certainly be made a completely automatic function. It requires little imagination to see how this principle can be extended to more complicated cases.

The testing of a complete piece of radar equipment such as a navigational or bombing computer and associated video circuitry is more to the point. Here, the input is a video signal, normally obtained from the target, antenna, and receiver. It is inconvenient to bring the target into the production facility and in the case of airborne radar, much expensive flight time may be required to test the assembly. It is possible, however, to simulate the target and its motion in a kind of analogue computer, which calculates the amplitude of the target signal for any altitude of the target antenna and for any target range and angle. (See AUTOMATIC CONTROL, Nov 1954, page 23.)

Since the target is completely synthetic, it can be controlled in any manner to simulate combined motion of the radar and/or target. Thus any desired test motion can be given the target, such as linear, accelerating, sinusoidal, circular, or completely arbitrary, and the ability of the computer or teacher to follow this motion can be made the basis of an automatic "go-no-go" test.

George A. Philbrick
November 1, 1951

Continuous Electronic Representation of Nonlinear Functions of n Variables

In analog computors, particularly in the fast electronic variety, there has been need for a simple way to embody functions of more than one variable. This includes mathematical functions along with those arising from empirical data. Components for this purpose should have no moving parts, and should accept and yield instantaneous voltages as input and output information. It should also be possible easily to adjust such components to the functions desired.

A number of techniques are popular for fitting nonlinear functions of one variable. One of these will be outlined briefly here since it can be generalised to cope with more than one independent variable. This is the method of segmental fitting, which has been available for some time in the form of a standard computing component.

Segmental Fitting for One Variable

Define the simple nonlinear function S(p,q) as having the value of whichever of the variables p and q is algebraically the greater, and the function S'(p,q) as having the value of the least. Note that $S(p,p) = S'(p,p) = p$, $S(p,-p) = |p|$, and that:

$$S(p,0) = \tfrac{1}{2}(p + |p|)$$

$$S'(p,0) = \tfrac{1}{2}(p - |p|)$$

The latter equations describe the familiar characteristics of idealised diodes, as shown in Figure 1.

Elements of the diode type may be combined to approximate functions by connected line segments. Thus consider:

$$F(p) = \sum_{i}^{n} G_i S(p-P_i,0)$$

where the G's and the P's may be assigned at will. Figure 2 shows a typical case for a few elements, which are added to form a segmental function. The P's determine the initial abscissae, and the G's determine the incremental slopes. A block diagram for a computor component based on this principle is exhibited in Figure 3. To adjust the fitting parameters, one begins with G_0 and proceeds to P_1, G_1, P_2, G_2 and so on, being guided by the local nature of the function being represented. In this way there is no influence of the adjustments on the segments already established.

Application to Functions of Two Variables

The diode operations S(p,0) and S'(p,0) are special cases of the "selection" operators S(p,q) and S'(p,q) defined above. These are shown more tangibly in Figure 4, and are mathematically described by the equations:

$$S(p,q) = \tfrac{1}{2}(p + q + |p - q|)$$
$$S'(p,q) = \tfrac{1}{2}(p + q - |p - q|)$$

This kind of selection may be made among as many variables as desired. Thus S(p,q,r) or S'(p,q,r), duplicating the greatest or the least of three variables, is obtained by a simple extension of the devices of Figure 4. A somewhat more complex selection is denoted by S(S'(p,q),r), which equals either the smaller of p and q, or r, whichever is the larger. The circuit and geometrical form, for zero r, are given in Figure 5.
= S'(S(p,0), S(q,0))

The operation S(S'(p,q),0) may be applied to construct an approximation to functions of two variables, just as the operation S(p,0) was applied in the case of one variable. The new combining expression is:

$$F(p,q) = \sum_{i,j}^{m,n} G_{ij} S(S'(p-P_i, q-Q_j), 0)$$

This is not quite as simple to visualise or to draw, but it can approximate to functional surfaces just as effectively as does the simpler segmental construction for curvilinear functions of one variable. A single element, for a particular choice of i and j, would resemble the surface shown in Figure 5, but displaced in the p and q directions by P_i and Q_j and dependent for steepness on G_{ij}. The number of such elements employed will depend on the degree of subdivision of p and q. In the symmetrical case, with p and q divided into the same number of equal (or similar) portions, the surface will contain $2n^2$ planar facets, each a triangle. Even with the P's and Q's fixed, the G's provide n^2 adjustments whereby a function of two variables may be straightforwardly fitted at that many points.

