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Nodal sets in mathematical physics

J. Brüning[a]

Institut für Mathematik der Humboldt-Universität, Unter den Linden 6, 10099 Berlin, Germany

Abstract. We describe the main lines of mathematical research dealing with nodal sets of eigenfunctions since the days of Chladni. We present the material in a form hopefully suited to a nonspecialized but mathematically educated audience.

1 Introduction

When Ernst Florens Friedrich Chladni published the discovery of his famous Klangfiguren in 1787, he aroused a lot of interest not only among his fellow physicists - or *Naturforscher* as they addressed themselves in those days - but also among the public at large. Notably with the work of Euler and Lagrange, Mathematical Physics had just come into being and the new phenomenon posed a great challenge to its protagonists. Nevertheless, it took more than 70 years until a satisfying model was formulated by Kirchhoff in spite of many previous attempts e.g. by Sophie Germain and Poisson. A reliable test for its predictions had to wait even until 1953.

The analysis of the Klangfiguren then requires us to find the characteristic vibrations of a plate and to determine their *nodes* i.e. the rest points of the plate. This amounts to solving an eigenvalue problem for the Bilaplacian and to find the zero sets of the eigenfunctions. This is a quite complicated problem which allows an explicit solution only for the circular plate ([8], p. 263). Therefore, mathematical physicists have preferred to study the conceptually analogous but technically simpler problem of the vibrating membrane on which we will focus in this review; we concentrate on results obtained by "classical" methods of Mathematical Physics, hence will exclude stochastic approaches from consideration which are well presented in other articles of this collection.

2 Vibrating membranes

A compact Riemannian manifold (M, g), of dimension m, possibly with boundary, ∂M, will be called a *membrane* in what follows; if $\partial M = \emptyset$, the membrane is called *closed*. We will require infinitely smooth data for simplicity, even though this assumption can be weakened considerably in many cases.

The vibrations we consider are described by the eigenfunctions of the Laplace-Beltrami operator which we introduce as an operator in $L^2(M, g)$ which is defined for $\sigma \in C_c^\infty(M)$ by

$$\Delta\sigma = -\frac{1}{\sqrt{g}} \sum_{i,j=1}^{m} \frac{\partial}{\partial x^i} \left(\sqrt{g} g^{ij} \frac{\partial}{\partial x^j} \sigma \right); \tag{1}$$

here (x^i) are local coordinates, (g^{ij}) denote the corresponding coefficients of the induced metric on T^*M, and $g = \det(g^{ij})^{-1}$. In order to obtain a symmetric operator (note that Δ becomes

[a] e-mail: `bruening@mathematik.hu-berlin.de`

nonnegative in our definition) we have to impose boundary conditions if $\partial M \neq \emptyset$; again, for simplicity (and frequency of occurence) we restrict attention to the Dirichlet and the Neumann boundary conditions which require σ or its normal derivative at the boundary to vanish. The domain of Δ will be generically denoted by $\mathcal{D}$.

The basis of the subsequent analysis is the following special instance of the Spectral Theorem, which was first conjectured by M. Ohm and Lord Rayleigh and eventually proved, in great generality, by Hilbert 1904.

Theorem 1 *There is a sequence (λ_n, σ_n), called the* spectral resolution *of Δ, of solutions of the eigenvalue equation*

$$\Delta\sigma = \lambda\sigma, \quad \sigma \in \mathcal{D}, \tag{2}$$

with the following properties.
(1) The sequence $(\lambda_n)_{n\in N}$ increases towards infinity.
(2) The sequence $(\sigma_n)_{n\in N}$ forms an orthonormal basis for $L^2(M,g)$.
In particular, Δ is a self-adjoint operator with domain $\mathcal{D}$ in $L^2(M,g)$.

We call the finite dimensional vector space

$$E(\lambda_n) := \{\sigma \in \mathcal{D} : \Delta\sigma = \lambda_n\sigma\}$$

the *eigenspace with eigenvalue* λ_n; λ_n is called *simple* if its multiplicity

$$\mu(\lambda_n) := \dim E(\lambda_n)$$

is equal to one, and otherwise *degenerate*.

Now we can introduce the *nodes* or the *nodal set* of σ_n as

$$N(\sigma_n) := (\sigma_n)^{-1}(0); \tag{3}$$

if $m = 2$, we talk about the *nodal lines*. There is no ambiguity about nodes if λ_n is simple but in the degenerate case, the nodal sets may vary greatly in the unit sphere of $E(\lambda_n)$. Since in this case we have many choices for an orthonormal basis, it is unclear how significant the knowledge of $N(\sigma_n)$ for any specific choice of basis can be.

