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A trace formula for the nodal count sequence

Towards counting the shape of separable drums

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Abstract. The sequence of nodal count is considered for separable drums. A recently derived trace formula for this sequence stores geometrical information of the drum. This statement is demonstrated in detail for the Laplace-Beltrami operator on simple tori and surfaces of revolution. The trace formula expresses the cumulative sum of nodal counts This sequence is expressed as a sum of two parts: a smooth (Weyl like) part which depends on global geometrical parameters, and a fluctuating part which involves the classical periodic orbits on the torus and their actions (lengths). The geometrical context of the nodal sequence is thus explicitly revealed.

1 Introduction

More than 200 years ago, when Ernst Florens Friedrich Chladni found the sound figures which now bear his name, he tried to classify the emerging patterns e.g. by the number of lines or the number of domains defined by the lines [1]. We are happy to dedicate a piece of work to E.F.F. Chladni whose 250th birthday we celebrate this year which follows his ideas closely. Chladni's work inspired research in physics and in mathematics until today. He not only considerably advanced the knowledge on resonance and wave phenomena but also introduced the nodal set as a new concept in the research of wave phenomena which is visualised in the sound figures. The nodal set is the zero set of a wave function. In the two dimensional case of vibrating plates or drums the nodal set consists of nodal lines which are the borders of nodal domains, i.e. maximally connected domains where the sign of the wave function does not change.

We consider drums (or quantum billiards) as the eigenproblem

$$-\Delta_{\mathcal{M}}\psi(\mathbf{x}) = E\psi(\mathbf{x}) \tag{1}$$

of the Laplace-Beltrami operator $-\Delta_{\mathcal{M}}$ on a compact Riemann surface $\mathcal{M}$ (if $\mathcal{M}$ has a boundary we consider Dirichlet boundary conditions on $\partial\mathcal{M}$). The spectrum $\{E_n\}_{n=1}^{\infty}$ is discrete and can be ordered $E_n \leq E_{n+1}$. The eigenfunction ψ_n corresponding to the eigenvalue E_n can be characterized by the number ν_n of its nodal domains. The intimate connection between the spectra, wave equations, and nodal sets is well known and frequently used or investigated in various branches of physics and mathematics (see [2] for a recent review). The relation between

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the nodal count and the spectrum is highlighted by Sturm's oscillation theorem [3] that states that in one dimension the n-th eigenfunction has exactly n nodal domains. In higher dimensions Courant proved that the number of nodal domains ν_n of the n-th eigenfunction cannot exceed n [3]. More recently, statistical properties of the nodal count have been investigated. It was shown that the fluctuations in the nodal count sequence $\{\nu_n\}_{n=1}^{\infty}$ display universal features which distinguish clearly between integrable (separable) and chaotic systems [4]. This statistical approach also lead to surprising connections with percolation theory [5] and the Schramm-Loewner evolution [6,7].

Numerical and analytical evidence lead to the belief that the sequence of nodal counts in a billiard contains a lot of information about the dynamics and the geometry, beyond the difference between integrable and chaotic. This lead to the question [8,9]: 'Can one count the shape of a drum?' This is a reformulation of the famous question by Kac [10]: 'Can one hear the shape of a drum?' More precisely, the question is: does the sequence of nodal counts determine the geometry of a drum (compact Riemmannian surface)? The independence of the inverse counting question from Kac' question has been shown in that nodal counting can distinguish between some isospectral systems [8,11]. For separable systems a trace formula for the nodal counting sequence has been established that explicitly shows the dependence of the nodal sequence on the geometry of the surface in both the smooth (Weyl-like) and the fluctuating parts. Thus, the nodal count trace formula is similar in structure to the corresponding spectral trace formula [12–15]. The sequence of nodal counts does not involve any spectral information (apart from the order) yet the nodal count trace formula contains all information on periodic orbits on e.g. surfaces of revolution. Thus it is an important step towards 'counting' the shape of such surfaces.

In this paper we give a more detailed account of the nodal count trace formulae for convex smooth surfaces of revolution and for simple two-dimensional tori. Generalizations to other Riemannian manifolds in two or more dimensions are possible, provided the wave equation is separable.