This discussion has been implicitly restricted to functions F(p,q) for which F(p,0) = F(0,q) = F(0,0) = 0, just as we assumed F(0) = 0 in the case of one variable. In that case it was evident that only the excess of F(p) over F(0) required representation, since the constant F(0) could be added separately. A similar process is employed here, but needs explaining. Consider any F(p,q) which does not in general vanish with p or q, and which incidentally is single-valued and otherwise well behaved. Obtain the related function H(p,q), defined by

$$F(p,q) = H(p,q) + F(p,0) + F(0,q) - F(0,0)$$

It is seen that H(p,0) = H(0,q) = 0, so that H(p,q) may be approximated by the above type of faceted surface. The constant F(0,0) may readily be subtracted out, while F(p,0) and F(0,q) may conveniently be represented by the one-variable segmental method and added in to complete the approximation of any practical F(p,q).

A block diagram of the fitting component for F(p,q), or rather for its principal sub-function H(p,q), is shown in Figure 6. Adjustment of the parameters would begin at the origin and proceed radially outward in an orderly manner. The local nature of the resulting surface is exhibited in Figure 7, made purposely irregular. The dotted contours, for fixed intermediate values of p and q, show the interpolation process. If the P's and Q's are made arithmetic series's from zero, and if all the G's are equal to zero except G_{i0} & G_{0j}, the component approximates to H(p,q) = Kpq and ~~will~~ may serve as a multiplier. The surface, corresponding to a hyperbolic paraboloid, is shown in Figure 8.

More Than Two Variables

This technique generalised concisely in passing from a single variable to two. It is interesting to notice that it can go further. For three variables the elemental operator is

$$G_{ijk}S(S'(p-P_i,q-Q_j,r-R_k),0)$$

which requires five diodes at each point. Geometrically, this element is a displaced octant comprising three contiguous triangular pyramids in each of which the space function is identical with one of the coordinate ~~p, q, and r~~ -differences. In Figure 9 an attempt is made to picture this element, intersected by a plane p + q + r = a.

Theoretically any number of variables may thus be handled, although the complexity of the machinery increases dramatically for many dimensions. It is thought that this functional technique may even find applications outside analog computation as such, owing to its ability to store, and to permit revision of, multidimensional information in a purely electrical structure.