The proof of Theorem 1 rests on the calculus of variations as applied to the Dirichlet integral. In particular, one obtains the following very useful non-recursive characterization of the eigenvalues, which is due to Courant ([8, p. 351]).

Theorem 2 *Denote by $\mathcal{V}^k$ the set of k-dimensional subspaces of $L^2(M,g)$. Then for all n,*

$$\lambda_n = \max_{V\in\mathcal{V}^{n-1}} \min_{\sigma\in\mathcal{D}\setminus\{0\}\cap V^\perp} \frac{\int_M |\nabla\sigma|^2}{\int_M |\sigma|^2}. \tag{4}$$

One of the advantages in dealing with mebranes is the existence of large families for which the spectral resolution can be given explicitly. Even though these classes are special in the sense that they consist of manifolds with integrable geodesic flow, they do provide interesting examples to build an intuition and to test conjectures. We restrict to the case of closed membranes and briefly discuss the case $m = 1$, the spheres, and the flat tori below.

Example 1: m = 1. 1 In dimension one, the isometry classes of closed membranes form a one-parameter family given by the circles of fixed radius R. For $R = 1$, we need to find the 2π-periodic solutions of the equation

$$\sigma''(t) + \lambda\sigma(t) = 0,$$

and get, for all $n \in N$,

$$\lambda_n = (n-1)^2, \quad E(\lambda_n) = \{\alpha\cos(\sqrt{\lambda_n}t) + \beta\sin(\sqrt{\lambda_n}t) : \alpha, \beta \in C\}.$$

The nodal set $N(\sigma_n)$ is given by $n-1$ equidistant points on S^1, and the set $S^1 \setminus N(\sigma_n)$ has exactly n connected components. In what follows, we will call the connected components of $M \setminus N(\sigma_n)$ the *nodal domains* of σ_n, and we define the number $NC(\sigma_n)$, the *nodal count* of σ_n, as the number of nodal domains of σ_n.

Example 2: Spheres

We equip the sphere

$$S^m := \{x \in R^{m+1} : |x| = 1\} \subset R^{m+1}$$

with the metric induced from the Euclidean metric on R^{m+1}. Next we introduce the space of homogeneous polynomials in $m+1$ variables, $\mathcal{P}^k = \mathcal{P}^k(R_{m+1})$, and the subspace of harmonic polynomials,

$$\mathcal{H}^k = \mathcal{H}^k(R^{m+1}) := \{\sigma \in \mathcal{P}^k : \Delta\sigma = 0\}.$$

The link with Δ_{S^m}, the Laplace-Beltrami operator on S^m, is provided by orthogonal symmetry since in polar coordinates, $x = r\omega$, we obtain for $\sigma \in \mathcal{P}^k$

$$\Delta_{R^{m+1}}\sigma(r\omega) = r^{k-2}(\Delta_{S^m} - k(k+m-1))\sigma(\omega). \tag{5}$$

As an easy consequence we find that

$$\mathcal{H}^k|S^m \subset E_{S^m}(k(k+m-1)),$$

and that the image of the map $r^2\Delta_{R^{m+1}} : \mathcal{P}^{k+2} \to \mathcal{P}^{k+2}$ contains the spaces

$$\oplus_{j=0}^{l} r^{2(l+1-j)}\mathcal{H}^{2j}$$

and

$$\oplus_{j=0}^{l-1} r^{2(l-j)}\mathcal{H}^{2j+1},$$

for $k = 2l$ and $k = 2l-1$, respectively. But then it follows inductively that

$$\mathcal{P}^{2l} = \oplus_{j=0}^{l} r^{2(l-j)}\mathcal{H}^{2j}, \tag{6}$$

$$\mathcal{P}^{2l+1} = \oplus_{j=0}^{l} r^{2(l-j)}\mathcal{H}^{2j+1}. \tag{7}$$

The direct sum in this decomposition is actually orthogonal if we equip $\mathcal{P}^k$ with the scalar product

$$\langle\sigma_1, \sigma_2\rangle := \int_{S^m} \sigma_1(\omega)\bar{\sigma_2}(\omega) dvol_{S^m}(\omega).$$