2 The cumulative nodal count

If the spectrum of the Laplace-Beltrami operator on a compact Riemannian manifold is void of degeneracies we may order the spectrum such that $E_n < E_{n+1}$. The corresponding nodal counts ν_n then form an ordered sequence which will be central to this discussion. Let $[\![K]\!]$ denote the largest integer smaller than $K \in \mathbb{R}$, then we define the *cumulative nodal count*

$$C(K) = \sum_{n=1}^{[\![K]\!]} \nu_n \qquad \text{for} \quad K > 0. \tag{2}$$

To generalize this definition to degenerate spectra one has to uniquely choose a basis of wave functions in the degeneracy eigenspace and also decide on the order in which they appear in the nodal counting sequence. There are several more or less natural ways to do this. For separable systems which are the focus of this paper we propose to choose the unique (real) basis in which the wave functions appear in product form. This still does not suffice to set a unique order within the degenerate states. Chosing an unambiguous order can be circumvented by modifying the definition of the cumulative nodal count in the following way. First define the function

$$\tilde{c}(E) = \sum_{n=1}^{\infty} \nu_n \Theta(E - E_n) \tag{3}$$

which is independent of the order of the nodal counts. Here, $\Theta(x)$ is Heaviside's step function. This comes at the price that now the function is based on information obtained from both the nodal counting sequence and the exact positions of the eigenvalues. To eliminate the dependence on the latter, we use the ϵ-smoothed spectral counting function

$$\mathcal{N}_\epsilon(E) = \sum_{n=1}^{\infty} \Theta_\epsilon(E - E_n), \tag{4}$$

where $\Theta_\epsilon(x)$ is a continuous, symmetric and monotonically increasing function with

$$\lim_{\epsilon \to 0} \Theta_\epsilon(x) = \Theta(x).$$

As a consequence, for finite ϵ, $\mathcal{N}_\epsilon(E)$ is a continuous strictly monotonically increasing function which can be inverted. Let $E_\epsilon(K)$ as the solution of $\mathcal{N}_\epsilon(E) = K$. Then we define the modified cumulative nodal count by

$$c(K) = \lim_{\epsilon \to 0} \tilde{c}(E_\epsilon(K)). \tag{5}$$

For non-degenerate systems this is equivalent to the original definition (2) up to a trivial shift $c(K) = C\left(K + \frac{1}{2}\right)$. In the limit $\epsilon \to 0$ the contribution of a g-times degenerate eigenvalue $E_n = E_{n+1} = \cdots = E_{n+g-1}$ reduces to a single step $\Theta\big(K - (n - 1 + \frac{g}{2})\big) \sum_{s=1}^{g} \nu_{n+s-1}$ at the central index $K = \left(n - 1 + \frac{g}{2}\right)$ where the cumulative nodal count increases by the sum of the nodal counts within the degeneracy class. We will derive a trace formula for this modified cumulative nodal count (omitting 'modified' in the sequel).

3 A trace formulae for the cumulative nodal count

Trace formulae for spectral functions like the spectral counting function $\mathcal{N}(E)$ have been derived for many classes of drums (and more general quantum systems) and they have been applied with great success. In the case of separable drums, we will show that the same methods that are used for spectral functions can be applied to spectral nodal counting function $\tilde{c}(E)$ which eventually leads to a trace formulae for $c(K)$.

The main ingredients of the derivation of spectral trace formulae for spectral functions are the Poisson summation formula (for finite sums)

$$\begin{aligned}\sum_{n=n_0}^{n_1} f(n) &= \sum_{N=-\infty}^{\infty} \int_{n_0}^{n_1} f(n) e^{2\pi i N n}\, dn + \frac{1}{2}\left[f(n_0) + f(n_1)\right] \\ &= \sum_{N=-\infty}^{\infty} \int_{n_0-1}^{n_1+1} f(n) e^{2\pi i N n}\, dn - \frac{1}{2}\left[f(n_0) + f(n_1)\right]\end{aligned} \tag{6}$$

and saddle point approximations to the resulting integrals.

3.1 Simple tori

Let us start with the simpler case of a 2-dim torus represented as a rectangle with side lengths a and b and periodic boundary conditions $\psi(0, y) = \psi(a, y)$ and $\psi(x, 0) = \psi(x, b)$. This leads to (real) eigenfunctions

$$\psi_{n,m}(x, y) = \begin{matrix}\cos \\ \sin\end{matrix}\left(\frac{2\pi n}{a}\right) \begin{matrix}\cos \\ \sin\end{matrix}\left(\frac{2\pi m}{b}\right) \tag{7}$$

for $m, n \in \mathbb{Z}$ (cosines apply for $m, n \geq 0$ and sines for negative m, n). The corresponding eigenvalues take the values

$$E_{n,m} = (2\pi)^2 \left[\frac{n^2}{a^2} + \frac{m^2}{b^2}\right]. \tag{8}$$

Due to the checkerboard like structure of the nodal set, it is straight forward to count the nodal domains in the wavefunction $\psi_{n,m}$ which gives

$$\nu_{m,n} = (2|n| + \delta_{n,0})(2|m| + \delta_{m,0}) . \tag{9}$$

The only free parameter of the nodal count sequence for tori is the aspect ratio $\tau = a/b$ because the number of nodal domains is invariant to rescaling of the lengths.