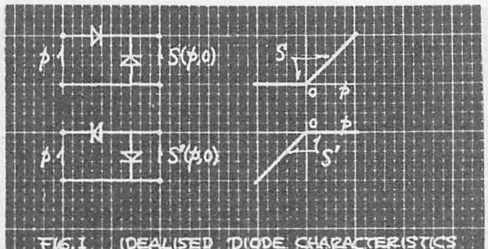


FIG. 1 IDEALISED DIODE CHARACTERISTICS

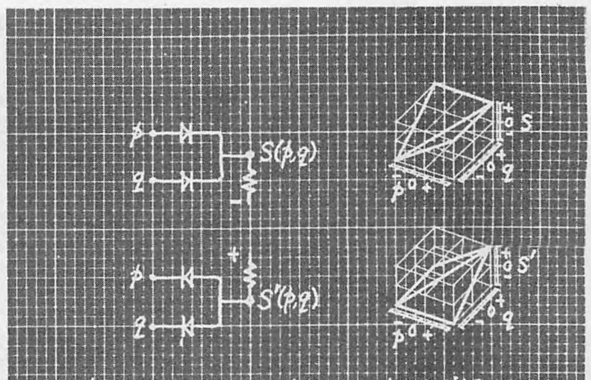


FIG. 4 DIODE SELECTION BETWEEN TWO VARIABLES

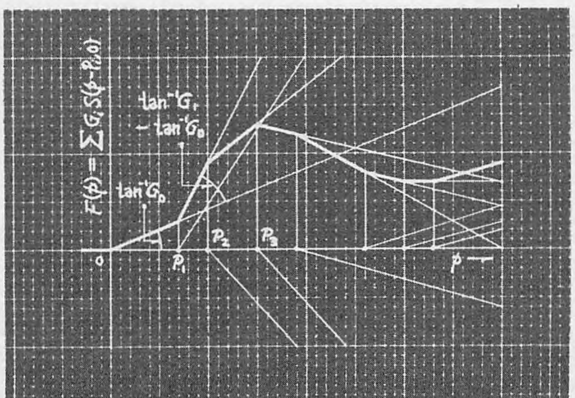


FIG. 2 SUMMATION OF DISPLACED DIODE CHARACTERISTICS

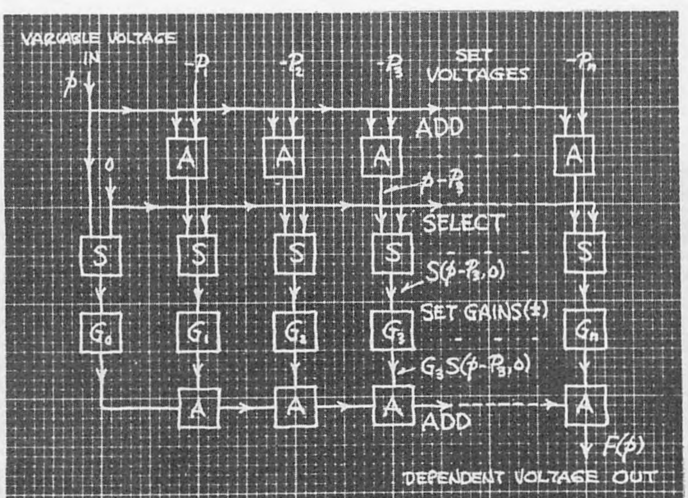


FIG. 3 BLOCK DIAGRAM FOR SEGMENTAL FITTING

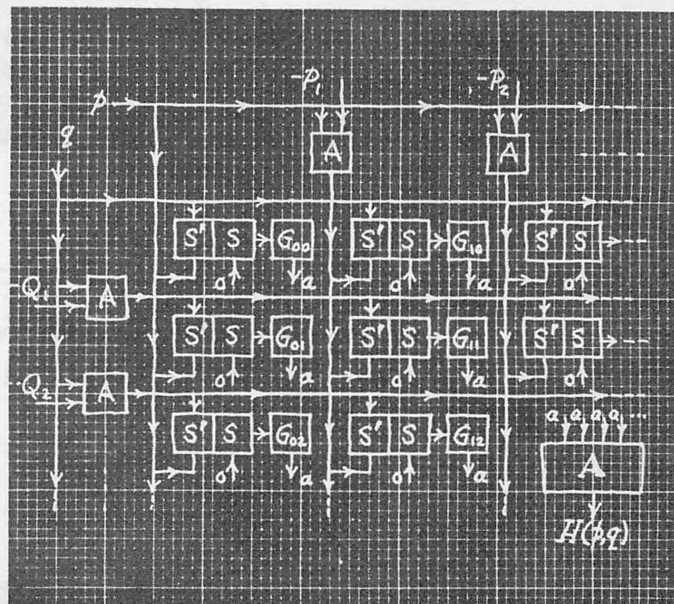


FIG. 6 BLOCK DIAGRAM FOR $H(p,q)$