In summary, we find that the spectral resolution of Δ_{S^m} is provided by the data

$$\lambda_n = n(n+m-1), \quad E(\lambda_n) = \mathcal{H}_n|S^m.$$

Moreover,

$$\mu(\lambda_n) = \binom{n+m}{m} - \binom{n-2+m}{m} = \frac{2}{m!}n^{m-1} + O(n^{m-2}) = \frac{2}{m!}\sqrt{\lambda_n}^{m-1} + O(\sqrt{\lambda_n}^{m-2}).$$

We also see that the eigenfunctions of Δ_{S^m} with eigenvalue λ_n are polynomials of degree $n \sim \sqrt{\lambda_n}$. Their nodal sets, however, are not at all easy to analyze.

Example 3: Flat Tori

A flat torus, $T = T_\Gamma$, is the quotient of R^m by a lattice Γ, where a *lattice* is the set of all integer linear combinations of a fixed basis, $(\gamma_j)_{j=1}^m$, of R^m. The torus is then metrically obtained from the Euclidean parallelepiped

$$\mathcal{F}_\Gamma := \left\{ \sum_{j=1}^{m} x^j\gamma_j \in R^m : 0 \le x^j \le 1 \right\},$$

by appropriately identifying the faces; in particular, $\mathrm{vol}\mathcal{F}_\Gamma = \mathrm{vol}T$.

We introduce the dual lattice, Γ^*, by

$$\Gamma^* := \{\gamma^* \in R^m : \langle \gamma^*, \gamma \rangle \in Z \text{ for all } \gamma \in \Gamma\} = \left\{ \sum_{j=1}^m k_j \gamma^j : k_j \in Z \right\},$$

where $(\gamma^j)_{j=1}^m$ denotes the dual basis to $(\gamma_i)_{i=1}^m$, $\langle \gamma^j, \gamma_i \rangle = \delta_i^j$. The functions

$$\sigma^{\pm}_{\gamma^*}(x) := \exp(\pm 2\pi i \langle \gamma^*, x \rangle) \tag{8}$$

satisfy the eigenvalue equation

$$\Delta_{R^m} \sigma^{\pm}_{\gamma^*} = 4\pi^2 |\gamma^*|^2 \sigma^{\pm}_{\gamma^*}, \tag{9}$$

and a well known completeness argument shows that all eigenvalues of Δ_T are given by (8), with corresponding eigenfunctions $\sigma^{\pm}_{\gamma^*}|\mathcal{F}_\Gamma$. The growth of the eigenvalues is related to a volume estimate as follows: if we denote the diameter of $\mathcal{F}_{\Gamma^*}$ by R_{Γ^*} and by $B^m_R(0)$ the ball of radius R around 0 in R^m, then we have

$$\mathrm{vol}\, B^m_R(0) \le N_\Gamma(4\pi^2 R^2)\, \mathrm{vol}\mathcal{F}_{\Gamma^*} \le \mathrm{vol}\, B^m_{R+R_{\Gamma^*}}(0),$$

if we write

$$N_\Gamma(t) = \sum_{\lambda_n \le t} \mu(\lambda_n).$$

Since $\mathrm{vol}\mathcal{F}_{\Gamma^*}\mathrm{vol}\mathcal{F}_\Gamma = 1$, it follows that

$$N_\Gamma(t) = \frac{\mathrm{vol}\, B^m_1(0)}{(2\pi)^m} \mathrm{vol}\, T\, t^{m/2} + O(t^{(m-1)/2}). \tag{10}$$

Again, the nodal sets of generic eigenfunctions seem hopelessly complicated, as a consequence of the high multiplicity of the eigenvalues. However, if we restrict attention to the linear combinations of the basic eigenfunctions $\sigma^{\pm}_{\gamma^*}$, the situation greatly simplifies. Their nodal sets in R^m are a union of hypersurfaces, e.g. for $(\sigma^{even}_{\gamma^*})(x) := \sin 2\pi \langle \gamma^*, x \rangle$ we obtain

$$(\sigma^{even}_{\gamma^*})^{-1}(0) = \bigcup_{k \in Z} \{x \in R^m : \langle \gamma^*, x \rangle = k/2\}$$

and it is easy to see that these are inequivalent mod Γ precisely for $k = 1, \ldots, \nu(\gamma^*)$, if we put

$$\nu(\gamma^*) := \min\{\langle \gamma^*, \gamma \rangle > 0 : \gamma \in \Gamma\}. \tag{11}$$