Applying Poisson's summation formula (6) to the spectral counting function

$$\mathcal{N}(E) = \sum_{n,m=-\infty}^{\infty} \Theta(E - E_{n,m}) \tag{10}$$

$$= \sum_{N,M=-\infty}^{\infty} \int_{-\infty}^{\infty} dn \int_{-\infty}^{\infty} dm\; \Theta\left(E - (2\pi)^2 \left[\frac{n^2}{a^2} + \frac{m^2}{b^2}\right]\right) e^{2\pi i (nN + mM)} \tag{11}$$

all appearing integrals can be performed exactly. Here we are only interested in the leading asymptotic behaviour obtained by saddle-point approximation of all oscillatory integrals which gives

$$\mathcal{N}(E) = \mathcal{A}E + \sqrt{\frac{8}{\pi}} \mathcal{A} E^{\frac{1}{4}} \sum_{\mathbf{r}} \frac{\sin(L_{\mathbf{r}}\sqrt{E} - \frac{\pi}{4})}{L_{\mathbf{r}}^{\frac{3}{2}}} + \mathcal{O}(E^{-\frac{3}{4}}) \tag{12}$$

The leading smooth term $\mathcal{A}E$ is obtained from the term $N = M = 0$ in (11) and $\mathcal{A} = ab/(4\pi)$ is proportional to the area of the torus. The sum in 12 runs over $\mathbf{r} = (N, M) \in \mathbb{Z}^2 \backslash (0,0)$ (in the sequel every sum over $\mathbf{r}$ will not include $(0,0)$ unless stated otherwise). These terms are oscillatory functions of E. Here, $L_{\mathbf{r}} = \sqrt{(Na)^2 + (Mb)^2}$ is the length of a periodic geodesic (periodic orbit) with winding numbers $\mathbf{r} = (N, M)$.

One can treat $\tilde{c}(E)$ analogously. Here a closed analytic expression for the integrals would be out of reach, but higher order corrections to the leading result can be obtained systematically. The leading asymptotic contributions are given by

$$\tilde{c}(E) = \tfrac{2\mathcal{A}^2}{\pi^2} E^2 + E^{\frac{5}{4}} \tfrac{2^{\frac{11}{2}} \mathcal{A}^3}{\pi^{\frac{5}{2}}} \sum_{\mathbf{r}} \frac{|MN|}{L_{\mathbf{r}}^{\frac{7}{2}}} \sin\left(L_{\mathbf{r}}\sqrt{E} - \frac{\pi}{4}\right) + \mathcal{O}(E). \tag{13}$$

We now have the leading asymptotic expressions for both $\tilde{c}(E)$ and $\mathcal{N}(E)$. The next step would be to invert $\mathcal{N}(E) = K$ and eliminate the dependence of $\tilde{c}(E)$ on the spectrum. However, the leading orders of the trace formula (12) for the spectral counting functions do *not* define a manifestly monotonically increasing function. Still, one may think of the exact inverse $E(K)$ as an asymptotic series itself. The leading orders of this series can formally be obtained from the trace formula (12)

$$E(K) = \frac{K}{\mathcal{A}} - K^{\frac{1}{4}} \frac{2^{\frac{3}{2}}}{\mathcal{A}\pi^{\frac{1}{2}}} \sum_{\mathbf{r}} \frac{\sin(l_{\mathbf{r}}\sqrt{K} - \frac{\pi}{4})}{l_{\mathbf{r}}^{\frac{3}{2}}} + \mathcal{O}(K^0). \tag{14}$$

Here, $l_{\mathbf{r}} = L_{\mathbf{r}}/\sqrt{\mathcal{A}}$ is the re-scaled (dimensionless) length of a periodic orbit. The above step definitely needs a more detailed justification. Here, we can only refer to the numerical tests that we will give below.

We may now replace E by $E(K)$ in (13) to obtain the leading orders of the cumulative nodal count $c(K) \equiv \tilde{c}(E(K))$. The latter can be written as

$$c(K) = \overline{c}(K) + c_{\mathrm{osc}}(K) \tag{15}$$

with a smooth part

$$\overline{c}(K) = \frac{2}{\pi^2} K^2 + \mathcal{O}(K) \tag{16}$$

and an oscillatory part

$$c_{\rm osc}(K) = K^{\frac{5}{4}} \sum_{\mathbf{r}} a_{\mathbf{r}} \sin(l_{\mathbf{r}}\sqrt{K} - \frac{\pi}{4}) + \mathcal{O}(K), \tag{17}$$

where we introduced the amplitudes

$$a_{\mathbf{r}} = \frac{2^{\frac{7}{2}}}{\pi^{\frac{5}{2}} l_{\mathbf{r}}^{\frac{3}{2}}} \left(\frac{4\pi^2 |NM|}{l_{\mathbf{r}}^2} - 1 \right). \tag{18}$$

Note, that the smooth part is independent of the geometry of the torus. However, the oscillating part depends explicitly on the aspect ratio $\tau = a/b$ and can distinguish between different geometries.