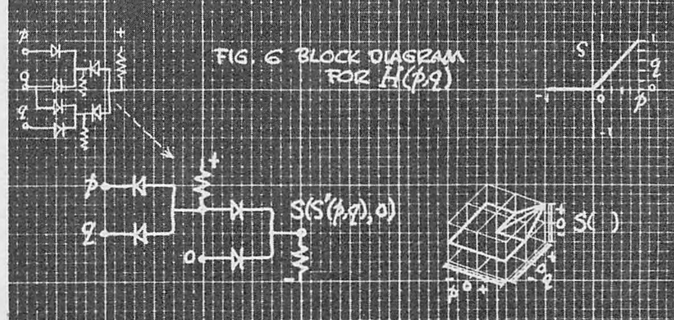


FIG. 5 SPECIAL SELECTION AMONG TWO VARIABLES AND A NULL

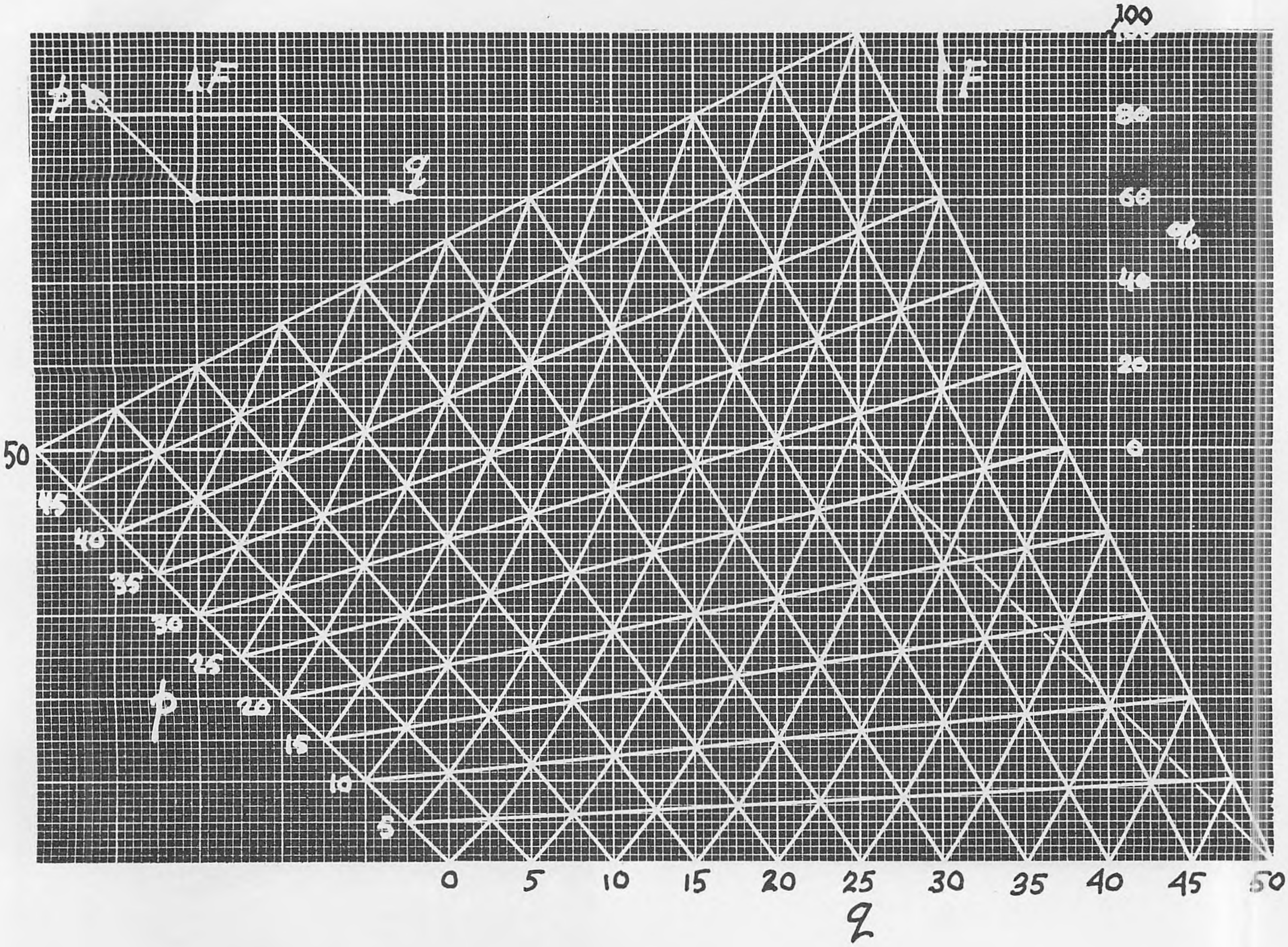


FIG. 8
FACETED SURFACE APPROXIMATING A HYPERBOLIC PARABOLOID

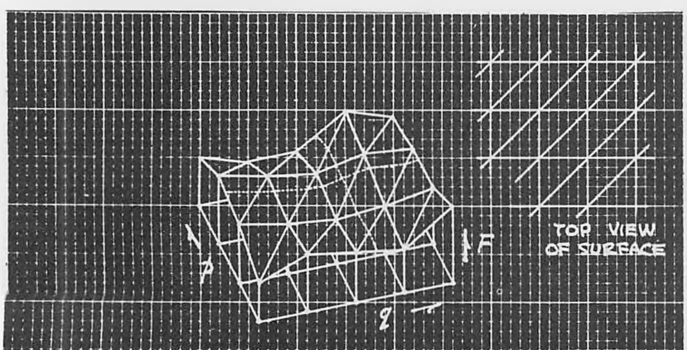


FIG. 7 PORTION OF SUMMED SURFACE, SHOWING FACETS

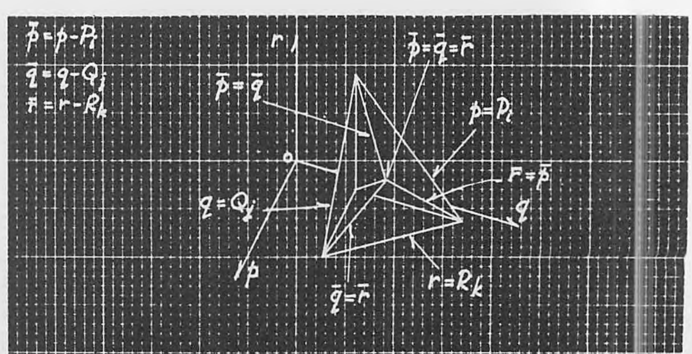


FIG. 9 SPACE-ELEMENT FOR FUNCTIONS OF THREE VARIABLES

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