We note that in terms of the basis representation of γ^*,

$$\gamma^* = \sum_{j=1}^m k_j \gamma^j,$$

$\nu(\gamma^*)$ equals the greatest common divisor of the integers $k_1, \ldots, k_m$. As a consequence, we see that the eigenfunction of T induced by $\sigma^{even}_{\gamma^*}$ has exactly $2\nu(\gamma^*)$ nodal domains. In this case we can even compute the *volume of the nodal set* since the geometry is so simple: we find with

$$L(\sigma^{even}_{\gamma^*}) := \mathrm{vol}\, N(\sigma_{\gamma^*}) \tag{12}$$

the relation

$$L(\sigma^{even}_{\gamma^*}) = \mathrm{vol}\, T(2|\gamma^*|) = \frac{\mathrm{vol}\, T}{\pi} \sqrt{\lambda(\gamma^*)}.$$

3 Eigenvalue estimates

The examples above are very special as we emphasized before. Therefore, one must be careful in generalizing phenomena observed there to more general mebranes like compact surfaces with negative curvature, for which the geodesic flow is known to be ergodic. In this section, we will discuss some results which are valid for all mebranes in order to see more clearly where our examples deviate from the generic structure. The first and certainly most important general result is due to Hörmander [16] for compact membranes and to Ivrii [18] for membranes with boundary, after a long history beginning with Hermann Weyl [30] in 1911. It concerns the eigenvalue asymptotics as exemplified in (10) and reads as follows.

Theorem 3 *For any compact membrane (M, g), we have the asymptotic relation*

$$N_\Delta(t) := \sum_{\lambda_n \le t} \mu(\lambda_n) = \frac{\operatorname{vol} B_1^m(0)}{(2\pi)^m} \operatorname{vol} M\, t^{m/2} + O(t^{(m-1)/2}). \tag{13}$$

We have seen that the sphere provides an example of a membrane where this estimate cannot be improved, but it is not known what the best possible remainder term looks like in other cases, like membranes with ergodic geodesic flow.

The asymptotic relation (13) leads to a relation between the eigenvalue and its number, to wit

$$n \sim \frac{\operatorname{vol} B_1^m(0)}{(2\pi)^m} \operatorname{vol} M\, \lambda_n^{m/2} + O(\lambda_n^{(m-1)/2}).$$

It is of interest to know whether these asymptotic relations can be turned into effective two-sided estimates, a result conjectured by Polya and proved in its probably most effective form - in terms of the dependence of the constants involbed on the geometric data – by Li and Yau [23]; in particular, one obtains the following generalization of a result by Faber and Krahn [21] for plane mebranes:

$$\operatorname{vol} M \ge C_{M,g} \lambda_1^{-m/2}. \tag{14}$$

The relation (13) does not tell us anything about the eigenfunctions, Hörmander's proof, however, does since it is based on the so called *spectral function* of Δ which is defined as

$$e_\Delta(p, q; t) := \sum_{\lambda_n \le t} \sigma_n(p) \bar{\sigma}_n(q). \tag{15}$$

In fact, Hörmander proves that this function satisfies the estimate (10), too, if the membrane is closed; the case with boundary is more complicated since the spectral function necessarily diverges near the boundary. Again, this universal estimate is sharp, with the sphere providing again a counterexample, since any improvement in the estimate of e_Δ implies the same improvement for N_Δ. For closed membranes, we easily deduce the pointwise estimate

$$\sup_{p \in M} |\sigma_n(p)| \le C_M \lambda_n^{(m-1)/4} ||\sigma_n||_{L^2(M,g)}. \tag{16}$$

This estimate can certainly be improved considerably for specific classes of membranes but the precise extent of this improvement remains largely unknown. For a thorough review of this question, see [19].

Coming back to the nodal sets, it has to be said that our knowledge is more restricted, mainly because eigenfunctions are much less accessible in general than eigenvalues. In two dimensions, the nodal lines are locally isometric to the nodal lines of harmonic polynomials, a fact apparently first proved by Bers [1]. In higher dimensions, no comparable results are known; but we do know that the $(m-1)$-dimensional Hausdorff measure of $N(\sigma_n)$ and the $(m-2)$-dimensional Hausdorff measure of its singularities, i.e. the set where also $d\sigma_n$ vanishes, are both finite (for more refined information, cf. [19, Sec. 2]).