When trying to calculate higher order corrections to the leading terms in the smooth and oscillatory parts of the cumulative nodal count one runs into some difficulties. Already in the next-to leading order products of sums over periodic orbits appear and it is no longer straight forward to discern the smooth from the oscillatory parts.

3.2 Surfaces of revolution

Let us now consider surfaces of revolution $\mathcal{M}$ which are created by the rotation of the line $y = f(x)$ for $x \in I \equiv [-1, 1]$ about the x-axis. We restrict our attention to smooth (analytic) and convex surfaces. In more detail we make the following assumptions:

(i) The function $q(x) \equiv f^2(x)$ is analytic in $I = [-1, 1]$, and vanishes at $x = \pm 1$ where $q(x) \approx a_{\pm}(1 \mp x)$, with $a_{\pm}$ positive constants. This requirement guarantees that the surface is smooth even at the points where $\mathcal{M}$ is intersected by the axis of rotation. In particular, $\mathcal{M}$ has no boundary.

(ii) The second derivative of $f(x)$ is strictly negative, so that $f(x)$ has a single maximum at $x = x_{\rm max}$, where f reaches the value $f_{\rm max}$. This requirement guarantees convexity of $\mathcal{M}$.

Surfaces which satisfy the requirements above are convex, mild deformations of ellipsoids of revolution. Below we will add a further technical requirement that will exclude the sphere among other surfaces – generic mild deformations of ellipsoids will not be affected.

The metric on the surface (induced from the Euclidian metric in $\mathbb{R}^3$) is given by

$$ds^2 = \left[1 + f'(x)^2\right] dx^2 + f(x)^2 d\theta^2, \tag{19}$$

where the prime denotes differentiation with respect to x, and θ is the azimuthal angle.

3.2.1 The wave equation on a surface of revolution

Considering a surface of revolution as a drum we have to discuss the solutions of the wave equation

$$-\Delta_{\mathcal{M}} \psi(x, \theta) = E\psi(x, \theta) \tag{20}$$

where the Laplace-Beltrami operator corresponding to the metric (19) for a surface of revolution is given by

$$\Delta_{\mathcal{M}} = \frac{1}{f(x)\sigma(x)} \frac{\partial}{\partial x} \frac{f(x)}{\sigma(x)} \frac{\partial}{\partial x} + \frac{1}{f(x)^2} \frac{\partial^2}{\partial \theta^2}, \tag{21}$$

where $\sigma(x) = \sqrt{1 + f'(x)^2}$.

Solutions $\Psi(x, \theta)$ to (20) can be found for a discrete spectrum of eigenvalues E and are doubly differentiable, 2π-periodic in θ and non singular on $[I \times S^1]$. The wave equation (20) is separable and the solutions can be written as a product

$$\Psi(x, \theta) = {}^{\cos}_{\sin}(m\theta)\ \phi_m(x) \tag{22}$$

where $m \in \mathbb{Z}$ to ensure 2π-periodicity in θ. In the seperation ansatz (22) we choose to use the cosine for $m \geq 0$ and the sine for $m < 0$.
For any fixed m, (21) now reduces to the ordinary differential equation

$$-\frac{1}{f(x)\sigma(x)}\frac{d}{dx}\frac{f(x)}{\sigma(x)}\frac{d}{dx}\phi_m(x) + \frac{m^2}{f(x)^2}\phi_m(x) = E\phi_m(x) \tag{23}$$

which is of the Sturm-Liouville type. Let us denote the eigenvalues $E_{n,m}$ and eigenfunctions $\phi_{n,m}(x)$, where $n = 0, 1, 2, \ldots$ and $E_{n,m} \leq E_{n+1,m}$. Sturm's oscillation theorem then implies that $\phi_{n,m}(x)$ has n nodes.

The nodal pattern of the wave $\psi_{n,m}(x,\theta) = \phi_{n,m}(x)\,{}^{\cos}_{\sin}(m\theta)$ is that of a checkerboard typical to separable systems and contains

$$\nu_{n,m} = (n+1)(2|m| + \delta_{m,0}) \tag{24}$$

nodal domains.

3.2.2 The semiclassical approach to the spectrum

To proceed further we also need to know the eigenvalues $E_{n,m}$. For $n, m \gg 1$ the latter can be replaced by the semiclassical eigenvalues using the Bohr-Sommerfeld approximation [13]

$$E^{\mathrm{scl}}_{n,m} = \boldsymbol{H}\left(n + \frac{1}{2}, m\right) + h(n,m) \quad , \; n \in \mathbb{N}, \; m \in \mathbb{Z}\,. \tag{25}$$

where $\boldsymbol{H}(n,m)$ is the classical Hamiltonian defined in terms of the action variables, and $h(n,m)$ is homogeneous of order 0. Neglecting $h(n,m)$ in the sequel, amounts to introducing an error which is bounded by a constant. As indicated by the notation the action variables m and n in (25) coincide with the integers m and n used in the separation ansatz above. Note, that in general classical integrability leads to analogous semiclassical approximations for the spectrum. However classical integrability does not imply quantum separability. In our approach we use the property of quantum separable drums that the nodal sets have a checkerboard structure which implies that the number of nodal domains is an explicit function $\nu_{n,m} = \nu(n,m)$ of the action variables n and m (basically a product). Since quantum separability implies classical integrability our approach can be generalized to all drums for which the wave equation is separable.

The classical Hamiltonian $\boldsymbol{H}(n,m)$ can be obtained from the observation that the classical trajectories are the geodesics on the surface. The latter can be derived from the Euler-Lagrange variational principle with the Lagrangian

$$\boldsymbol{L} \equiv \frac{v^2}{4} = \frac{1}{4}\left(\left[1 + f'(x)^2\right]\dot{x}^2 + f(x)^2\dot{\theta}^2\right). \tag{26}$$

where a dot above denotes time derivative (the factor 1/4 in front of the squared velocity is consistent with our choice of energy and action units). The angular momentum along the axis $p_\theta = f(x)^2\dot{\theta}/2$ is conserved and we shall use it as the first action variable $m = \frac{1}{2\pi}\int_0^{2\pi} p_\theta d\theta \equiv p_\theta$. The momentum conjugate to x is

$$p_x = \frac{1}{2}\left[1 + f'(x)^2\right]\dot{x}, \tag{27}$$

and the conserved kinetic energy is obtained by a Legendre transformation

$$E \equiv \boldsymbol{H}(p_x, x, m) = p_\theta\dot{\theta} + p_x\dot{x} - \boldsymbol{L} = \frac{p_x^2}{1 + f'(x)^2} + \frac{m^2}{f(x)^2} \tag{28}$$

We may now introduce the action variable n,

$$n(E;m) = \frac{1}{2\pi}\oint p_x(E,x)\,dx = \frac{1}{\pi}\int_{x_-}^{x_+} p_x(E,x)dx \tag{29}$$

where

$$p_x(E,x) = \sqrt{[Ef(x)^2 - m^2]\,[1+f'(x)^2]}/f(x) \tag{30}$$

and $x_\pm$ are the classical turning points where $Ef(x)^2 - m^2 = 0$, with $x_- \le x_{\max} \le x_+$. Real classical trajectories exist only if $E > (m/f_{\max})^2$. The classical Hamiltonian $\boldsymbol{H}(n,m)$ in the action-angle representation is obtained by inverting (29) to express the energy in terms of n and m.

The classical Hamiltonian $\boldsymbol{H}(n,m)$ is a homogenous function of order 2

$$\boldsymbol{H}(\lambda n, \lambda m) = \lambda^2 \boldsymbol{H}(n,m). \tag{31}$$

For the discussion of the classical dynamics and the structure of phase space it is therefore sufficient to consider unit energy $E = 1$. Any other energy can be obtained from simple rescaling (and trajectories remain the same upto a rescaling of the time). All dynamic content is thus stored in the function

$$n(m) \equiv n(E=1,m) \tag{32}$$

which defines a line Γ in the (n,m) plane and is one of the main building blocks of the semi-classical theory which will be used throughout this work. We shall list therefore its relevant properties:

(i) $n(m)$ is defined on the interval $-f_{\max} \le m \le f_{\max}$.
(ii) The reflection symmetry, $n(-m) = n(m)$, follows from the definition (29).
(iii) In the interval $0 < m \le f_{\max}$ the function $n(m)$ is monotonically decreasing from its maximal value $n(0)$ to $n(m = f_{\max}) = 0$.
(iv) At $m = 0$ the function $n(m)$ is not analytic.
(v) Some authors (e.g., [15]) prefer to use the Clairaut integral $\mathcal{I}$ instead of the angular momentum. They are related by

$$\mathcal{I} = \frac{m}{\sqrt{2E}}\,. \tag{33}$$

Let us now turn to periodic motion on the surface of revolution. Periodic geodesics apear if the angular velocities

$$\omega_n = \frac{\partial \boldsymbol{H}(m,n)}{\partial n} \qquad \omega_m = \frac{\partial \boldsymbol{H}(m,n)}{\partial m} \tag{34}$$

have a rational ratio. Since $\frac{dn(m)}{dm} = -\frac{\omega_m}{\omega_n}$ this is equivalent to the condition

$$M + N\frac{dn(m)}{dm} = 0 \tag{35}$$

for $M, N \ne 0$. The integers $\mathbf{r} = (M,N) \in \mathbb{Z}^2 \backslash (0,0)$ are the winding numbers in the θ and x directions.