The variational characterization of the eigenvalues in Theorem 2 allows some very useful conclusions which are due to Courant ([8, Ch. 6]). In their formulation, we denote by $B_\epsilon^X(p)$ the ball around p of radius ϵ in an arbitrary metric space (which is here the Riemannian manifold (M, g)).

Theorem 4 *(1) The union of all nodal sets is dense in M. More precisely, for some constant C_M and all $p \in M$ and $n \in N$,*

$$N(\sigma_n) \cap B^M_{C_M/\sqrt{\lambda_n}}(p) \neq \emptyset. \tag{17}$$

(2) For all $n \in N, \sigma_n$ has at most n nodal domains i.e.

$$NC(\sigma_n) \leq n.$$

We note that part 2 of the Theorem has been somewhat improved for flat membranes by Pleijel [28]. We have seen in Example 1, that $NC(\sigma_n) = n$ if $m = 1$, but in Example 3 we found that we may have $NC(\sigma_n) = 2$ for infinitely many n if M is a flat torus of dimension greater than one; the same fact has been established by Lewy for the 3-sphere [22]. At any rate, there does not seem to exist an easy correlation between nodal count and eigenvalue; this may seem disappointing at first sight, but also hints at a possibly interesting fluctuation of the nodal count for a fixed eigenvalue when the manifold is varied – which may encode finer geometric information; we will return to this issue in the last section of this review.

4 The geometry of nodal sets

Since the nodal count does not correlate with the eigenvalues we may return to our examples for inspiration; then it becomes apparent that the next likely candidate for such a correlation should be the volume of the nodal set, $L(\sigma_n)$. This was conjectured and proved in the two-dimensional case by Brüning and Gromes ([3,4]).

Theorem 5 *Let (M, g) be s smooth closed membrane in dimension two. Then there is a constant C_M such that*

$$L(\sigma_n) \geq C_M^{-1}\sqrt{\lambda_n}. \tag{18}$$

It is not hard to see that this estimate extends to membranes with boundary and suitable boundary value problems, and that it can be formulated with minimal smoothness. In our examples, we can establish with some work also upper estimates of the same type, that is,

$$L(\sigma_n) \leq C_M\sqrt{\lambda_n}, \tag{19}$$

but, so far, no such estimate could be established under the same natural smoothness assumptions, nor could any of these estimates be extended to higher dimensions. There are estimates, however, in terms of different functions of the eigenvalue, among which we mention here only the following result in the surface case, due to Donnelly and Fefferman [10].

Theorem 6 *If M is closed and $\dim M = 2$, then*

$$L(\sigma_n) \leq C_M\lambda_n^{3/4}.$$

If, however, the assumption of sufficient differentiability is replaced by the requirement that both the membrane and its metric be real-analytic, then the best possible estimate holds, as was shown by Donnelly and Fefferman [11].

Theorem 7 *If M is a real-analytic closed membrane with real-analytic metric g, then there is a constant C_M such that for all $n \in N$*

$$C_M^{-1}\sqrt{\lambda_n} \leq L(\sigma_n) \leq C_M\sqrt{\lambda_n}.$$

The analyticity is used here to exploit complex – analytic methods by analytic extension, notably a fairly straightforward upper estimate for the volume of the nodal set of a complex polynomial; the harder work consists in making explicit the analogy between σ_n and a polynomial in m complex variables of degree $\sqrt{\lambda_n}$ suggested by example 2 above. The said inequality rests on an integral geometric formula which asserts for a polynomial σ in C^m and $V := N(\sigma) \cap B_1(0)$ the identity

$$\mathcal{H}^{m-1}(V) = \int_{L\in\mathcal{L}} \sharp(V \cap L),$$

where $\sharp$ denotes the cardinality of a set, $\mathcal{L}$ is the (compact) space of lines in C^m, and $\mathcal{H}^{m-1}$ denotes $(m-1)$-dimensional Hausdorff measure.

Thus the upper estimate - which seems so elusive in the smooth case! - is quite plausible under analyticity assumptions. The lower bound, however, is very difficult to obtain, even under these stronger assumptions.

The volume is certainly only the simplest geometric invariant of the nodal set, and one would like to proceed and to analyze the curvature. This seems to be a very difficult task since already on a flat torus we can see that the curvature is not bounded in eigenspaces of high multiplicity. Consequently, almost nothing is known in this direction; the following curious result in two dimensions, due to Brüning [5], may be worth noting, though.