The classical motion is considerably simplified if the *twist condition* [15]

$$n''(m) \equiv \frac{d^2 n(m)}{dm^2} \ne 0 \qquad \text{for} \quad 0 < m \le f_{\max} \tag{36}$$

is obeyed. This excludes, for example, the sphere but includes all mild deformations of an ellipsoid of revolution. We will assume the twist condition for the rest of this work. It guarantees that there is a unique solution to (35) which we will call $m_{\mathbf{r}}$.

Note, that $n'(m)$ has a finite range that we will denote by Ω. A solution to (35) only exists if $-M/N \in \Omega$. Periodic motion with winding numbers $N = 0$, $M \ne 0$ or with $M \ne 0$, $N = 0$ are not described by solutions of (35). The first case, $N = 0$ describes a pure rotation in the

θ-direction at constant $x = x_{\max}$ where $m_{0,\pm|M|} = \pm f_{\max}$ and the second case $M = 0$ is a periodic motion through the two poles at fixed angle θ mod π such that $m_{|N|,0} = 0$.

The length of a periodic geodesic can be obtained by observing that $E = v^2/4$ is a constant of motion the metric length $L = \oint v^2 dt/v$ of a periodic geodesic is given by

$$L_{\mathbf{r}} = 2\pi \left| N n(m_{\mathbf{r}}) + M m_{\mathbf{r}} \right|. \tag{37}$$

Returning to the spectrum, we note that the leading terms in the trace formula for the spectral counting function $N(E) = \sum_{m,n} \Theta(E - E_{n,m})$ can be obtained by using (25) and Poisson's summation formula [15].

$$\mathcal{N}(E) = \mathcal{A}E + E^{\frac{1}{4}} \sum_{\mathbf{r}} \mathcal{N}_{\mathbf{r}}(E) \tag{38}$$

where

$$\mathcal{A} = \int_{-f_{\max}}^{f_{\max}} n(m)\ dm = ||\mathcal{M}||/4\pi \tag{39}$$

and $\|\mathcal{M}\|$ is the area of the surface. The oscillating parts contain integrals

$$\mathcal{N}_{\mathbf{r}} \propto \int_{-f_{\max}}^{f_{\max}} dm\ e^{2\pi i \sqrt{E}[Nn(m)+Mm]}. \tag{40}$$

We will calculate these to leading order in $E^{\frac{1}{2}}$ using the stationary phase approximation. The stationary phase condition turns out to be identical to equation (35) which describes periodic motion. As a consequence the stationary points are $m = m_{\mathbf{r}}$. Note that the range of contributing $\mathbf{r}$ values is restricted to the classically accessible domain $-M/N \in \Omega$. For $-M/N \notin \Omega$ the integral does not have a stationary point and contributes only to higher orders in $1/\sqrt{E}$. Eventually one obtains, in stationary phase approximation [15]

$$\mathcal{N}_{\mathbf{r}}(E) = (-1)^N \frac{\sin(L_{\mathbf{r}} E^{\frac{1}{2}} + \sigma \frac{\pi}{4})}{2\pi |N^3 n''_{\mathbf{r}}|^{\frac{1}{2}}} + \mathcal{O}(E^{-\frac{1}{2}}) \tag{41}$$

where $n''_{\mathbf{r}} = n''(m = m_{\mathbf{r}})$ and $\sigma = \mathrm{sign}(n''_{\mathbf{r}})$ which is the same for all values of $\mathbf{r}$. The contributions of the terms with either $N = 0$ or $M = 0$ or with $-M/N \notin \Omega$ are of higher order in $1/E$ and will not be considered here.

3.2.3 The cumulative nodal count

We have now all ingredients to derive an asymptotic trace formula for the cumulative nodal count

$$c(K) = \tilde{c}(E(K)) = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \nu_{mn}\ \Theta\left(E(K) - E_{m,n}\right). \tag{42}$$

Inverting the asymptotic trace formula (38) for the spectral counting function $\mathcal{N}(E) = K$ one obtains

$$E(K) = \frac{K}{\mathcal{A}} - \left(\frac{K}{\mathcal{A}}\right)^{\frac{1}{4}} \sum_{\mathbf{r}} \frac{\mathcal{N}_{\mathbf{r}}(\frac{K}{\mathcal{A}})}{\mathcal{A}} + \mathcal{O}(K^0) \tag{43}$$

to leading order in $1/K$.

The function $c(E)$ can be obtained as an asymptotic trace formula by the same approach that we used for $\mathcal{N}(E)$ in the preceding section 3.2.2. Expanding the result in $\delta E = E(K) - K/\mathcal{A}$ such that

$$c(K) = \tilde{c}(K/\mathcal{A}) + \tilde{c}'(K/\mathcal{A})\delta E + \mathcal{O}(\tilde{c}''\delta E^2) \tag{44}$$

is consistent if we neglect all orders smaller than $\mathcal{O}(K)$. In almost complete analogy to the trace formula (15) for simple tori, this can be expressed as a sum

$$c(K) = \overline{c}(K) + c_{\mathrm{osc}}(K) \tag{45}$$

of a smooth part $\overline{c}(K)$ and an oscillatory part, $c_{\mathrm{osc}}(K)$. Defining

$$\overline{m^p n^q} = \frac{1}{\mathcal{A}} \int_{E(m,n)<1} dm\ dn\ |m|^p n^q \tag{46}$$

as the action moments (averaged over the area under the curve Γ) the smooth part can be expressed as

$$\overline{c}(K) = 2\frac{\overline{mn}}{\mathcal{A}} K^2 + \frac{\overline{m}}{\mathcal{A}^{\frac{1}{2}}} K^{\frac{3}{2}} + \mathcal{O}(K) \tag{47}$$

which, compared to the trace formula of the torus (15), has an additional term $\propto K^{3/2}$ which can be traced back to the different way of counting nodal domains in tori (9) and surfaces of revolution (24). Likewise, the oscillatory part can be expressed as

$$c_{\mathrm{osc}}(K) = K^{\frac{5}{4}} \sum_{\mathbf{r}:-\frac{M}{N}\in\Omega} a_{\mathbf{r}} \sin\left(l_{\mathbf{r}}\sqrt{K} + \frac{\sigma\pi}{4}\right) + \mathcal{O}(K) \tag{48}$$

with the amplitude

$$a_{\mathbf{r}} = (-1)^N \frac{m_{\mathbf{r}} n(m_{\mathbf{r}}) - 2\overline{mn}}{\mathcal{A}^{\frac{5}{4}} \pi |N^3 n''_{\mathbf{r}}|^{\frac{1}{2}}} \tag{49}$$

and rescaled length

$$l_{\mathbf{r}} = \frac{L_{\mathbf{r}}}{\sqrt{\mathcal{A}}} \tag{50}$$

of a periodic geodesic $\mathbf{r}$ with $-\frac{M}{N} \in \Omega$. Note, that for $m_{\mathbf{r}} = 0$ or $m_{\mathbf{r}} = \pm f_{\max}$ only one half of the stationary phase integral contributes and the amplitude $a_{\mathbf{r}}$ has to be multiplied by 1/2. If the (finite) interval $\Omega \subset \mathbb{R}$ is bounded by rational numbers, then the amplitudes $a_{\mathbf{r}}$ for periodic geodesics with winding numbers satisfying $-\frac{M}{N} \in \partial\Omega$ also have to be multiplied by 1/2.

The above trace formula reveals a quite astonishing relation between the nodal count sequence and the geometry of the surface. So far, similar relations have only been derived for spectral functions. Yet the cumulative nodal count does not contain any spectral information apart from the ordering inherited from the spectrum and still the oscillatory part can be written as a sum over all different periodic geodesics.

For ellipsoids defined by the rotation of the curve

$$f(x) = R\sqrt{1 - x^2} \tag{51}$$

with maximal radius $f_{\max} = R$ at the equator the curve $n(m)$ can be expressed explicitly in terms of elliptic integrals.

4 Application of the trace formula and comparison to numerical results

We have tested the approximations in the above calculations numerically on four different systems for which we built up a large data base which will be denoted as data sets (a) to (d). We chose two different ellipsoids of revolution with $R = 2$ (for data set (a)) and $R = 1/2$ (for data set (b)). These parameters provide us with data sets for an oblate ($R = 2$) and prolate ($R = 1/2$) ellipsoid. We also considered two different tori with $\tau^2 = 2$ (for data set (c)) and $\tau^2 = \sqrt{2}$ (for data set (d)). For rational τ^2 the spectrum contains growing number theoretic degeneracies which are absent in the irrational case. Our parameters cover both cases.

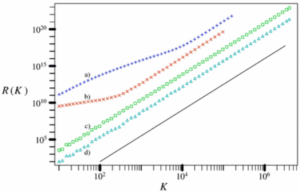

Fig. 1. The integrated variance $R(K)$ (double logarithmic plot, the plots have been shifted for better visibility) for the two ellipsoids (data set (a) with $R = 2$ and data set (b) with $R = 1/2$), and the two tori (data set (c) with $\tau^2 = 2$ and data set (d) with $\tau^2 = \sqrt{2}$). The full line has slope 7/2.

For the ellipsoids the first 10^5 eigenvalues and eigenfunctions have been calculated, from which we constructed the sequence of nodal counts. For the tori obtaining the spectrum and the corresponding eigenfunctions is straight forward – in our numerics we used the lowest $\approx 4 \times 10^6$ eigenvalues.

To obtain the fluctuating part the numerically computed $c(K)$ were fitted to a fourth order polynomial in $\kappa = \sqrt{K}$. Not surprisingly, the numerically obtained two leading coefficients ($\propto K^2$ and $K^{3/2}$) fitted extremely well with the corresponding analytically obtained coefficients in the smooth parts of the corresponding trace formulae.