Theorem 8 *Assume that a membrane $M \subset R^2$ admits a sequence of eigenfunctions $(\sigma_n)_{n\in N}$ with the property that all nodal lines have constant curvature. Then M is contained in the following list:*

(1) sectors of circles,
(2) sectors of annuli,
(3) membranes that arise from a triangle with angles $(\pi/2, \pi/4, \pi/4)$ or $(\pi/2, \pi/3, \pi/6)$ by finitely many reflections in a side.

Another natural question would be to ask for the topological significance of nodal sets. As far as we can see, such results exist again only in two dimensions. A rather well developed research direction was initiated by Payne in [27] when he asked whether the second eigenfunction of a plane membrane can be closed, and gave an negative answer in the case that the membrane is, in addition, symmetric with respect to the coordinate axes. The question was settled in the negative, after several intermediate steps, by Melas 1992 [24] for convex membranes while a counterexample in the non simply connected case was provided in [17].

Another interesting question arises from the fact that the multiplicities of the eigenvalues are bounded by the topology of M in two dimensions, but not in any higher dimension. We can therefore ask for the extremal metrics g on a surface M which maximize the multiplicity of any given eigenvalue; cf. [19] for a discussion of work on this problem.

5 Isospectrality and the nodal count

In 1966, Mark Kac [20] posed the now famous question "Can one hear the shape of a drum?" His formulation paraphrases the intuitive idea that the fundamental frequencies of a drum, or of any other vibrating system, should characterize it up to isometries, that from "hearing" the system we can reconstruct its shape. In spite of its immediate appeal, the point that Kac made was by no means new but a rather inevitable consequence of the success of Gustav Kirchhoff's spectral analysis, the prototypical inverse problem, cf. for example the report given by Sir Arthur Schuster to the British Association in 1882, as quoted in [12, p. XI].

The answer to the celebrated question is also long known to be negative; with a counter example in John Milnor's paper of 1964 [25] began a long list of articles which provide counterexamples to the inverse spectral problem for membranes, in pointing out membranes $(M_i, g_i)_{i\in I}$ which are *isospectral* i.e. have the same spectrum for the Laplace-Beltrami operator but are mutually not isometric. The first general construction (of pairs) was given by Sunada [29], the first examples parametrized by a continuum by DeTurk and Gordon [9].

These examples tend to disguise the expectation that, generically, isospectrality should imply isometry. A precise statement in this direction, however, seems to be available so far only in two dimensions, cf. [26], so it makes sense to look for additional spectral data which might imply isospectrality directly. In this direction, Smilansky has proposed to use the nodal count as additional information [2], and together with his coworkers he has corroborated this conjecture in various examples (cf. e.g. [15]). As we have seen, the high fluctuations in the nodal count indicate that there could be indeed a classifying potential, the extent of which is certainly worth exploiting. These authors even proposed that the nodal count alone would characterize a suitable class of systems, i.e. that one "can count the shape of a drum" [13]. In spite of the appeal of this formulation, it runs into trouble in the framework of membranes as soon as eigenvalues degenerate, requiring an "uncountable nodal count" since all eigenfunctions ought to be considered. This case occurs, as we know, notably for the flat tori (the class of membranes where Milnor's counterexample comes from), and a beautiful test case is provided by two mutually isospectral four-parameter families of flat tori in four dimensions, constructed by Conway and Sloane [7]. Smilansky and [14] have analyzed some members of this family numerically, proposing a different way of counting nodal domains which avoids multiplicities and considers only the basic functions (8). Brüning and Klawonn [6] have taken up this question and have shown that this way of counting actually distinguishes the two families while the knowledge of the spectrum and the true nodal count of the basic functions, that is, the knowledge of the numbers (cf. (8) and (10))

$$\{|\gamma^*|, \nu(\gamma^*) : \gamma^* \in \Gamma^*\}$$

will not distinguish them. In order to achieve this, they go on and introduce certain extremal values of the nodal count in each eigenspace which allow a tractable algebraic representation and hence can be seen to distinguish the lattices definig the tori.

Coming back to Chladni, whose memory is honoured by this conference, we recall that he became famous in his day for making people "see the sound". To paraphrase the result of Brüning and Klawonn, we are hence tempted to state (or rather: conjecture) that we can deduce the shape of a flat torus from hearing it and seeing its Klangfiguren (in Euclidean space).

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