The more critical tests, which we will present here, involve the fluctuations decribed by the oscillatory part of the trace formulae. The latter has been obtained numerically by subtracting the best polynomial fit from the exact $c(K)$.

The tests of on the oscillatory part of the trace formulae give us also the opportunity to discuss some aspects of the fluctuations of the cumulative nodal count sequence.

4.1 The integrated variance

The simplest measure of the fluctuations is the variance given by the squared oscillatory part averaged over some interval – or its integral

$$R(K) \approx \int_0^K dK' \; c_{\rm osc}(K')^2. \tag{52}$$

Substituting the trace formula this expression consists of a double sum over periodic geodesics. The main contribution can be expected from the diagonal pairs. Neglecting all non-diagonal terms one obtains

$$R(K) = \frac{2}{7} K^{\frac{7}{2}} \sum_{\mathbf{r}} |a_{\mathbf{r}}|^2 \tag{53}$$

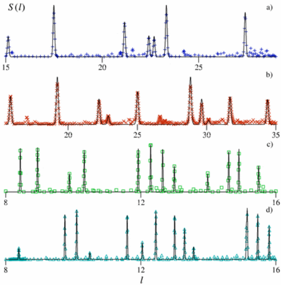

Fig. 2. Absolute value of length spectra of the cumulative nodal counts 54 for the two ellipsoids (data set (a) with $R = 2$ and data set (b) with $R = 1/2$), and the two tori (data set (c) with $\tau^2 = 2$ and data set (d) with $\tau^2 = \sqrt{2}$). The full line is obtained from the trace formulae (47) (for the ellipsoids) and (17) (for the tori). Points represent the numerical data.

which scales like $K^{7/2}$. This scaling has been tested and the results are shown in Fig. 1. Clearly, the expected power law is reached for sufficiently large values of the counting index K. The prefactor $\frac{7}{2}\sum_{\mathbf{r}}|a_{\mathbf{r}}|^2$ in the diagonal approximation cannot be expected to fit the numerical data because the non-diagonal parts will shift the result considerably.

4.2 The length spectrum

The integrated variance is still a quite rough test of the variance. A much more elaborate test is provided by computing the length spectrum, which we define roughly as the Fourier transform of $c_{\mathrm{osc}}(K)$ with respect to $\kappa = \sqrt{K}$. In more detail, before the Fourier transformation we multiply c_{osc} by a Gaussian window function which defines a finite interval of width $\sqrt{\omega}$ centered at $\kappa = \kappa_0$ (in practice all numerical nodal count sequences are finite – a Gaussian window is the appropriate way to deal with that). To obtain a result which does not scale with κ_0 we also multiply with $\kappa^{-5/2}$ such that the amplitude of a periodic geodesic is independent of κ.

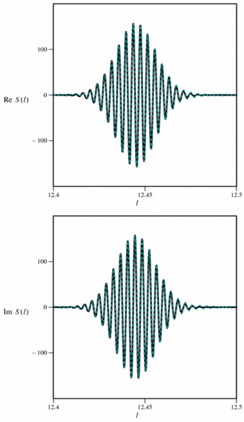

Fig. 3. Real and imaginary part for the length spectra of the cumulative nodal count (54) for the torus with irrational $\tau^2 = \sqrt{2}$ (data set (d)) near the $|N| = 2$, $|M| = 3$ peak (with scaled length $l_{N,M} = 12.445$). The black full line is obtained from the trace formula (17) neglecting contributions with $|N| \neq 2$ and $|M| \neq 3$. The blue dashed line is obtained numerically from the exact cumulative nodal. There is a phase shift of $\pi/2$ between the real and imaginary parts as can be checked by plotting both in a single graph.

Altogether we define the length spectrum by

$$S(l) = l^{3/2} \int_0^\infty d\kappa\ \kappa^{-5/2} c_{\rm osc}(K = \kappa^2) e^{-\frac{(\kappa-\kappa_0)^2}{\omega} + i\kappa l}. \qquad (54)$$

The final multiplication with $l^{3/2}$ is not necessary but improves visibilty of peaks in a plot over a large range of lengths l.

The trace formula for the cumulative nodal count predicts pronounced peaks at the scaled lengths $l = l_{\mathbf{r}}$ of the periodic geodesics. For the absolute value of the length spectrum these can be seen very nicely in Fig. 2 which shows a remarkable agreement of the numerical data with the theoretical predictions.

Not only the absolute value of the length spectrum is recovered by the trace formula but also its phase. This can be seen in fig. 3 where the real and imaginary parts of the length spectrum of the torus with $\tau^2 = \sqrt{2}$ (data set (d)) are plotted near the peak corresponding to periodic motion with winding numbers $(|N|, |M|) = (2, 3)$.

This excellent agreement provides further support for the validity of the approximations which were used in the derivation of the two versions of the nodal counts trace formula.

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